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On the Positive Solvability of one Class of Nonlinear Multidimensional Integral Equations of Hammerstein Type

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Abstract: This paper investigates a class of nonlinear multidimensional integral equations on $\mathbb{R}^n$ with non-compact Hammerstein operator. These equations arise in various fields of mathematical physics and mathematical epidemiology. A distinguishing feature of the studied equations is the lack of complete continuity of the associated nonlinear operator in the space of bounded functions on $\mathbb{R}^n$, the presence of a trivial (zero) solution, and the non-reflexivity of the corresponding function space, within which the existence of a nontrivial fixed point is considered. Under appropriate conditions on the kernel and the nonlinear term, a constructive theorem is established for the existence of a positive, bounded, and continuous solution. Moreover, the method of successive approximations is shown to converge uniformly to the solution at a rate an infinitely decreasing geometric progression. Within a sufficiently broad subclass of nonnegative, bounded functions on $\mathbb{R}^n$, the uniqueness of the solution is also proven. The integral asymptotic behavior of the constructed solution is examined under additional constraints on the kernel and nonlinearity. Finally, explicit examples of kernels and nonlinearities satisfying all the assumptions of the theorems are provided.

Keywords: monotonicity, iterations, concavity, bounded solution, integral asymptotic

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Научная статья

О положительной разрешимости одного класса нелинейных многомерных интегральных уравнений типа Гаммерштейна

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Аннотация: Исследуется класс нелинейных многомерных интегральных уравнений на $\mathbb{R}^n$ с некомпактным оператором Гаммерштейна. Указанный класс уравнений имеет непосредственное применение в различных направлениях математической физики и математической эпидемиологии. Отличительной особенностью исследуемого уравнения является отсутствие полной непрерывности соответствующего нелинейного многомерного интегрального оператора в пространстве ограниченных на $\mathbb{R}^n$ функций, наличие тривиального (нулевого) решения и нерефлексивность соответствующего пространства, где рассматривается вопрос построения нетривиальной неподвижной точки нелинейного интегрального оператора Гаммерштейна. При определённых условиях на ядро и нелинейность доказывается конструктивная теорема существования положительного ограниченного и непрерывного решения. Кроме того, устанавливается равномерная сходимость специально выбранных последовательных приближений к решению со скоростью бесконечно убывающей геометрической прогрессии. В достаточно широком подклассе класса неотрицательных и ограниченных на $\mathbb{R}^n$ функций доказывается также единственность решения уравнения. Исследуется интегральная асимптотика построенного решения при дополнительных ограничениях на ядро и на нелинейность уравнения. В конце работы приводятся конкретные примеры ядер и нелинейностей, удовлетворяющих всем условиям доказанных теорем.

Ключевые слова: монотонность, итерации, вогнутость, ограниченное решение, интегральная асимптотика

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1. Introduction

1.1. Statement of the Problem

We consider the following class of nonlinear integral equations of Hammerstein type on $\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$, $\mathbb{R} := (-\infty, +\infty)$:

$$f(x_1,\ldots,x_n)= \\ =\int\limits_{\mathbb{R}^n} K(x_1,\ldots,x_n,t_1,\ldots,t_n)G(t_1,\ldots,t_n,f(t_1,\ldots,t_n))\,dt_1\ldots dt_n, \quad (1.1)$$

$\mathrm{x}:=(x_1,\ldots,x_n)\in\mathbb{R}^n$ with respect to the unknown function $f(x_1,\ldots,x_n)$ which is assumed to be non-negative, bounded and continuous on $\mathbb{R}^n$. In equation (1.1) the kernel K satisfies the following **key conditions**:

a) $K(x_1,\ldots,x_n,t_1,\ldots,t_n)>0$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $(t_1,\ldots,t_n)\in\mathbb{R}^n$, $K\in C(\mathbb{R}^{2n})$,

b) $\gamma(x_1,\ldots,x_n):=\int\limits_{\mathbb{R}^n} K(x_1,\ldots,x_n,t_1,\ldots,t_n)\,dt_1\ldots dt_n\le 1, \gamma\in C(\mathbb{R}^n)$,
$\lim\limits_{|\mathrm{x}|\to\infty}\gamma(x_1,\ldots,x_n)=1$, where $|\mathrm{x}|=(x_1^2+\cdots+x_n^2)^{1/2}$.

The nonlinearity $G(x_1,\ldots,x_n,u)$ is defined on the set $\mathbb{R}^n\times\mathbb{R}^+$, $\mathbb{R}^+:=[0,+\infty)$, takes non-negative values and satisfies the **following assumptions**:

A) $G\in C(\mathbb{R}^n\times\mathbb{R}^+)$, $G(x_1,\ldots,x_n,0)\equiv 0$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$ and for every fixed $(x_1,\ldots,x_n)\in\mathbb{R}^n$, the function G is monotone increasing in u on $\mathbb{R}^+$,

B) there exists a continuous, monotone increasing, and strictly concave mapping $\Phi:[0,1]\to[0,1]$, with the properties $\Phi(0)=0$, $\Phi(1)=1$, $\Phi'(+0)=+\infty$, such that $G(x_1,\ldots,x_n,\sigma u)\ge\Phi(\sigma)G(x_1,\ldots,x_n,u)$, $(x_1,\ldots,x_n,u)\in\mathbb{R}^n\times\mathbb{R}^+,\sigma\in[0,1]$,

C) there exists a constant $\eta>0$ such that

$$G(x_1,\ldots,x_n,\eta)\le\eta, (x_1,\ldots,x_n)\in\mathbb{R}^n \text{ and } \lim_{|\mathrm{x}|\to\infty} G(x_1,\ldots,x_n,\eta)=\eta.$$

It follows immediately from condition A) that the zero function

$$f(x_1,\ldots,x_n)\equiv 0$$

is a trivial solution of equation (1.1). The principal objective of this paper is to construct a second, nontrivial solution to equation (1.1) which is bounded and continuous on $\mathbb{R}^n$. Additionally, we investigate the uniqueness and the asymptotic behavior at infinity of the constructed solution.

1.2. Possible Applications and Brief Historical Background

Equation (1.1), with various forms of the kernel K and nonlinearity G, arises in numerous applications in mathematical physics and mathematical biology. In particular, when

$$K(x_1,\ldots,x_n,t_1,\ldots,t_n)=\frac{1}{\pi^{n/2}}\cdot\exp\left(-\sum_{i=1}^n(x_i-t_i)^2\right), x_i,t_i\in\mathbb{R}, i=1,\ldots,n,$$

$$G(x_1,\dots,x_n,u)=u^{\alpha},\quad (x_1,\dots,x_n)\in\mathbb{R}^n,\ u\in\mathbb{R}^+,\ \alpha\in(0,1),$$

equation (1.1) appears in the theory of p-adic open-closed string models for scalar tachyon fields (see [1; 14; 16]), where the desired solution f repressing the tachyon field of a string. When $n=1$, the kernel K depends on the difference of its arguments and is given as a superposition of exponentials, and when the nonlinearity $G(x,u)$ has the form $G(x,u)=u-\omega(x,u)$, $x\in\mathbb{R}$, $u\in\mathbb{R}^+$, $\omega\in C(\mathbb{R}\times\mathbb{R}^+)$, $\omega(x,0)\equiv 0$, $x\in\mathbb{R}$, $\omega(x,u)\downarrow$ in u on $[A,+\infty)$, $A>0$, $\sup\limits_{u\in\mathbb{R}^+}\omega(x,u)\in L_1(\mathbb{R})$, the equation (1.1) finds applications in the theory of radiative transfer in spectral lines (see [5], [6]). Finally, in the case $K(x_1,\dots,x_n,t_1,\dots,t_n)=\overset{\circ}{K}(x_1-t_1,\dots,x_n-t_n)$, $(x_1,\dots,x_n)\in\mathbb{R}^n,(t_1,\dots,t_n)\in\mathbb{R}^n$, $G(x_1,\dots,x_n,u)=\gamma(1-e^{-u})$, $\gamma>1$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $u\in\mathbb{R}^+$, equation (1.1) models the spatio-temporal spread of epidemic diseases within the frameworks of the Diekmann–Kaper and Atkinson–Reuter models (see [2–4; 7; 10]).

Historically, research on equation (1.1) has focused mainly on scenarios, where the kernel K depends on the difference of its arguments or admits a majorant or minorant of this type, and the nonlinearity G is independent of $x_1,\dots,x_n$, while being monotone, continuous, and concave (see [3; 7; 10]). Under various additional assumptions on the function K and G, these works established the existence and uniqueness of nontrivial bounded solutions. Some qualitative properties of these solutions have also been analyzed (see [4; 8]).

Furthermore, the one-dimensional analogue of equation (1.1), has been extensively studied under various conditions on the kernel $K(x,t)$ and the nonlinearity $G(x,u)$. For example, in papers [15] and [16], iterative methods were developed for constructing a sign-preserving solution of a one-dimensional analogue of equation (1.1) in the particular case when $K(x,t)=\frac{1}{\pi}e^{-(x-t)^2},(x,t)\in\mathbb{R}\times\mathbb{R}$, $G(u)=u^{\frac{1}{p}},p>2-$odd number, and in paper [14], the absence of a sign-preserving non-trivial solution was proven. In papers [2] and [3], the existence of a positive and bounded solution of the one-dimensional equation (1.1) was studied in the case where the kernel depends on the difference of its arguments, and the nonlinearity satisfies sufficiently strict constraints. In papers [7] and [8], the questions of existence and uniqueness of a nontrivial bounded solution of the one-dimensional equation (1.1) were discussed, where the kernel is majorized by a difference kernel, and the nonlinearity is a monotone concave function satisfying weaker conditions than those considered in papers [2] and [3]. These studies have primarily relied on results from the theory of linear integral operators of convolution type and monotone nonlinear operators in regular cones.

It is interesting to note that nonlinear integral equations of a detailed nature with loads and bifurcation parameters have been studied in sufficient detail in the works [12; 13].

1.3. Summary of the Results Obtained

In this paper, under conditions $a), b)$ and $A)-C)$, we prove a constructive theorem on the existence of a second nontrivial, bounded, continuous, and positive solution to equation (1.1). Furthermore, we establish the uniform convergence of the corresponding successive approximations to this solution, with a rate characterized by a geometrically decreasing progression (see Theorem 1). Under the same conditions, we also prove a uniqueness theorem for the solution within a specific subclass of the space of nonnegative, bounded functions:

$$\begin{aligned}\mathfrak{M} := \{f \in M(\mathbb{R}^n) : f(x_1, \dots, x_n) \geq 0,\ (x_1, \dots, x_n) \in \mathbb{R}^n \\ \textit{and } \exists r > 0,\ \textit{s.t.} \inf_{|\mathrm{x}|>r} f(x_1, \dots, x_n) > 0\},\end{aligned} \tag{1.2}$$

where $M(\mathbb{R}^n)$ the space refers to functions bounded on $\mathbb{R}^n$ (see Theorem 2). As a special case, this result implies that if $\gamma(x_1, \dots, x_n) \equiv 1$, $G(x_1, \dots, x_n, \eta) \equiv \eta$, $(x_1, \dots, x_n) \in \mathbb{R}^n$, then equation (1.1) in the class $\mathfrak{M}$ has only the solution $f(x_1, \dots, x_n) \equiv \eta$, $(x_1, \dots, x_n) \in \mathbb{R}^n$.

Under the following **additional conditions**:

c) there exist continuous functions λ, $\overset{\circ}{K}_i$, $i = 1, 2, \dots, n$ on $\mathbb{R}$ with properties:

$$0 < \delta := \inf_{x \in \mathbb{R}^+} \lambda(x) \leq \lambda(x) \leq 1, \quad x \in \mathbb{R}^+ := [0, +\infty),\ 1 - \lambda \in L_1(\mathbb{R}),$$

$$0 < \overset{\circ}{K}_i(-t) = \overset{\circ}{K}_i(t), t \in \mathbb{R}^+, \quad \overset{\circ}{K}_i \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}),$$

$$\int_0^\infty \overset{\circ}{K}_i(x)dx = \frac{1}{2},\ i = 1, 2, \dots, n,$$

λ is monotonically non-decreasing on $\mathbb{R}^+$, such that

$$K(x_1, \dots, x_n, t_1, \dots, t_n) \geq \lambda(|\mathrm{x}|) \prod_{i=1}^n K_i(x_i - t_i),$$

for all $\mathrm{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $(t_1, \dots, t_n) \in \mathbb{R}^n$.

D) $G(x_1, \dots, x_n, u) = G_0(u)$, where $G_0 \in C(\mathbb{R}^+)$ – is a monotonically increasing strictly concave function on $\mathbb{R}^+$ and $G_0(0) = 0$, $G_0(\eta) = \eta$,

we will demonstrate that the constructed solution possesses the properties

$$\begin{aligned}\sup_{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}} \int_{-\infty}^{\infty} (\eta - f(x_1, \dots, x_n))dx_i < +\infty, \\ \int_{\mathbb{R}^n} (\eta - f(x_1, \dots, x_n))^n dx_1 \dots dx_n < +\infty,\end{aligned} \tag{1.3}$$

outlined in Theorem 3.

At the end of the paper, we provide specific examples of the kernel K and the nonlinearity G, that satisfy conditions a)-c) and A)-D) respectively.

2. Existence Theorem

One of the main results of this paper is the following

Theorem 1. *Under conditions a), b) and A)-C), equation (1.1) has a positive continuous and bounded on* $\mathbb{R}^n$ *solution* $f(x_1,\dots,x_n)$, *where*

$$f(x_1,\dots,x_n)\le \eta\gamma(x_1,\dots,x_n),\quad (x_1,\dots,x_n)\in\mathbb{R}^n. \tag{2.1}$$

Moreover, there exist numbers $C>0$ *and* $k\in(0,1)$ *such that*

$$0\le f_m(x_1,\dots,x_n)-f(x_1,\dots,x_n)\le C\cdot k^m, (x_1,\dots,x_n)\in\mathbb{R}^n, m=1,2,\dots, \tag{2.2}$$

where the sequence of functions $\{f_m(x_1,\dots,x_n)\}_{m=0}^{\infty}$ *is determined from the following recurrence relations:*

$$f_{m+1}(x_1,\dots,x_n)=$$
$$=\int\limits_{\mathbb{R}^n} K(x_1,\dots,x_n,t_1,\dots,t_n)G(t_1,\dots,t_n,f_m(t_1,\dots,t_n))dt_1\dots dt_n, \tag{2.3}$$
$$f_0(x_1,\dots,x_n)\equiv\eta,\quad (x_1,\dots,x_n)\in\mathbb{R}^n,\quad m=0,1,\dots$$

Proof. First, note that using conditions a), b), A) and B) it is easy to verify the following facts by induction on m:

$$f_m\in C(\mathbb{R}^n),\quad m=0,1,\dots \tag{2.4}$$

$$f_m(x_1,\dots,x_n)>0,\quad m=0,1,\dots,\quad (x_1,\dots,x_n)\in\mathbb{R}^n, \tag{2.5}$$

$$f_{m+1}(x_1,\dots,x_n)\le f_m(x_1,\dots,x_n),\quad m=0,1,\dots,\quad (x_1,\dots,x_n)\in\mathbb{R}^n. \tag{2.6}$$

In the iterations (2.3) we estimate $f_1(x_1,\dots,x_n)$ from below. To this end, we first verify that

$$\varepsilon_0:=\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n}\gamma(x_1,\dots,x_n)>0. \tag{2.7}$$

Indeed, taking into account condition b), we can assert that there exists a number $r_0>0$ such that for $|\mathrm{x}|>r_0$

$$\gamma(x_1,\dots,x_n)>\frac{1}{2},\quad \mathrm{x}=(x_1,\dots,x_n). \tag{2.8}$$

On the other hand, taking into account (2.5) and condition a), according to the Weierstrass theorem, we have

$$\min_{(x_1,\dots,x_n)\in B_{r_0}} \gamma(x_1,\dots,x_n) > 0, \tag{2.9}$$

where $B_{r_0} := \{(x_1,\dots,x_n)\in\mathbb{R}^n : |\mathrm{x}| = (x_1^2+\cdots+x_n^2)^{1/2} \le r_0\}$ is a closed ball in $\mathbb{R}^n$ with radius r_0. From (2.8) and (2.9) we arrive at (2.7).

Now, using conditions A), C) and repeating similar reasoning as above, we arrive at the inequality

$$\varepsilon_1 := \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} G(x_1,\dots,x_n,\eta) > 0. \tag{2.10}$$

Thus, taking into consideration A), (2.3), (2.7) and (2.10), we have

$$f_1(x_1,\dots,x_n) \ge \varepsilon_1 \int\limits_{\mathbb{R}^n} K(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n \ge \varepsilon_0\varepsilon_1, \quad (x_1,\dots,x_n)\in\mathbb{R}^n, \tag{2.11}$$

where

$$\varepsilon_0\in(0,1], \quad \varepsilon_1\in(0,\eta]. \tag{2.12}$$

Thus, based on (2.11) and (2.6), taking into account (2.12) we arrive at the following two-sided inequality:

$$\sigma_0 f_0(x_1,\dots,x_n) \le f_1(x_1,\dots,x_n) \le f_0(x_1,\dots,x_n), \quad (x_1,\dots,x_n)\in\mathbb{R}^n, \tag{2.13}$$

where

$$\sigma_0 := \frac{\varepsilon_0\cdot\varepsilon_1}{2\cdot\eta} \in (0,1). \tag{2.14}$$

Now, using conditions A) and B) from (2.13), taking into account (2.14), we obtain

$$\begin{aligned}\Phi(\sigma_0)G(t_1,\dots,t_n,f_0(t_1,\dots,t_n)) \le \\ \le G(t_1,\dots,t_n,\sigma_0 f_0(t_1,\dots,t_n)) \le G(t_1,\dots,t_n,f_1(t_1,\dots,t_n)) \le \\ \le G(t_1,\dots,t_n,f_0(t_1,\dots,t_n)),(t_1,\dots,t_n)\in\mathbb{R}^n.\end{aligned} \tag{2.15}$$

Multiplying both parts of (2.15) by the function $K(x_1,\dots,x_n,t_1,\dots,t_n)$ and integrating over $t_1,\dots,t_n$ on $\mathbb{R}^n$ the obtained inequalities, due to conditions a) and (2.3) we obtain:

$$\Phi(\sigma_0)f_1(x_1,\dots,x_n) \le f_2(x_1,\dots,x_n) \le f_1(x_1,\dots,x_n),(x_1,\dots,x_n)\in\mathbb{R}^n. \tag{2.16}$$

Since $\Phi(\sigma_0)\in(0,1)$ (for $\sigma_0\in(0,1)$), then again using conditions A) and B) from (2.16), we arrive at the following chain of inequalities:

$$\begin{aligned}\Phi(\Phi(\sigma_0))G(t_1,\dots,t_n,f_1(t_1,\dots,t_n)) \le \\ \le G(t_1,\dots,t_n,\Phi(\sigma_0)f_1(t_1,\dots,t_n)) \le G(t_1,\dots,t_n,f_2(t_1,\dots,t_n)) \le \\ \le G(t_1,\dots,t_n,f_1(t_1,\dots,t_n)),(t_1,\dots,t_n)\in\mathbb{R}^n.\end{aligned} \tag{2.17}$$

Now, multiplying both sides of (2.17) by the kernel $K(x_1,\dots,x_n,t_1,\dots,t_n)$ and integrating the resulting inequalities over $t_1,\dots,t_n$ on $\mathbb{R}^n$ by virtue of condition a) and (2.3) we arrive at the estimates

$$\Phi(\Phi(\sigma_0))f_2(x_1,\dots,x_n)\le f_3(x_1,\dots,x_n)\le f_2(x_1,\dots,x_n),(x_1,\dots,x_n)\in\mathbb{R}^n.$$

Continuing this process, at the m-th step we obtain that

$$\underbrace{\Phi(\Phi\dots\Phi(\sigma_0))}_{m}f_m(x_1,\dots,x_n)\le f_{m+1}(x_1,\dots,x_n)\le f_m(x_1,\dots,x_n),$$

$(x_1,\dots,x_n)\in\mathbb{R}^n$, from which, in view of (2.6) and (2.3) in particular, it follows that

$$0\le f_m(x_1,\dots,x_n)-f_{m+1}(x_1,\dots,x_n)\le\eta\left(1-\underbrace{\Phi(\Phi\dots\Phi(\sigma_0))}_{m}\right),$$
$$(x_1,\dots,x_n)\in\mathbb{R}^n,\quad m=1,2,\dots. \tag{2.18}$$

Now, using inequality (3.16) from [9] and (2.18) we have

$$\begin{gathered}0\le f_m(x_1,\dots,x_n)-f_{m+1}(x_1,\dots,x_n)\le\\ \le\eta(1-\sigma_0)k^m,\ m=1,2,\dots,\ (x_1,\dots,x_n)\in\mathbb{R}^n,\end{gathered} \tag{2.19}$$

where

$$k:=\frac{1-\Phi(\sigma_0)}{1-\sigma_0}\in(0,1). \tag{2.20}$$

From (2.19) and (2.20), taking into consideration (2.4), (2.5) it follows that the sequence of continuous and positive functions $f_m(x_1,\dots,x_n)$, $m=0,1,\dots: f_m(x_1,\dots,x_n):\underset{m\to\infty}{\overset{\mathbb{R}^n}{\rightrightarrows}}f(x_1,\dots,x_n)$, on $\mathbb{R}^n$ where $f\in C(\mathbb{R}^n)$ and, by virtue of conditions a), b), A), C), satisfies the inequalities

$$0\le f(x_1,\dots,x_n)\le\eta\gamma(x_1,\dots,x_n),\quad(x_1,\dots,x_n)\in\mathbb{R}^n. \tag{2.21}$$

Taking into account the continuity of the functions K and G with respect to all their arguments, as well as conditions (2.5) and (2.6), we can easily verify—using B. Levy's theorem (see [11]) that the limit function f satisfies equation (1.1). However, these considerations alone do not guarantee the non-triviality of the constructed solution. To establish this, we prove by induction on m the validity of the following inequality from below for each element of the sequence of functions $\{f_m(x_1,\dots,x_n)\}_{m=0}^{\infty}$:

$$f_m(x_1,\dots,x_n)\ge\tau^*\eta,\quad(x_1,\dots,x_n)\in\mathbb{R}^n,\quad m=0,1,\dots, \tag{2.22}$$

where the number $\tau^*\in(0,\sigma_0)$ is uniquely determined from the characteristic equation

$$\Phi(\tau)=\frac{\tau}{\sigma_0}. \tag{2.23}$$

First, to ensure correctness, we prove the existence and uniqueness of the solution τ^* of equation (2.23) on the interval $(0, \sigma_0)$. Indeed, consider the function $\chi(\tau) := \dfrac{\Phi(\tau)}{\tau} - \dfrac{1}{\sigma_0}, \quad \tau \in (0, \sigma_0]$, from the properties of the mapping Φ (see condition B)) and by applying L'Hôpital's Rule, it follows that

$$\chi \in C(0, \sigma_0], \quad \chi \text{ is monotonically decreasing on } (0, \sigma_0], \tag{2.24}$$

$$\chi(+0) = \lim_{\tau \to 0^+} \chi(\tau) = \Phi'(+0) - \frac{1}{\sigma_0} = +\infty, \quad \chi(\sigma_0) = \frac{\Phi(\sigma_0) - 1}{\sigma_0} < 0. \tag{2.25}$$

From (2.24) and (2.25) the existence and uniqueness of the number $\tau^* \in (0, \sigma_0)$, satisfying equation (2.23) is established. Now, let us return to the proof of inequality (2.22). For the number $m = 0$ estimate (2.22) follows directly from the definition of the initial approximation in the iteration scheme (2.3) taking into account that $0 < \tau^* < \sigma_0 < 1$. Assuming inequality (2.22) holds for some $m \in \mathbb{N}$ and using conditions a), A) and B), along with equation (2.13), we deduce from (2.3) that:

$$\begin{aligned} &f_{m+1}(x_1, \ldots, x_n) \geq \\ &\qquad \geq \int\limits_{\mathbb{R}^n} K(x_1, \ldots, x_n, t_1, \ldots, t_n) G(t_1, \ldots, t_n, \tau^* \eta) dt_1 \ldots dt_n \geq \\ &\geq \Phi(\tau^*) f_1(x_1, \ldots, x_n) \geq \Phi(\tau^*) \sigma_0 f_0(x_1, \ldots, x_n) = \tau^* \eta, (x_1, \ldots, x_n) \in \mathbb{R}^n. \end{aligned}$$

Finally, passing to the limit in (2.22) as $m \to \infty$ and taking into account (2.21), we obtain that:

$$\tau^* \eta \leq f(x_1, \ldots, x_n) \leq \eta \gamma(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{2.26}$$

Thus, the positivity of the constructed solution is proven. Now writing down inequalities (2.19) for numbers $m+1, \ldots, m+p$, and then adding the resulting inequalities and (2.19), we obtain:

$$\begin{aligned} 0 &\leq f_m(x_1, \ldots, x_n) - f_{m+p+1}(x_1, \ldots, x_n) \leq \eta(1 - \sigma_0) k^m (1 + k + \cdots + k^p) \leq \\ &\leq \frac{\eta(1 - \sigma_0)}{1 - k} k^m, \quad m = 1, 2, \ldots, p = 1, 2, \ldots, (x_1, \ldots, x_n) \in \mathbb{R}^n. \end{aligned} \tag{2.27}$$

In (2.27) passing $p \to \infty$, we arrive at (2.2). The theorem is proved. □

Remark 1. It should also be noted that the estimate (2.26) obtained in the course of the proof of Theorem 1, will play an important role in the study of the integral asymptotic behavior of the constructed solution (see §4).

3. Uniqueness of the solution

We now turn to the question of the uniqueness of the constructed solution of equation (1.1) in the class of functions $\mathfrak{M}$.

The following theorem holds

Theorem 2. *Under the conditions of Theorem 1, equation (1.1) has a unique solution in the class $\mathfrak{M}$, where the class of functions $\mathfrak{M}$ is defined according to (1.2).*

Proof. First, for correctness, we note that the solution f of equation (1.1) constructed using successive approximations (2.3) belongs to the class of functions $\mathfrak{M}$. This fact immediately follows from estimate (2.26). Now we assume that equation (1.1), in addition to the solution f, also has another solution $f^* \in \mathfrak{M}$. Let us verify that

$$\varepsilon^* := \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f^*(x_1,\dots,x_n) > 0. \tag{3.1}$$

Since $f^* \in \mathfrak{M}$, there exists a number $r > 0$ such that

$$c^* := \inf_{|\mathrm{x}|>r} f^*(x_1,\dots,x_n) > 0, \quad \mathrm{x} = (x_1,\dots,x_n). \tag{3.2}$$

Taking into account conditions a) and A) from (1.1) and (3.2), we have

$$f^*(x_1,\dots,x_n) \geq$$
$$\geq \int\limits_{\mathbb{R}^n\setminus B_r} K(x_1,\dots,x_n,t_1,\dots,t_n)G(t_1,\dots,t_n,c^*)dt_1\dots dt_n > 0,\ \mathrm{x}\in\mathbb{R}^n, \tag{3.3}$$

where $B_r := \{(x_1,\dots,x_n)\in\mathbb{R}^n : |\mathrm{x}| \leq r\}$.

Taking into consideration $a), b)$ and $A)$ from (3.3) according to the Weierstrass theorem for all points $\mathrm{x} = (x_1,\dots,x_n)$ from the ball B_r we obtain

$$f^*(x_1,\dots,x_n) \geq$$
$$\geq \min_{|\mathrm{x}|\leq r} \int\limits_{\mathbb{R}^n\setminus B_r} K(x_1,\dots,x_n,t_1,\dots,t_n)G(t_1,\dots,t_n,c^*)dt_1\dots dt_n =: \tilde{C} > 0,$$

$\mathrm{x} = (x_1,\dots,x_n) \in B_r$. Therefore,

$$\inf_{|\mathrm{x}|\leq r} f^*(x_1,\dots,x_n) \geq \tilde{C} > 0. \tag{3.4}$$

From (3.2) and (3.4) we arrive at (3.1). We introduce the following notations:

$$\delta^* := \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} f^*(x_1,\dots,x_n) < +\infty, \tag{3.5}$$

$$\alpha_1 := \min\left\{\frac{\varepsilon^*}{\eta}, \frac{1}{2}\right\} \in (0, \frac{1}{2}], \alpha_2 := \max\left\{\frac{\delta^*}{\tau^*\eta}, 2\right\} \in [2, +\infty). \tag{3.6}$$

Using (2.26), (3.1), (3.5), (3.6) and conditions a), b) we have

$$\alpha_1 \leq \frac{\varepsilon^*}{\eta} \leq \frac{f^*(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n)} \leq \frac{\delta^*}{\tau^*\eta} \leq \alpha_2, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{3.7}$$

whence it follows that

$$\alpha_1 f(x_1, \ldots, x_n) \leq f^*(x_1, \ldots, x_n) \leq \alpha_2 f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{3.8}$$

Now, using the double inequality (3.8) by induction on m, it is easy to verify that the following chain of inequalities is valid

$$\underbrace{\Phi(\Phi \ldots \Phi(\alpha_1))}_{m} f(x_1, \ldots, x_n) \leq f^*(x_1, \ldots, x_n) \leq$$

$$\frac{1}{\underbrace{\Phi\left(\Phi \ldots \Phi\left(\frac{1}{\alpha_2}\right)\right)}_{m}} f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots. \tag{3.9}$$

Thus, inequalities (3.9) are proved. From (3.9), taking into account (3.6), (3.6), (2.26) and inequality (3.16) from [9], we obtain

$$|f(x_1, \ldots, x_n) - f^*(x_1, \ldots, x_n)| \leq$$

$$\leq \left(\frac{1}{\underbrace{\Phi\left(\Phi \ldots \Phi\left(\frac{1}{\alpha_2}\right)\right)}_{m}} - \underbrace{\Phi\left(\Phi \ldots \Phi\left(\alpha_1\right)\right)}_{m}\right) \cdot \eta \leq$$

$$\leq \left(\alpha_2(1 - \underbrace{\Phi\left(\Phi \ldots \Phi\left(\frac{1}{\alpha_2}\right)\right)}_{m}) + 1 - \underbrace{\Phi\left(\Phi \ldots \Phi\left(\alpha_1\right)\right)}_{m}\right) \cdot \eta \leq$$

$$\leq \eta\left((\alpha_2 - 1)\left(\frac{1 - \Phi(\frac{1}{\alpha_2})}{1 - \frac{1}{\alpha_2}}\right)^m + (1 - \alpha_1)\left(\frac{1 - \Phi(\alpha_1)}{1 - \alpha_1}\right)^m\right),$$

$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots,$$

whence it follows that

$$|f(x_1, \ldots, x_n) - f^*(x_1, \ldots, x_n)| \leq \eta(\alpha_2 - \alpha_1) \cdot k_0^m,$$

$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots, \tag{3.10}$$

where

$$k_0 := \max\left\{\frac{1 - \Phi(\frac{1}{\alpha_2})}{1 - \frac{1}{\alpha_2}}, \frac{1 - \Phi(\alpha_1)}{1 - \alpha_1}\right\} \in (0, 1). \tag{3.11}$$

In (3.10) passing $m \to \infty$ and taking into account (3.11) we obtain that $f(x_1, \dots, x_n) = f^*(x_1, \dots, x_n)$, $(x_1, \dots, x_n) \in \mathbb{R}^n$. Thus, the theorem is proved. $\square$

Remark 2. From the proven Theorem 2 it immediately follows that if $\gamma(x_1, \dots, x_n) \equiv 1$, $G(x_1, \dots, x_n, \eta) \equiv \eta$, $(x_1, \dots, x_n) \in \mathbb{R}^n$, then equation (1.1) in the class $\mathfrak{M}$ has only the solution $f(x_1, \dots, x_n) \equiv \eta$, $(x_1, \dots, x_n) \in \mathbb{R}^n$.

4. Integral asymptotic of the constructed solution. Examples

4.1. Theorem on the integral asymptotic of the solution.

The following theorem is true

Theorem 3. *Under conditions a)-c) and A)-D) the solution $f(x_1, \dots, x_n)$ of equation (1.1) constructed using successive approximations (2.3) has the properties (1.3):*

Proof. Along with equation (1.1) we consider the following auxiliary one-dimensional nonlinear integral equations on the entire axis:

$$\psi_i(x_i) = \lambda(|x_i|) \int_{-\infty}^{\infty} \mathring{K}_i(x_i - t_i) G_0(\psi_i(t_i))\, dt_i, \quad x_i \in \mathbb{R}, \tag{4.1}$$

with respect to unknown non-negative functions ψ_i, $i = 1, 2, \dots, n$.

It follows from the results of [8] that if $\lambda_i(x_i) \not\equiv 1$, $i = 1, \dots, n$, then equations (4.1) have non-negative, non-trivial, continuous, and bounded solutions on $\mathbb{R}$ where $0 \le \eta - \psi_i \in L_1(\mathbb{R})$, $i = 1, 2, \dots, n$. However, if for some $i_0 \in \{1, 2, \dots, n\}$ $\lambda_{i_0}(x_{i_0}) \equiv 1$, $x_{i_0} \in \mathbb{R}$, then in the class of non-negative, non-zero, and bounded on $\mathbb{R}$ functions, we have $\psi_{i_0}(x) \equiv \eta$, $x \in \mathbb{R}$.

Now, by induction on m we verify the validity of the following inequalities from below:

$$f_m(x_1, \dots, x_n) \ge \psi_i(x_i), \quad i = 1, 2, \dots, n, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, m = 0, 1, \dots \tag{4.2}$$

In the case $m = 0$ estimate (4.2) follows immediately from the definition of the zero approximation in iterations (2.3) taking into consideration the inequalities $\psi_i(x_i) \le \eta$, $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Let us assume that (4.2) is satisfied for some natural m. Then, taking into account conditions c), D) and relations (4.1) from (2.3), we obtain:

$$f_{m+1}(x_1, \dots, x_n) \ge \int_{\mathbb{R}^n} K(x_1, \dots, x_n, t_1, \dots, t_n) G_0(\psi_i(t_i)) dt_1 \dots dt_n \ge$$

$$\geq \lambda\left(\sqrt{x_1^2+\cdots+x_n^2}\right)\cdot\int_{-\infty}^{\infty}\mathring{K}_1(x_1-t_1)\ldots\int_{-\infty}^{\infty}\mathring{K}_{i-1}(x_{i-1}-t_{i-1})\cdot$$

$$\cdot\int_{-\infty}^{\infty}\mathring{K}_i(x_i-t_i)G_0(\psi_i(t_i))\int_{-\infty}^{\infty}\mathring{K}_{i+1}(x_{i+1}-t_{i+1})\ldots\int_{-\infty}^{\infty}\mathring{K}_n(x_n-t_n)dt_1\ldots dt_n=$$

$$=\lambda\left(\sqrt{x_1^2+\cdots+x_n^2}\right)\cdot\int_{-\infty}^{\infty}\mathring{K}_i(x_i-t_i)G_0(\psi_i(t_i))dt_i\geq$$

$$\geq\lambda(|x_i|)\int_{-\infty}^{\infty}\mathring{K}_i(x_i-t_i)G_0(\psi_i(t_i))dt_i=\psi_i(x_i), i=1,2,\ldots,n, (x_1,\ldots,x_n)\in\mathbb{R}^n.$$

In (4.2) passing $m\to\infty$, we arrive at the following inequality:

$$f(x_1,\ldots,x_n)\geq\psi_i(x_i),\quad i=1,2,\ldots,n,\quad (x_1,\ldots,x_n)\in\mathbb{R}^n. \tag{4.3}$$

Therefore, taking into account (2.1),condition b) and (4.3), we obtain that $0\leq\eta-f(x_1,\ldots,x_n)\leq\eta-\psi_i(x_i)$, $i=1,2,\ldots,n$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$, whence, by virtue of the inclusions $\eta-\psi_i\in L_1(\mathbb{R})$, $i=1,2,\ldots,n$, (1.3) follows. Thus, the theorem is proved. □

4.2. Examples of the kernel K and the nonlinearity G

At the end of the paper, we present several illustrative examples of the kernel K and the nonlinearity G, that satisfy the conditions of the theorems established above. We begin with examples of the kernel K:

1) $K(x_1,\ldots,x_n,t_1,\ldots,t_n)=\frac{1}{\pi^{n/2}}e^{-\sum\limits_{i=1}^{n}(x_i-t_i)^2}\left(1-\varepsilon e^{-\sum\limits_{i=1}^{n}x_i^2}\right)$,
   $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $(t_1,\ldots,t_n)\in\mathbb{R}^n$, where $\varepsilon\in[0,1)$ — is a numerical parameter.

2) $K(x_1,\ldots,x_n,t_1,\ldots,t_n)=\int\limits_a^b e^{-s\sum\limits_{i=1}^{n}|x_i-t_i|}L(s)ds\cdot\left(1-\varepsilon e^{-\sqrt{\sum\limits_{i=1}^{n}x_i^2}}\right)$,
   $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $(t_1,\ldots,t_n)\in\mathbb{R}^n$, where $\varepsilon\in[0,1)$ — is a numerical parameter, $L(s)>0$, $s\in[a,b)$, $L\in C[a,b)$, $0<a<b\leq+\infty$, and $\int\limits_a^b\frac{L(s)ds}{s}=\frac{1}{2^n}$.

Let us move on to examples for nonlinearity G:

I) $G(x_1,\dots,x_n,u)=\left(1-\varepsilon e^{-\sum\limits_{i=1}^{n}x_i^2}\right)u^\alpha$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $\varepsilon\in[0,1)$ — is a numerical parameter, and $\alpha\in(0,1)$.

II) $G(x_1,\dots,x_n,u)=\left(1-\varepsilon e^{-\sqrt{\sum\limits_{i=1}^{n}x_i^2}}\right)\dfrac{u^\alpha+u^\beta}{2}$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $\varepsilon\in[0,1)$, $\alpha,\beta\in(0,1)$ — are the parameters.

III) $G(x_1,\dots,x_n,u)=\left(1-\varepsilon e^{-\sum\limits_{i=1}^{n}x_i^2}\right)\cdot\gamma\left(1-e^{-u^\alpha}\right)$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $\varepsilon\in[0,1)$, $\alpha\in(0,1)$, $\gamma\in(1,+\infty)$ — are the parameters.

It should be noted that in the particular case, when in examples I) and 1) the parameter $\varepsilon=0$, the equation

$$f(x_1,\dots,x_n)=\frac{1}{\pi^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}e^{-\sum\limits_{i=1}^{n}(x_i-t_i)^2}f^\alpha(t_1,\dots,t_n)dt_1\dots dt_n,\alpha\in(0,1)$$

describes the interaction (rolling) of p-adic open-closed strings for the scalar field of tachyons (see [16]). It is also interesting to note that if in examples 1) and III) we take $\varepsilon=0$, then the equation (1.1) arises in the mathematical theory of epidemic diseases within the framework of the Diekmann–Kaper model (see [3; 4]).

5. Conclusion

In this paper, we proved constructive theorems on the existence and uniqueness of a bounded solution for a class of nonlinear multidimensional Hammerstein-type integral equations on . Furthermore, it was shown that the corresponding iterations converge uniformly to the solution at the rate of an infinitely decreasing geometric progression. Furthermore, in one particular case, the integral asymptotic behavior of the constructed solution was studied.

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