Серия «Математика»
2026. Т. 56. С. 145—159

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 512.554
MSC 12K99, 15A04, 17A35, 17D99
DOI https://doi.org/10.26516/1997-7670.2026.56.145

Dihedral and Quaternion Autotopism Subgroups of Semifield Projective Planes of Order p^4

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Abstract: In 1959, D.R. Hughes conjecture that the full collineation group of any finite non-Desarguesian semifield projective plane is solvable (see also the question 11.76 by N.D. Podufalov in Kourovka notebook). The spread set method is useful to exclude some simple non-Abelian groups from the list of possible autotopism subgroups (collineations fixing a triangle) or for constructing the examples of semifield planes with certain autotopism subgroup. The present paper continues the series of results on 2-subgroups and 2-elements in an autotopism group. The natural restrictions from previous papers allow us to complete the description of dihedral and quaternion autotopism subgroup of order 8, together with their geometrical sense. For a semifield projective plane of odd order and 4-dimensional over the center, the matrix representation of the spread set is determined, depending on the characteristic of prime field. It is proven that the dihedral autotopism group of order 8 necessarily contains the perspectivities and therefore cannot be a subgroup of any simple non-Abelian group. For the case of a quaternion subgroup without perspectivities, examples of semifield projective planes of order 81 and 2401 are constructed, up to isomorphism. The list of exceptions complements the classical results of H. Lüneburg etc. on projective special linear collineation groups. The method used and the described algorithms allow for generalization to the case of a different dimension or a different order.

Keywords: semifield plane, semifield, spread set, autotopism group, quaternion group, dihedral group, Hughes problem

Acknowledgements: This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2026-1314).

For citation: Kravtsova O. V., Skok D. S. Dihedral and Quaternion Autotopism Subgroups of Semifield Projective Planes of Order p^4. *The Bulletin of Irkutsk State University. Series Mathematics*, 2026, vol. 56, pp. 145–159.
https://doi.org/10.26516/1997-7670.2026.56.145

Научная статья

Диэдральные и кватернионные подгруппы автотопизмов полуполевых проективных плоскостей порядка p^4

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Аннотация: Отмечается, что в 1959 г. Д. Р. Хьюз выдвинул предположение, что полная группа коллинеаций всякой конечной недезарговой полуполевой проективной плоскости разрешима (см. также вопр. 11.76 Н. Д. Подуфалова в «Коуровской тетради»). Метод регулярного множества предлагается использовать для исключения некоторых простых неабелевых групп из списка возможных подгрупп автотопизмов (коллинеаций, фиксирующих треугольник). Настоящая статья продолжает цикл результатов о 2-подгруппах и 2-элементах в группе автотопизмов. Естественные ограничения, полученные ранее, позволяют завершить описание диэдральных и кватернионных подгрупп порядка 8 в группе автотопизмов, с указанием их геометрического смысла. Получено матричное представление регулярного множества для полуполевой проективной плоскости нечетного порядка и размерности 4 над центром, в зависимости от характеристики его простого подполя. Доказано, что диэдральная подгруппа автотопизмов порядка 8 обязательно содержит перспективности и поэтому не может быть подгруппой никакой простой неабелевой группы. Для случая кватернионной подгруппы автотопизмов без перспективностей построены примеры полуполевых проективных плоскостей порядков 81 и 2401 с точностью до изоморфизма. Список исключений дополняет классические результаты Х. Люнебурга и других о проективных специальных линейных подгруппах коллинеаций. Используемый метод и описанные алгоритмы допускают обобщение на случай другой размерности или другого порядка.

Ключевые слова: полуполевая плоскость, полуполе, регулярное множество, группа автотопизмов, группа кватернионов, группа диэдра, проблема Хьюза

Благодарности: Работа поддержана Красноярским математическим центром, финансируемым Минобрнауки РФ (Соглашение 075-02-2026-1314).

Ссылка для цитирования: Kravtsova O. V., Skok D. S. Dihedral and Quaternion Autotopism Subgroups of Semifield Projective Planes of Order p^4 // Известия Иркутского государственного университета. Серия Математика. 2026. Т. 56. С. 145–159.
https://doi.org/10.26516/1997-7670.2026.56.145

1. Introduction

A classical problem in projective geometry is the description of the collineation group (automorphisms) of a finite projective plane, as well as the classification of planes of fixed order that admit a certain collineation group (for translation planes, see [5]). The first significant result concerning projective planes of order n admitting a collineation group isomorphic to $PSL(2,q)$ was obtained by H. Lüneburg in 1964; it was a characterization of Desarguesian projective planes of order $n=q$ [9]. In 1989, G.E. Moorhouse enumerated [12] projective planes that admit $PSL(2,q)$ for $n<q$. In a 2007 survey, A. Montinaro [11] provided a solution for $n \leq q^2$ along with some examples of translation planes admitting $PSL(2,q)$.

Closely related to this problem is D. Hughes' 1959 conjecture [3] on the solvability of the collineation group of a finite non-Desarguesian semifield plane (Question 11.76 by N.D. Podufalov in the Kourovka Notebook). All currently known projective planes coordinatized by non-associative finite semifields possess this property. We propose an approach based on the classification of finite simple groups and J. Thompson's theorem on minimal simple groups. The spread set method allows us to determine conditions for the existence of a semifield plane with a given autotopism subgroup (collineations fixing a triangle) and to construct examples, including computational ones. By excluding certain simple non-Abelian groups from possible autotopism subgroups, we can make significant progress in solving this problem.

The results on projective linear groups imply that a non-Desarguesian semifield plane of order n does not admit a collineation group isomorphic to $PSL(2,q)$, $q \geq n$. In previous publications by the first author, other useful results have been proved on the autotopism group Λ of a non-Desarguesian semifield projective plane of order p^N, where p is prime and $N=2^m \cdot s$ with s odd. In particular, for $p>2$ group Λ contains no subgroups isomorphic to $PSL(2,2^{2n})$ and $PSL(2,5^n)$. For $p \equiv 1 \pmod 4$ it contains no subgroups isomorphic to $PSL(2,q)$, where $q \equiv 1 \pmod{2^{m+2}}$ or $q \not\equiv \pm 3 \pmod 8$.

These results are based [6] on the boundedness of the order of cyclic or elementary Abelian autotopism 2-subgroup without central collineations. The spread set approach involves first constructing matrix representations of autotopism subgroups isomorphic to the dihedral group D_8 or the quaternion group Q_8 of order 8. For $p \equiv 1 \pmod 4$ such a subgroup necessarily contains central collineations [6; 7] and thus cannot be embedded in any simple non-Abelian autotopism subgroup.

The case $p \equiv -1 \pmod 4$ requires special consideration due to the different geometric meaning of an autotopism of order 4. In [8], the authors derive matrix representations for generators of autotopism subgroups isomorphic to D_8 or Q_8. For dimension $N \not\equiv 0 \pmod 4$, this excludes most simple non-Abelian groups as possible autotopism subgroups, except possibly: $PSL(2,2^n)$, $n \geq 2$, $PSU(3,2^n)$, $n \geq 2$, $Sz(2^n)$, n is odd, $n>1$, $PSL(2,q)$, $q \equiv \pm 3 \pmod 8$, J_1 or ${}^2G_2(3^n)$, n is odd, $n>1$ (due to [2]).

This paper focuses on the case $|\pi| = p^4$, $p \equiv -1 \pmod 4$, aiming to construct matrix representations of spread sets for semifield planes with autotopism subgroups $H \simeq D_8$ or $H \simeq Q_8$. We show (theorem 1) that the first case is impossible. For the second case (theorem 2) the existence is confirmed by examples of orders 3^4 and 7^4. Note that for "good" characteristics, examples of semifield planes of orders 5^4 and 13^4 were constructed by the first author earlier. For $H \simeq Q_8$ we exclude the existence of a subgroup $F \simeq SL(2,3)$ without central collineations.

2. Main definitions and preliminary discussion

The study of finite semifields and semifield planes started more than a century ago with the first examples constructed by L.E. Dickson [1]. A *semifield* (term from 1965) is called a non-associative ring $Q = (Q, +, \cdot)$ with identity where the equations $ax = b$ and $ya = b$ are uniquely solved for any $a, b \in Q$, $a \neq 0$. The absence of an associative law in a semifield leads to a number of anomalous properties in comparison with a field or a skewfield or even a near-field. Moreover, the coordinatization of points and lines of a finite projective plane by the semifield elements provides special geometric properties.

We use main definitions, according [4; 5], see also [6], for more detail.

Consider a linear space W, n-dimensional over the finite field $GF(q)$ and the subset of linear transformations $R \subset GL_n(q) \cup \{0\}$ such that:

1) R consists of q^n square $(n \times n)$-matrices over $GF(q)$;

2) R contains the zero matrix 0 and the identity matrix E;

3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a nonsingular matrix.

The set R is called a *spread set* [4]. Consider a bijective mapping θ from W onto R and present the spread set as $R = \{\theta(m) \mid m \in W\}$. Define the multiplication on W by the rule $x * m = x \cdot \theta(m)$ $(x, m \in W)$. Then $\langle W, +, * \rangle$ is a right quasifield of order q^n [4]. Moreover, if R is closed under addition then $\langle W, +, * \rangle$ is a semifield.

To construct and study finite semifields, we use a prime field $\mathbb{Z}_p$ as a basic field, p is prime. In this case the mapping θ is presented using only linear functions; it greatly simplifies reasoning and calculations (also computer).

A semifield W coordinatizes the semifield projective plane π of order $p^n = |W|$ such that:

1) the affine points are the elements (x, y) of the space $W \oplus W$;

2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in W\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in W\} \quad (m \in W);$$

3) the set of all cosets to the subgroup is the singular point;

4) the set of all singular points is the singular line;

5) the incidence is set-theoretical.

The finite plane π is Desarguesian (classical) if W is a field, then $R \simeq W \simeq GF(p^n)$.

The solvability of a collineation group $Aut\,\pi$ for a semifield plane is reduced [4] to the solvability of an autotopism group Λ (collineations fixing a triangle). Without loss of generality, we can assume that autotopisms are determined by linear transformations of the space $W \oplus W$:

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

here the matrices A and B satisfy the *collineation condition* (see also [10])

$$A^{-1}\theta(m)B \in R \quad \forall\theta(m) \in R. \tag{2.1}$$

To shorten the record, we will often use the notation $diag(A, B)$ for the block-diagonal matrix. Note that throughout the article, the blocks-submatrices have the same dimension by default.

The collineations fixing a closed Baer subset have special properties. It is well known [4], that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central* (or *perspectivity*), if it fixes a line pointwise (*the axis*) and a point linewise (*the center*). If the center is incident to the axis then a collineation is called *an elation*, and *a homology* in another case. All the perspectivities in an autotopism group are homologies and form the normal cyclic subgroups [6]:

$$\begin{aligned} H_r &= \{diag(E, M) \mid M \in R_r^*\}, \\ H_m &= \{diag(M, E) \mid M \in R_m^*\}, \\ H_l &= \{diag(M, M) \mid M \in R_l^*\}. \end{aligned}$$

The matrix subsets R_l, R_m, R_r are defined by a spread set:

$$\begin{aligned} R_l &= \{M \in GL_n(p) \cup \{0\} \mid M\theta(m) = \theta(m)M \ \forall\theta(m) \in R\}, \\ R_m &= \{M \in R \mid M\theta(m) \in R \ \forall\theta(m) \in R\}, \\ R_r &= \{M \in R \mid \theta(m)M \in R \ \forall\theta(m) \in R\}, \end{aligned}$$

they are called *left, middle and right nuclei* of the plane π respectively [4].

An autotopism group of a semifield plane of odd order contains three involutory homologies (clear, in the center of Λ):

$$h_1 = diag(-E, E), \quad h_2 = diag(E, -E), \quad h_3 = diag(-E, -E). \tag{2.2}$$

A collineation of a projective plane π of order m is called *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|} = \sqrt{m}$ (*Baer subplane*). We use the following results on the matrix representation of a Baer involution $\tau \in \Lambda$ and of a spread set obtained earlier, for more detail, see [6].

Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ is prime). If its autotopism group Λ contains the Baer involution τ then $N = 2n$ is even and we can choose the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \tag{2.3}$$

where the matrix $L \in GL_{2n}(p)$ is block-diagonal $L = diag(-E, E)$. The spread set R in $GL_{2n}(p) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}, \tag{2.4}$$

where $V \in Q$, $U \in K$, Q, K are the spread sets in $GL_n(p) \cup \{0\}$ (but maybe $E \notin Q$), m, f are additive injective functions from K and Q into $GL_n(p) \cup \{0\}$, $m(E) = E$.

3. Matrix representation of spread sets

Theorem 1. *Let π be a non-Desarguesian semifield plane of order p^4, where $p > 2$ is prime, and let Λ be its autotopism group. Then any dihedral subgroup $H < \Lambda$ of order 8 necessarily contains central collineations.*

For $p \equiv 1 \pmod 4$, this statement was proved in [7]. For the remaining case $p \equiv -1 \pmod 4$ this proof approach is inapplicable due to the particular geometric meaning of autotopisms of order 4, see [6]. We shall find the matrix representation of the spread set for the plane π in this case.

Lemma 1. *Let π be a non-Desarguesian semifield plane of order p^4, where $p \equiv -1 \pmod 4$ is prime, and let its autotopism subgroup H without homologies is isomorphic to D_8. Then the base of the 4-dimensional linear space over $\mathbb{Z}_p$ can be chosen such that the spread set $R \subset GL_4(p) \cup \{0\}$ consists of matrices ($a, b, c, d, j \in \mathbb{Z}_p$)*

$$\theta(x, y, z, t) = \begin{pmatrix} t & az & by & cx \\ -az & t & -bx & cy \\ jy & -jx & t & dz \\ x & y & z & t \end{pmatrix}, \quad x, y, z, t \in \mathbb{Z}_p.$$

Proof. As proved in [8] for $|\pi| = p^4$, the subgroup H can be generated by the autotopism α of order 4 and the Baer involution β:

$$\alpha = diag(S, E, S, E), \qquad \beta = diag(L, L, L, L),$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Here $\tau = \alpha^2 = diag(L, L)$ (2.3) is a Baer involution. The spread set matrices can be written, therefore, as (2.4), with $K \simeq GF(p^2)$.

The collineation condition (2.1) allows us to determine all submatrices. Since α and β are collineations, the products

$$\begin{pmatrix} S & 0 \\ 0 & E \end{pmatrix}^{-1} \theta(V,U) \begin{pmatrix} S & 0 \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}^{-1} \theta(V,U) \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$$

must belong to the spread set R for any $V \in Q$ and $U \in K$. Matrix multiplication yields:

a) $LVL \in Q$,
b) $VS \in Q$,
c) $f(LVL) = Lf(V)L$,
d) $f(VS) = -Sf(V), \ \forall V \in Q$;

e) $LUL \in K$,
f) $m(LUL) = Lm(U)L$,
g) $Sm(U) = m(U)S, \ \forall U \in K$.

From the conditions (a) and (e), respectively, we have the form of blocks V and U:

$$V = \begin{pmatrix} \nu_1(y) & \nu_2(x) \\ x & y \end{pmatrix}, \quad U = \begin{pmatrix} \mu_1(t) & \mu_2(z) \\ z & t \end{pmatrix}, \quad x, y, z, t \in \mathbb{Z}_p. \tag{3.1}$$

Since ν_i, μ_i are linear functions, with $VS \in Q$ and $E \in K$, we obtain:

$$V = \begin{pmatrix} jy & -jx \\ x & y \end{pmatrix}, \quad U = \begin{pmatrix} t & dz \\ z & t \end{pmatrix}. \tag{3.2}$$

Note that the coefficient d of the linear function $\mu_2(z)$ is necessarily not a square in $\mathbb{Z}_p$, since any nonzero matrix U must be non-degenerate.

Let us express the block $m(U)$ as:

$$m(U) = \begin{pmatrix} m_1(z,t) & m_2(z,t) \\ m_3(z,t) & m_4(z,t) \end{pmatrix} \tag{3.3}$$

From the condition (f) we have:

$$m_i(z,t) = m_i(-z,t), \ i = 1,4; \qquad -m_k(z,t) = m_k(-z,t), \ k = 2,3,$$

and, given linearity and $m(E) = E$, it follows to $U = \begin{pmatrix} t & az \\ \overline{a}z & t \end{pmatrix}$. Condition (g) gives $\overline{a} = a$. Similarly for block $f(V)$, from (c) we get:

$$f(V) = \begin{pmatrix} f_1(y) & f_2(x) \\ f_3(x) & f_4(y) \end{pmatrix}, \tag{3.4}$$

and from (d): $f_1(y) = by$, $f_2(x) = cx$, $f_3(x) = -bx$, $f_4(y) = cy$. Additionally, all the blocks are either zero or non-degenerate. □

To complete the proof of the theorem 1, we show that among the nonzero matrices of the obtained set R, there are necessarily degenerate ones. Computing the determinant:

$$\det\,\theta(x,y,z,t) = t^4 + bcjx^4 - ct^2x^2 - bjt^2x^2 + bcjy^4 - ct^2y^2 - \\ - bjt^2y^2 + 2bcjx^2y^2 - a^2dz^4 + a^2t^2z^2 - dt^2z^2 + abdx^2z^2 - \\ - acjx^2z^2 + abdy^2z^2 - acjy^2z^2.$$

Let $M = x^2 + y^2$, then:

$$\det\,\theta(x,y,z,t) = bcjM^2 + M(abdz^2 - acjz^2 - bjt^2 - ct^2) + (t^2 - z^2d)(t^2 + a^2z^2).$$

Let us consider the case $z = 0$ and decompose this quadratic polynomial into factors:

$$\det\,\theta(x,y,0,t) = bcj\left(M - \frac{t^2}{bj}\right)\left(M - \frac{t^2}{c}\right).$$

As is known, in a finite field of odd characteristic, any element can be represented as the sum of two squares. Therefore, choosing an arbitrary $t \neq 0$, we can find x and y so that

$$M = x^2 + y^2 = \frac{t^2}{bj} \quad \text{or} \quad M = x^2 + y^2 = \frac{t^2}{c}.$$

For the chosen nonzero row $(x,y,0,t)$, the determinant of the matrix vanishes, so R cannot be a spread set of the semifield plane π. The resulting contradiction proves the theorem.

Thus, the proved result completely excludes for the semifield plane of order p^4 the situation when the Sylow 2-subgroup of autotopisms $\Sigma < \Lambda$ contains two non-commuting involutions. The bound on the number of commuting Baer involutions in Λ was proved earlier: for $|\pi| = p^4$ the order of an elementary Abelian 2-subgroup without homologies does not exceed 4 (for more detail, [6]). Taking into account the presence in Λ of homologies h_1, h_2, $h_3 = h_1h_2$ of order 2 from the center of the group, we have the corollary.

Corollary 1. *If Σ is a Sylow 2-subgroup of the autotopism group Λ for a non-Desarguesian semifield plane π of order p^4, where $p > 2$ is prime, then its elementary abelian 2-subgroup $F < \Sigma$ has order 4, 8 or 16, with $|F \cap Z(\Lambda)| = 4$.*

Now consider the autotopism subgroup $H \simeq Q_8$, where the only involution is τ. For $p \equiv 1 \pmod 4$ this involution must be a homology [8]. For $p \equiv -1 \pmod 4$, $\tau \in H$ could be Baer. Using the result [8] on the matrix representation of the subgroup H generating elements, we find the matrix representation of the spread set.

Theorem 2. *Let π be a non-Desarguesian semifield plane of order p^4, $p \equiv -1 \pmod 4$ is prime, the autotopism subgroup H is isomorphic to Q_8 and does not contain homologies. Then the base of the 4-dimensional linear space over $\mathbb{Z}_p$ can be chosen such that the spread set $R \subset GL_4(p) \cup \{0\}$ consists of matrices ($a, b, c, d, j \in \mathbb{Z}_p$)*

$$\theta(x,y,z,t) = \begin{pmatrix} t & az & bx & cy \\ -az & t & by & -cx \\ jy & -jx & t & dz \\ x & y & z & t \end{pmatrix}, \quad x,y,z,t \in \mathbb{Z}_p.$$

Proof. According to the results of [8], the subgroup $H \simeq Q_8$ can be generated by autotopisms α and β of order 4:

$$\alpha = diag(S,E,S,E), \qquad \beta = diag(C,L,C,L), \tag{3.5}$$

where S and L are as above, $C^2 = -E$, $CS = SC$, $\alpha^2 = \beta^2 = \tau$. It's not hard to show that for $|\pi| = p^4$ the matrix $C = \begin{pmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{pmatrix}$, $c_1^2 + c_2^2 = -1$.

Repeating the reasoning from the proof of Theorem 1, from the collineation condition for α and β we obtain:

a) $LVC \in Q$,
b) $VS \in Q$,
c) $f(LVC) = -Cf(V)L$,
d) $f(VS) = -Sf(V)$, $\forall V \in Q$;
e) $LUL \in K$,
f) $m(LUL) = -Cm(U)C$,
g) $Sm(U) = m(U)S$, $\forall U \in K$.

From conditions (a) and (e), respectively, similarly to the previous one, we have the form (3.1) of the blocks V and U. Further, from the linearity of the functions ν_i, μ_i and the conditions $VS \in Q$ and $E \in K$ we again see (3.2). For the block $m(U)$ of the form (3.3), from conditions (f) and (g) we obtain: $m(U) = \begin{pmatrix} t & az \\ -az & t \end{pmatrix}$. For $f(V)$, unlike (3.4), the condition (c) gives $f(V) = \begin{pmatrix} bx & cy \\ by & -cx \end{pmatrix}$. □

Remark 1. Note that the coefficient d of the linear function $\mu_2(z)$ is necessarily not a square in $\mathbb{Z}_p$, since any nonzero matrix U must be nondegenerate. Since -1 is not a square in $\mathbb{Z}_p$ and non-squares form a coset in the multiplicative group of the field, then we will write $-d^2$ instead of d, now assuming that the element $d \in \mathbb{Z}_p$ is arbitrary:

$$\theta(x,y,z,t) = \begin{pmatrix} t & az & bx & cy \\ -az & t & by & -cx \\ jy & -jx & t & -d^2z \\ x & y & z & t \end{pmatrix}, \quad x,y,z,t \in \mathbb{Z}_p. \tag{3.6}$$

It should be noted that in the case of $H \simeq Q_8$ this subgroup cannot be contained in $F \simeq SL(2,3)$. Indeed, in this case there would exist an autotopism γ given by the block-diagonal matrix $diag(D_1, D_2, D_3, D_4)$, commuting with the involution τ and satisfying the condition $\gamma^{-1}\alpha\gamma = \beta$, and for matrices (3.5) this is obviously impossible.

Corollary 2. *Let π be a non-Desarguesian semifield plane of order p^4, $p \equiv -1 \pmod 4$ is prime, the autotopism subgroup F does not contain homology. Then F cannot be isomorphic to $SL(2,3)$.*

4. Examples

We now construct examples of semifield planes of order p^4 that admit the autotopism subgroup $H \simeq Q_8$ without homologies.

As noted in [8], the complete list of semifield planes of order 81 admitting a Baer involution contains exactly one plane with a homology-free autotopism subgroup $H \simeq Q_8$, and none with $H \simeq D_8$, consistent with Theorem 1. The spread sets for order-81 examples were previously written in an another form, so we recomputed them for matrices $\theta(x, y, z, t)$ (3.6), for both $p = 3$ and $p = 7$. Computer calculations show that in the first case there are 8 coefficient rows (a, b, c, d, j):

$$(1,1,2,1,1), \quad (1,1,2,2,1), \quad (1,2,1,1,1), \quad (1,2,1,2,1),$$
$$(2,1,1,1,2), \quad (2,1,1,2,2), \quad (2,2,2,1,2), \quad (2,2,2,2,2),$$

in the second we obtain 792 tuples. Note that distinct parameter rows may correspond to isomorphic planes, so we must to classify them into isomorphism classes. It is clear that for $p = 3$ we should end up with unique class.

The first step in solving the isomorphism issue will be to calculate the nuclei orders for the constructed semifield planes.

Lemma 2. *Let π be a semifield plane of order p^4, $p \equiv -1 \pmod 4$ with the spread set (3.6). The orders of its right, middle, and left nuclei are determined as follows.*

1. *If $a^2 – d^2 = 0$, $ab + c = 0$, then $|R_r| = p^2$, otherwise $|R_r| = p$.*
2. *If $a^2 - d^2 = 0$, $a = j$, then $|R_m| = p^2$, otherwise $|R_m| = p$.*
3. *If $j^2 – d^2 = 0$, $jb + c = 0$, then $|R_l| = p^2$, otherwise $|R_l| = p$.*

Proof. It will be convenient for us to write down an arbitrary matrix of a spread set as a decomposition $\theta(x, y, z, t) = xX + yY + zZ + tE$, where

$$X = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & -c \\ 0 & -j & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & b & 0 \\ j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -d^2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We detail the proof for the right nucleus (others follow similarly). If M is a matrix from the right nucleus R_r, then for any matrix from R all products $\theta(x,y,z,t)M$ must remain in R. It is sufficient to check this condition only for the base elements X, Y, Z, E of the spread set: $XM, YM, ZM, M \in R$.

Let's write M as $xX + yY + zZ + tE$ and calculate the product XM:

$$XM = \begin{pmatrix} bjy & -bjx & bt & -bd^2z \\ -cx & -cy & -cz & -ct \\ jaz & -jt & -jby & jcx \\ t & az & bx & cy \end{pmatrix} = \begin{pmatrix} cy & abx & bt & caz \\ -abc & cy & baz & -ct \\ jaz & -jt & cy & -d^2bx \\ t & az & bx & cy \end{pmatrix},$$

because it is $\theta(t, az, bx, cy) \in R$. For the corresponding elements, we obtain the equations:

$$\begin{aligned} &(ab+bj)x = 0, \\ &(c-ab)x = 0, \\ &(jc+bd^2)x = 0, \end{aligned} \qquad \begin{aligned} &(ac+bd^2)z = 0, \\ &(ab+c)z = 0, \end{aligned} \qquad \begin{aligned} &(bj-c)y = 0, \\ &2cy = 0, \\ &(c+bj)y = 0. \end{aligned}$$

Since $c \neq 0$, then $y = 0$, and only the first five conditions remain. For the product YM with $y = 0$ we similarly obtain $x = 0$ and $(ac - bd^2)z = (ab+c)z = 0$, and for the product ZM with $x = y = 0$ we add the condition $(a^2 - d^2)z = 0$. Consequently, if at least one of the elements $a^2 - d^2$, $ab + c$, $ac + bd^2$ is nonzero, then $z = 0$ and $M = \theta(0,0,0,t) = tE$, then $R_r \simeq \mathbb{Z}_p$. When $a^2 - d^2 = ab + c = ac + bd^2 = 0$ we obtain $M = \theta(0,0,z,t)$, $|R_r| = p^2$.

For the matrix M from the middle kernel R_m we check the conditions $MX, MY, MZ, M \in R$, and the conditions $MX = XM$, $MY = YM$, $MZ = ZM$ for $M \in R_l$. □

For order-81 planes, calculation result is consistent with [8]: all nuclei have order $p^2 = 9$. For order 7^4 results are summarized in Table 1.

We can divide any class K_i into isomorphic subclasses, taking into account that the base replacement with the transition matrix $diag(T, W)$ makes the spread set R to the form $T^{-1}RW$ (for more details, see [10]). We will choose the transition matrix so that the subgroup H is preserved. Therefore, we need to find the centralizer and normalizer of H in the group $GL_8(p)$. Let us present the reasoning for the centralizer.

Table 1

Class	$\lvert R_r\rvert$	$\lvert R_m\rvert$	$\lvert R_l\rvert$	Number of spread sets
K_1	p^2	p^2	p^2	72
K_2	p^2	p	p	144
K_3	p	p^2	p	144
K_4	p	p	p^2	144
K_5	p	p	p	288

Since the matrix $diag(T, W)$ must commute with $\alpha^2 = \beta^2 = \tau$, it is block-diagonal of the form $diag(T_1, T_2, W_1, W_2)$. The commutativity imply the equalities $T_1S = ST_1$, $W_1S = SW_1$, $T_1C = CT_1$, $W_1C = CW_1$, $T_2L = LT_2$, $W_2L = LW_2$. From the first four conditions, we obtain scalar matrices $T_1 = t_1E$, $W_1 = w_1E$, and from the last two the diagonal matrices $T_2 = diag(t_2, t_3)$, $W_2 = diag(w_2, w_3)$. Thus, the centralizer of the subgroup H is

$$C(H) = \{diag(t_1, t_1, t_2, t_3, w_1, w_1, w_2, w_3) \mid t_i, w_i \in \mathbb{Z}_p^*,\ i = 1, 2, 3\}.$$

Let us compose the matrix of the new spread set as $T\theta(x, y, z, t)W$, replacing T^{-1} with T to simplify the notation (it possibly due to arbitrariness):

$$\begin{pmatrix} t_1tu_1 & t_1azu_1 & t_1bxu_2 & t_1cyu_3 \\ -t_1azu_1 & t_1tu_1 & t_1byu_2 & -t_1cxu_3 \\ t_2jyu_1 & -t_2jxu_1 & t_2tu_2 & -t_2d^2zu_3 \\ t_3xu_1 & t_3yu_1 & t_3zu_2 & t_3tu_3 \end{pmatrix} = \theta'(x', y', z', t'),$$

where $x' = t_3xu_1$, $y' = t_3yu_1$, $z' = t_3zu_2$, $t' = t_3tu_3$. Comparing the coefficients a', b', c', d', j' of the matrix of the new spread set with the corresponding coefficients for the original plane, we obtain:

$$a' = a\,\frac{t_2}{t_3},\ b' = b\,\frac{t_1}{t_2}\,\frac{t_1}{t_3},\ c' = c\,\left(\frac{t_1}{t_3}\right)^2,\ d'^2 = d^2\,\left(\frac{t_2}{t_3}\right)^2,\ j' = j\,\frac{t_2}{t_3}.$$

Let's make the substitution $t_2/t_3 = u$, $t_1/t_3 = v$ and rewrite the result.

Lemma 3. *Let π and π' be the semifield planes of order p^4 with the spread sets (3.6) for the coefficient rows (a, b, c, d, j) and (a', b', c', d', j'), respectively. If there exist nonzero elements $u, v \in \mathbb{Z}_p$ such that*

$$a' = au, \quad b' = b\,\frac{v^2}{u}, \quad c' = cv^2, \quad d'^2 = d^2u^2, \quad j' = ju,$$

then the planes π and π' are isomorphic.

Direct comparison of the coefficients in the case $p = 3$ shows the isomorphism of all the constructed planes. For the case $p = 7$, the classes K_i from the Table 1 are divided into 22 isomorphism classes, with 36 spread sets in each. The final list is presented in Table 2 with the choice of one of the class representatives.

Remark 2. By the conditions of lemma 3, we may assume without loss of generality that, for instance, the coefficient a (or the coefficient j) in (3.6) is equal to 1. Further, we can assume that the coefficient c is equal to 1 if it is a square, and -1 otherwise.

Clearly, a complete classification up to isomorphism requires determining the transition matrices $diag(T, W)$ from the normalizer of the subgroup H. We omit these computations, as extensive calculations in our cases yield the same result.

Table 2

Class	Plane	a	b	c	d	j	Class	Plane	a	b	c	d	j
K_1	π_1	1	1	6	1	1	K_4	π_{11}	1	1	6	1	2
K_1	π_2	1	6	1	1	1	K_4	π_{12}	1	1	6	1	4
K_2	π_3	1	2	6	3	4	K_4	π_{13}	1	6	1	1	2
K_2	π_4	1	4	6	2	2	K_4	π_{14}	1	6	1	1	4
K_2	π_5	1	3	1	2	2	K_5	π_{15}	1	2	6	2	1
K_2	π_6	1	5	1	3	4	K_5	π_{16}	1	4	6	3	2
K_3	π_7	1	2	6	1	1	K_5	π_{17}	1	1	6	2	4
K_3	π_8	1	4	6	1	1	K_5	π_{18}	1	1	6	3	1
K_3	π_9	1	3	1	1	1	K_5	π_{19}	1	3	1	3	2
K_3	π_{10}	1	5	1	1	1	K_5	π_{20}	1	5	1	2	1
							K_5	π_{21}	1	6	1	2	4
							K_5	π_{22}	1	6	1	3	1

Theorem 3. *There exists a unique semifield plane of order 81 whose autotopism group contains a perspectivity-free subgroup isomorphic to Q_8. There exist exactly 22 pairwise non-isomorphic semifield planes of order 7^4 satisfying this condition.*

5. Conclusion

If D. Hughes' assumption on the solvability of the autotopism group for any non-trivial finite semifield is incorrect, then the composition series of the autotopism group must contain simple non-Abelian factors. Having excluded the possibility of the existence of a simple non-Abelian autotopism subgroup, we will have to proceed further to the development of computational technique for the quotient-group. Given the complexity of the problem, it seems natural to continue searching for solutions in the case of the smallest possible dimension over the center, that is, for the order p^4 of a semifield plane.

References

1. Dickson L.E. Linear algebras in which division is always uniquely possible. *Trans. Amer. Math. Soc.*, 1906, vol. 7, no. 3, pp. 370–390. https://doi.org/10.1090/S0002-9947-1906-1500755-5

2. Goldschmidth D.M. 2-fusion in finite groups. *Ann. Math.*, 1974, vol. 99, no. 1, pp. 70–117.
3. Hughes D.R. Review of some results in collineation groups. *Proc. Sympos. Pure Math., American Mathematical Society, Providence, R. I.*, 1959, vol. 1, pp. 42–55.
4. Hughes D.R., Piper F.C. *Projective planes.* New York, Springer-Verlag, 1973, 292 p.
5. Johnson N.L., Jha V., Biliotti M. *Handbook of finite translation planes.* London, Chapman and Hall, 2007, 861 p. ISBN 9781584886051
6. Kravtsova O.V. 2-elements in an autotopism group of a semifield projective plane. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 39, pp. 96–110. https://doi.org/10.26516/1997-7670.2022.39.96
7. Kravtsova O.V. Dihedral group of order 8 in an autotopism group of a semifield projective plane of odd order. *Journal of Siberian Federal University. Mathematics & Physics*, 2022, vol. 15, no. 3, pp. 378–384. https://doi.org/10.17516/1997-1397-2022-15-3-378-384
8. Kravtsova O.V. Skok D.S., Non-Abelian autotopism subgroups of order 8 of semifield projective planes. *Trudy Inst. Mat. i Mekh. UrO RAN*, 2025, vol. 31, no. 1, pp. 90–100. https://doi.org/10.21538/0134-4889-2025-31-1-90-100
9. Lüneburg H. Charakterisierungen der endlichen Desarguesschen projectiven Ebenen. *Math. Z.*, 1964, vol. 85, pp. 419–450. https://doi.org/10.1007/BF01115362
10. Maduram D.M. Matrix representation of translation planes. *Geom Dedicata*, 1975, vol. 4, pp. 485–492. https://doi.org/10.1007/BF00148776
11. Montinaro A. The general structure of the projective planes admitting $PSL(2, q)$ as a collineation group. *Innovations in Incidence Geometry*, 2007, vol. 5, pp. 35–116. DOI: 10.2140/iig.2007.5.35
12. Moorhouse G.E. $PSL(2, q)$ as a collineation group of projective planes of small order. *Geom. Dedicata*, 1989, vol. 31, pp. 63–88. https://doi.org/10.1007/BF00184160

Список источников

1. Dickson L. E. Linear algebras in which division is always uniquely possible // Trans. Amer. Math. Soc. 1906. Vol. 7, N 3. P. 370–390. https://doi.org/10.1090/S0002-9947-1906-1500755-5
2. Goldschmidth D. M. 2-fusion in finite groups // Ann. Math. 1974. Vol. 99, N 1. P. 70–117.
3. Hughes D. R. Review of some results in collineation groups // Proc. Sympos. Pure Math., American Mathematical Society, Providence, R. I. 1959. Vol. 1. P. 42–55.
4. Hughes D.R., Piper F.C. Projective planes. New York : Springer-Verlag, 1973. 292 p.
5. Johnson N. L., Jha V., Biliotti M. Handbook of finite translation planes. London : Chapman and Hall, 2007. 861 p. ISBN 9781584886051
6. Кравцова О. В. 2-элементы в группе автотопизмов полуполевой проективной плоскости // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 39. С. 96–110. https://doi.org/10.26516/1997-7670.2022.39.96
7. Kravtsova O. V. Dihedral group of order 8 in an autotopism group of a semifield projective plane of odd order // Journal of Siberian Federal University. Mathematics & Physics. 2022. Vol. 15, N 3. P. 378–384. https://doi.org/10.17516/1997-1397-2022-15-3-378-384

8. Kravtsova O. V., Skok D. S. Non-Abelian autotopism subgroups of order 8 of semifield projective planes // Труды ИММ УрО РАН. 2025. Т. 31, № 1. С. 90–100. https://doi.org/10.21538/0134-4889-2025-31-1-90-100
9. Lüneburg H. Charakterisierungen der endlichen Desarguesschen projectiven Ebenen // Math. Z. 1964. Vol. 85. P. 419–450. https://doi.org/10.1007/BF01115362
10. Maduram D. M. Matrix representation of translation planes // Geom Dedicata. 1975. Vol. 4. P. 485–492. https://doi.org/10.1007/BF00148776
11. Montinaro A. The general structure of the projective planes admitting $PSL(2,q)$ as a collineation group // Innovations in Incidence Geometry. 2007. Vol. 5. P. 35–116. DOI: 10.2140/iig.2007.5.35
12. Moorhouse G. E. $PSL(2,q)$ as a collineation group of projective planes of small order // Geom. Dedicata. 1989. Vol. 31. P. 63–88. https://doi.org/10.1007/BF00184160

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Поступила в редакцию / Received 18.08.2025
Поступила после рецензирования / Revised 01.10.2025
Принята к публикации / Accepted 06.10.2025