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On Optimal Control Problems with Active Infinite Horizon

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Abstract: The article proposes a new formulation of the optimal control problem on an infinite horizon. Usually in such problems, if a Bolza type problem is considered, then its terminal cost depends only on the initial state, additionally, one or another asymptotic requirement can be presented to the right end of the system. A feature of the proposed formulation is the ability to set control not only as a function of time, but also to choose an action-control at the completion of the process itself. This is primarily interesting from the point of view of economic applications, since it is the endless postponement of a generally unprofitable action (for example, ”debt repayment”) that often leads to a lack of optimal control. In addition to this new formulation, some necessary optimality conditions for the case of the simplest dynamics are presented. With these conditions the example of optimizing consumption under various borrowing restrictions is investigated.

Keywords: infinite horizon control problem, active infinite horizon, Pontryagin maximum principle, weakly overtaking optimality

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Научная статья

О задачах оптимального управления с активным бесконечным горизонтом

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Аннотация: Предлагается новая постановка задачи оптимального управления на бесконечном горизонте. В теории управления обычно, если и рассматривается задача типа Больца, то ее конечная стоимость зависит только от начального состояния, а к правому концу системы может быть предъявлено только то или иное асимптотическое требование. Вводится дополнительное управление в терминальное слагаемое, отвечающее за выбор действия после завершения траектории. Это в первую очередь интересно с точки зрения экономических приложений, поскольку именно бесконечное откладывание в целом убыточного действия (например, «погашения долга») часто приводит к отсутствию оптимального управления. Для такой формулировки доказаны необходимые условия оптимальности для случая простейшей динамики. На основе этих условий ищется оптимальное управление в примере оптимизации потребления при различных ограничениях на заимствование.

Ключевые слова: задача управления на бесконечном горизонте, активный бесконечный горизонт, принцип максимума Понтрягина, обгоняющая оптимальность

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To the memory of
Vladimir Aleksandrovich Dykhta

1. Introduction

We propose a novel formulation of the optimal control problem on an infinite horizon. Typically, in such problems with a Bolza-type formulation the terminal cost depends only on the initial state [9; 12; 13], additionally, one or another asymptotic requirement can be presented to the right end of the system [10; 11]. There are other formulations, and we note some very general constructions in [5], as well as particular cases in [2]. In our approach, we introduce an additional term to the terminal cost that accounts for the choice of action following the completion of the trajectory. For economic applications, such as the problem of optimal consumption, our statement is particularly relevant unlike the discounted utilitarian approach, which clearly emphasizes the immediate future at the expense of the long run. For instance, when a consumer's discount rate is lower than the interest rate on their assets, postponing consumption indefinitely can yield greater cumulative utility, leading to continuous asset accumulation without optimal control. Our statements accommodates such infinite delays in consumption by incorporating a finite utility from consumption after the trajectory ends, making such behavior possible. An illustrative example of this scenario will be analyzed in Section 3. Alongside this new formulation, we also present some optimality conditions for the simplest case of system dynamics in Section 2.

2. The general statement

We introduce the general statement of active infinite horizon control problem:

$$\text{maximize } g_{02}(T, x(0), w, x(T), v(T)) + \int_0^T g_1(\tau, x(\tau), u(\tau))\, d\tau \text{ as } T \uparrow \infty$$
$$\text{s.t. } \frac{dx(t)}{dt} = f(t, x(t), u(t)),\ x(t) \in \mathbb{R}^n,\ u(t) \in U(t) \text{ a.e.;}$$
$$v(T) \in V(T), (x(0), w, x(T), v(T)) \in Y(T)\ \forall T \geq 0.$$

We assume that:

(H1) the multimaps U, V, and Y are Lebesgue measurable on $\mathbb{R}_+$ and take values subsets of some finite-dimensional Euclidean spaces;

(H2) the maps f, g_1, g_{02} are LB-measurable; f is Lipschitz continuous in x and satisfies the condition of the sublinear growth.

We say that (x, w, u, v) is admissible process if (Lebesgue selectors) $u(t) \in U(t)$, $v(T) \in V(T)$, point w, and the arc $x(T)$ in $(AC)(\mathbb{R}_+; \mathbb{R}^n)$ satisfy all requirement to the problem above. in particular, the maps $t \mapsto f(t, x(t), u(t))$ and $t \mapsto g_1(t, x(t), u(t))$ are summable on any compact interval.

Similar to [4], call an admissible process $(\hat{x}, \hat{w}, \hat{u}, \hat{v})$ weakly overtaking optimal if, for each admissible process (x, u, v, w),

$$\limsup_{T\to\infty} \Big[g_{02}(T, \hat{x}(0), \hat{w}, \hat{x}(T), \hat{v}(T)) + \int_0^T g_1(\tau, \hat{x}(\tau), \hat{u}(\tau))\, d\tau$$
$$- g_{02}(T, x(0), w, x(T), v(T)) - \int_0^T g_1(\tau, x(\tau), u(\tau))\, d\tau \Big] \geq 0.$$

Note that the considered statement is general enough to accommodate the general approach of [5].

3. Necessary conditions

Further, we consider necessary conditions for a much less general statement of infinite horizon problem:

$$\text{maximize } w'' + \int_0^T u''(t)\,dt + v''(T) \text{ as } T \uparrow \infty \tag{3.1}$$

$$\text{s.t. } \dot{x}(t) = u'(t), (u', u'')(t) \in U(t) \subset \mathbb{R}^{n+1} \text{ a.e.}; \tag{3.2}$$

$$x(0) = w',\ (w', w'') \in W \subset \mathbb{R}^{n+1}, \tag{3.3}$$

$$(v', v'')(T) \in V(T) \subset \mathbb{R}^2,\ x(T) + v'(T) \leq 0 \quad \forall T \geq 0. \tag{3.4}$$

Define $H(p, q, u', u'') \stackrel{\triangle}{=} pu' + qu''$ for all $p, u' \in \mathbb{R}^n$ and $q, u'' \in \mathbb{R}$.

Theorem 1. *Under hypothesis* $(H1)$ *for* U *and* V*, assume also that the set* W *as well as all values of multimaps* U *and* V *are convex and closed. Let* $(\hat{x}, \hat{w}, \hat{u}, \hat{v})$ *be weakly overtaking optimal in* (3.1)–(3.4)*.*

Then, for a natural T*, there exist some non-decreasing in each variable function* $\hat{p} : [0; T] \to \mathbb{R}^n$ *and nonnegative* $\hat{\lambda}$ *such that* $\hat{\lambda} + |\hat{p}(0)| + |\hat{p}(T)| > 0$*,*

$$\sup_{(v', v'') \in W} H\big(\hat{p}(0), \hat{\lambda}, v', v''\big) = H\big(\hat{p}(0), \hat{\lambda}, \hat{w}', \hat{w}''\big), \tag{3.5}$$

$$\sup_{v', v'' \in U(t)} H\big(\hat{p}(t), \hat{\lambda}, v', v''\big) = H\big(\hat{p}(t), \hat{\lambda}, \hat{u}'(t), \hat{u}''(t)\big) \tag{3.6}$$

for almost all $t \in [0; T]$*, and* p *is constant on every subinterval* $(T'; T'')$ *of* $[0; T]$ *for which* $\hat{x}(\tau) + \limsup_{s \to \tau} \hat{v}'(s) < 0$ *for all* $\tau \in (T'; T'')$*.*

If in addition, $\hat{u}(t) + \mathbb{R}^n_+ \times (-\mathbb{R}_+) \subset U(t)$ *for almost all positive* t*, a unit normal to* W *at* $\hat{w}$ *is unique, and* $\hat{w}'' \neq \sup_{(v', v'') \in W} v''$*; then, we can also propose that* $\|\hat{p}(0)\| \neq 0$*,* $\hat{p}(t)$ *is defined on* $\mathbb{R}_+$ *and non-positive in each variable.*

Remark 1. All assumptions on U, W are fulfilled if W and all sets $U(t)$ are the hypographs of some concave smooth and increasing on each variable functions defined on $\mathbb{R}^n_+$.

The proof of Theorem 1 follows Halkin's method [7].

Proof. Define a function $\tilde{v}'(T)$ by the rule $\tilde{v}'(\tau) \stackrel{\triangle}{=} \limsup_{s \to \tau} \hat{v}'(s)$ for all $\tau \geq 0$. This function is upper semicontinuous; furthermore, $x(\tau) + \hat{v}'(\tau) \leq 0$ for all $\tau \geq 0$ leads to $x(\tau) + \tilde{v}'(\tau) \leq 0$.

Fix a natural T. Then, by the Dynamic Programming Principle, the pair $(w, u|_{[0;T]})$ has to be optimal in the following problem:

$$\begin{aligned} &\text{minimize } -z(T) \\ &\quad \text{s.t. } \dot{x}(t) = u'(t), \dot{z}(t) = u''(t),\ (u', u'')(t) \in U(t) \text{ a.e.}; \\ &\qquad w = (x(0), z(0)) \in W,\ x(T) = \hat{x}(T),\ x(\tau) + \tilde{v}'(\tau) \leq 0\ \forall \tau \in [0; T]. \end{aligned}$$

Note that the state constraints in this problem is pure and upper semicontinuous in t. Due to the Pontryagin maximum principle [3, Theorem 10.4.1], [8, Theorem 5], there exist some

– nonnegative number $\hat{\lambda}$;
– functions $\hat{p} \in BV([0;T];\mathbb{R}^n)$ and $\hat{q} \in BV([0;T];\mathbb{R})$;
– nonnegative Borel measure η under $[0;T]$;

for which $\hat{\lambda} + \|\hat{p}(0)\| + |\hat{q}(0)| + \eta([0;T]) \neq 0$,

$$(\hat{p}(t),\hat{q}(t)) = \frac{\partial H}{\partial u}\big(\hat{p}(t),\hat{q}(t),\hat{u}(t)\big) \in N(\bar{U}(t);\hat{u}(t)) \text{ a.e.}, \tag{3.7}$$

$$\hat{p}_1(\tau)-\hat{p}_1(0) = -\int_0^\tau \frac{\partial H}{\partial x}\big(\hat{p}(s),\hat{q}(s),\hat{u}(s)\big)ds + \int_{[0,\tau]} 1\,d\eta = \int_{[0,\tau]} d\eta, \tag{3.8}$$

$$\hat{p}_i(\tau)-\hat{p}_i(0) = -\int_0^\tau \frac{\partial H}{\partial x_i}\big(\hat{p}(s),\hat{q}(s),\hat{u}(s)\big)ds + \int_{[0,\tau]} 0\,d\eta = 0\,(i>1), \tag{3.9}$$

$$\hat{q}(\tau)-\hat{q}(0) = -\int_0^\tau \frac{\partial H}{\partial z}\big(\hat{p}(s),\hat{q}(s),\hat{u}(s)\big)ds + \int_{[0,\tau]} 0\,d\eta = 0\ \forall\tau>0, \tag{3.10}$$

$$\hat{q}(T) = \lambda, \qquad (\hat{p},\hat{q})(0) \in N(W;\hat{x}(0),\hat{z}(0)), \tag{3.11}$$

$$\operatorname{supp}\eta \subset \operatorname{cl}\{\tau \in [0;T] \,|\, x(\tau) + \tilde{v}'(\tau) = 0\}, \tag{3.12}$$

here by $N(A;a)$ we denote the normal cone (of convex analysis) to a set A at a point a.

Note that from (3.8)–(3.10) it follows that $\hat{p}$ is non-decreasing in each variable and $\hat{q}$ is constant. This together (3.11) entails $\hat{q} \equiv \lambda$. Further, in the case of $(\hat{p}(0),\hat{p}(T)) \equiv 0$, by (3.8) and $\hat{\lambda} + \|p(0)\| + |q(0)| + \eta([0;T]) \neq 0$ we obtain $\eta \equiv 0$ with $\hat{q} \equiv \hat{\lambda} = 1$. So, $(\hat{p}(0),\hat{p}(T),\hat{q} = \hat{\lambda}) \neq 0$.

Secondly, since W is convex, for all $(w',w'') \in W$, the second relation in (3.11) yields $\hat{p}(0)(w'-\hat{w}') + \hat{\lambda}(w''-\hat{w}'') \leq 0$; this gives (3.5). Similarly, by the convexity of $U(t)$, from (3.7) it follows (3.6). Finally, (3.12) with (3.8) yield that p is constant on any interval with negative $\hat{x} + \tilde{v}'$.

To prove the second part, note that (3.6) and $\hat{u}(t) + \mathbb{R}^n_+ \times (-\mathbb{R}_+) \subset U(t)$ entail

$$\hat{p}(t)\hat{u}' + \hat{\lambda}\hat{u}'' \geq \hat{p}(t)(\hat{u}'+s) + \lambda(\hat{u}''-r),$$

hence $r \geq \hat{\lambda}r \geq \hat{p}(t)s$, for all $s \in \mathbb{R}^n_+$ and positive r. This means that $\hat{p}_i(t)$ is nonpositive on $[0;T)$.

Suppose that $\hat{p}(0) = 0$. Then, (3.5) and $\hat{w}'' \neq \sup_{(v',v'')\in W} v''$ would yield $\hat{\lambda} = 0$. Then, $\|\hat{p}_i(T)\|$ as well as some $\hat{p}_i(T)$ would be positive. Contradiction. So, $\|\hat{p}(0)\| > 0$.

Since a unit vector in the cone $N(W;\hat{w})$ is unique, the relation (3.5) has to hold true with a unique quotient $\hat{p}(0) : \hat{\lambda}$. Put $\|\hat{p}(0)\| = 1$ when $\hat{\lambda} = 0$. Then, $\hat{p}(0)$ is determined by $\hat{\lambda}$ and is independent of the choice of T. Due to Helly's theorem, increasing $T \uparrow \infty$, passing to the subsequence if necessary, consider its limit instead of $\hat{p}(t)$. This limit is defined on $\mathbb{R}_+$, is also non-decreasing for each variable, and satisfy all needed conditions. $\square$

4. Example

Consider consumer's problem:

$$\begin{aligned}
\text{maximize } J(T) &\triangleq \frac{w^{1-\theta}}{1-\theta} + \int_0^T e^{-\varrho t}\frac{u^{1-\theta}(t)}{1-\theta}\,dt + e^{-\varrho T}\frac{v^{1-\theta}(T)}{1-\theta} \quad \text{as } T\uparrow\infty,\\
\text{s.t. } &\dot{x}(t) = e^{-r\,t}u(t),\ u(t)\ge 0 \text{ a.e.},\\
&x(0) = w \ge 0,\\
&v(T)\ge 0,\ x(T)\le y(T) - e^{-r\,T}v(T) \quad \forall T\ge 0;
\end{aligned}$$

where

- r is constant interest rate;
- ϱ is the individual discount rate;
- $\theta \ge 0$, $\theta \neq 1$ is the constant relative risk aversion coefficient;
- $y(T)$ is the wealth available for spending at time T.

We will also confine our research to the case of a particular borrowing constraint

$$y(t) \triangleq At + B$$

with a given positive B and nonnegative A.

The special case of such problem was considered in [2], with $\theta \to 1$, $A = 0$, $\varrho = 0$, modified constraint $x(T) = y(T) - e^{-r\,T}v(T)$, and the absence of control w.

Theorem 2. *If $(\hat{x}, \hat{w}, \hat{u}, \hat{v})$ is weakly overtaking optimal process in this problem, then*

$\underline{(1-\theta)r < \varrho,\ \theta \neq 0}$:

$$(\hat{w},\hat{u},\hat{v}) = \Big(\frac{(\varrho-(1-\theta)r)B}{\theta+\varrho-(1-\theta)r}, \frac{e^{(r-\varrho)\,t/\theta}(\varrho-(1-\theta)r)B}{\theta+\varrho-(1-\theta)r}, e^{r\,T}\delta(T)\Big) \ \textit{if } A = 0;$$

$$(\hat{w},\hat{u},\hat{v}) = (B, e^{r\,t}A, 0) \ \textit{if } A \ge B, \theta\in(0;1);$$

$$(\hat{w},\hat{u},\hat{v}) = (\hat{w}, e^{r\,t}\max(e^{-(\varrho-(1-\theta)r)\,t/\theta}\hat{w}, A), 0) \ \textit{if } 0 < A < B, \theta \in (0;1),$$
$$(A-\hat{w}+A\ln(\hat{w}/A))\theta = (\varrho-(1-\theta)r)(\hat{w}-B) \ \textit{for } \hat{w}\in(A;B);$$

no weakly overtaking process if $A > 0, \theta > 1$;

$\underline{r < \varrho,\ \theta = 0}$:

$$(\hat{w},\hat{u},\hat{v}) = (B, e^{r\,t}A, 0);$$

$\underline{r = \varrho, \theta = 0}$:

any admissible control with $\hat{\upsilon}(T) = e^{-r\,T}y(T) - \hat{x}(T) - \delta(T)$;

$\underline{(1-\theta)r = \varrho,\ \theta \neq 0}$:

$(\hat{w}, \hat{u}, \hat{\upsilon}) = (A, e^{r\,t}A, e^{r\,T}A - \delta(T))$ *if* $2A = B$,

no weakly overtaking process if $2A \neq B$;

$\underline{(1-\theta)r > \varrho}$:

$(\hat{w}, \hat{u}, \hat{\upsilon}) = (0, 0, e^{r\,T}y(T) - e^{(\varrho+r\theta)\,T}y^{\theta}(T)\delta(T))$ *if* $\theta \in [0; 1)$;

no weakly overtaking process if $\theta > 1$;

for some nonnegative function $\delta(T)$ *such that its lower limit is zero.*

Furthermore, $\hat{x}$ *is concave if* $\theta + |r - \varrho| > 0$.

Proof. Fix $(\hat{x}, \hat{w}, \hat{u}, \hat{\upsilon})$. Denote by $\hat{J}$ its quality function.

Define $h_t(\upsilon) \overset{\triangle}{=} e^{-\varrho\,t}(e^{r\,t}\upsilon)^{1-\theta}/(1-\theta)$ for all $t, \upsilon > 0$ and put $h_t(0) = 0$ if $0 \leq \theta < 1$ and $-\infty$ otherwise. Set $n = 1$. For all nonnegative t denote by $\overline{U}(t)$ the hypograph of the map $h_t(\cdot)$:

$$\overline{U}(t) \overset{\triangle}{=} \{(\upsilon, q) \in \mathbb{R}^2 \,|\, \upsilon \geq 0, q \leq h_t(\upsilon)\}.$$

This set is convex, closed and its boundary is smooth. Put $\overline{W} \overset{\triangle}{=} \overline{U}(0)$, $\overline{V}(T) \overset{\triangle}{=} \overline{U}(T) - (y(T), 0)$ for all $T > 0$. Notice that $\sup_{(\upsilon', \upsilon'') \in \overline{W}} \upsilon'' = \sup_{\upsilon \geq 0} h_0(\upsilon)$ is not reached.

Now, in the problem (3.1)–(3.4) with this data, the process

$$(\hat{x}, (\hat{w}, h_0(\hat{w})), (e^{-r\,t}\hat{u}(t), h_t(e^{-r\,t}\hat{u}(t))), (e^{-r\,T}\hat{\upsilon}(T) - y(T), h_T(e^{-r\,T}\hat{\upsilon}(T))))$$

is weakly overtaking optimal. Then, conditions of Theorem 1 are fulfilled. Applying this Pontryagin Maximum Principle, for a natural $\bar{T}$, there exists a non-decreasing function $p : \mathbb{R}_+ \to \mathbb{R}$ with $\lambda \in \{0, 1\}$ that satisfies $|p(0)| + |p(T)| + \lambda > 0$,

$$p(0)\hat{w} + \lambda \frac{\hat{w}^{1-\theta}}{1-\theta} = \max_{\upsilon \geq 0} \Big[p(0)\upsilon + \lambda \frac{\upsilon^{1-\theta}}{1-\theta} \Big], \tag{4.1}$$

$$p(t)e^{-r\,t}\hat{u}(t) + \lambda e^{-\varrho\,t} \frac{\hat{u}^{1-\theta}(t)}{1-\theta} = \max_{\upsilon \geq 0} \Big[p(t)e^{-r\,t}\upsilon + \lambda e^{-\varrho\,t} \frac{\upsilon^{1-\theta}}{1-\theta} \Big] \tag{4.2}$$

for almost all $t \in [0; \bar{T}]$. In the case of $\theta \neq 0$, all additional assumptions are fulfilled, and we can choose p defined and non-positive on $\mathbb{R}_+$ with $p(0) < 0$. So, either $(\hat{w} - B)\theta = 0$, or $(\hat{w} - B)\theta \neq 0$, $p(0) < 0$, $p(t) \leq 0$ and, for almost all $t \geq 0$ for which $p(t) < 0$,

$$\hat{u}^{\theta}(t)|p(t)| = \lambda e^{(r-\varrho)\,t}. \tag{4.3}$$

We claim that

$$\text{the arc } \hat{x}(t) \text{ is concave if } (1-\theta)r \le \varrho,\ \ \theta \neq 0. \tag{4.4}$$

Assume that $\hat{x}(t) < y(t)$ on some interval $(T';T'') \subset \mathbb{R}_+$. In the case of $\hat{x}|_{(T';T'')} \equiv 0$ this is trivial. Let $\hat{x}|_{(T';T'')} \not\equiv 0$. Applying the Pontrygin Maximum principle for the auxiliary problem with fixed end-points on any $[\tau';\tau''] \subset (T';T'')$, we pick a non-zero pair of some non-positive constant $\bar{p}$ and $\bar{\lambda} \in \{0,1\}$ for which (4.2) holds on $[\tau';\tau'']$. By $\theta \neq 0$, (4.2) gives (4.3) on $[\tau';\tau'']$, hence on $(T';T'')$; (4.3) with $\hat{x} \neq 0$ yields $\bar{\lambda} = 1$ and, for all t', t'', $T' < t' < t'' < T''$,

$$\frac{d\hat{x}(t'')}{dt} = e^{((1-\theta)r-\varrho)t''/\theta}|\bar{p}|^{-1/\theta} \le e^{((1-\theta)r-\varrho)t'/\theta}|\bar{p}|^{-1/\theta} = \frac{d\hat{x}(t')}{dt}. \tag{4.5}$$

Since $d\hat{x}/dt$ is non-decreasing, the non-zero arc $\hat{x}$ must be concave on $(T';T'')$ too. Thus, the arc $\hat{x}$ is concave on a interval $(T';T'')$ for which $\hat{x}(t) < y(t)$. Then, since continuous $\hat{x}$ is no more than y and y is concave, the arc $\hat{x}$ is also concave on $\mathbb{R}_+$ and (4.4) is verified.

The case 1: $(1-\theta)r < \varrho$**.**

We claim that

$$\limsup_{T\to\infty} \frac{\hat{x}(T)}{y(T)} = 1. \tag{4.6}$$

Indeed, were this not so and one would pick a natural k for which $\frac{k+1}{k}\hat{x}(t) \le y(t)$ for all $t > k$. Set $\check{u}(t) = \hat{u}(t)$ if $t < k$ and $\check{u}(t) = \hat{u}(t) + e^{-k\,t}$ otherwise. Increasing k if necessary, we can assume that the control $(\check{w} \overset{\triangle}{=} \hat{w}, \check{u}, \check{v} \overset{\triangle}{=} 0)$ is admissible. Hence its quality $\check{J}$ should satisfy

$$\check{J}(T) - \hat{J}(T) = \int_k^T e^{-\varrho t}\frac{(\hat{u}(t) + e^{-k\,t})^{1-\theta} - \hat{u}^{1-\theta}(t)}{1-\theta}dt - e^{-\varrho T}\frac{\hat{v}^{1-\theta}(T)}{1-\theta}.$$

Since $e^{((1-\theta)r-\varrho)\,T}y^{1-\theta}(T)/(1-\theta) \to 0$, the last term in the right side is the same. But the first term is positive and increasing in T. Therefore $(\hat{x}, \hat{w}, \hat{u}, \hat{v})$ should not be weakly overtaking. Thus, (4.6) is verified.

The case 1a): $\theta \neq 0$, $\hat{x}(t) \neq y(t)$ for all positive t and $(1-\theta)r < \varrho$. Since $\theta \neq 0$ and $\hat{x}(0) = \hat{w} \neq B$, we obtain $p(0) < 0$ and (4.3) for all t with $p(t) < 0$. Further, p is constant because $\hat{x}(t) \neq y(t)$ on $\mathbb{R}_+$. Therefore (4.3) holds for all positive t. Hence (4.1) and (4.3) give

$$\hat{x}(T) = \hat{w} + \hat{w}\int_0^T e^{-(\varrho-(1-\theta)r)\,t/\theta}\,dt \to \Big(1 + \frac{\theta}{\varrho-(1-\theta)r}\Big)\hat{w}.$$

By (4.6), we obtain that y is bounded, i.e. $A = 0$. Now, (4.6) with $A = 0$ yield $\hat{x}(T) \to B$, which means $e^{-r\,T_n}\hat{v}(T_n) \to 0$,

$$\hat{w} = \frac{(\varrho-(1-\theta)r)B}{\theta+\varrho-(1-\theta)r}, \quad \hat{u}(t) = \frac{(\varrho-(1-\theta)r)B}{\theta+\varrho-(1-\theta)r}e^{(r-\varrho)\,t/\theta}.$$

The case 1b): $\hat{x}(0) = B$, $\theta \neq 0$, and $(1-\theta)r < \varrho$.
By $\hat{x}(T_n)/y(T_n) \to 1$ and the concavity (4.4), $\hat{x}(0) = B$ entails $\hat{x} \equiv y$. This also yields $\hat{v}(T) \equiv 0$, $\hat{w} = B$ and $A = \frac{d\hat{x}(t)}{dt} = e^{-r\,t}\hat{u}(t)$ for all t. Thus,

$$\hat{w} = B,\ \hat{u}(t) = e^{r\,t}A,\ \hat{v} = 0. \tag{4.7}$$

Note that, from $\hat{v}(T) = 0$ for all sufficiently large T, it follows that $\theta < 1$ because $h_T(0) = -\infty$ in the case of $\theta > 1$. So, $\theta < 1$.

In addition, we claim that $(1-\theta)r < \varrho$, $\theta \neq 0$, and (4.7) imply $A \geq B$. Indeed, for all sufficiently small $\varepsilon > 0$ define $\check{w} \overset{\triangle}{=} B + A\varepsilon - A\theta(e^{\sigma\,\varepsilon/\theta} - 1)/\sigma$, $\check{v} \overset{\triangle}{=} 0$, $\check{u}(t \overset{\triangle}{=} Ae^{\sigma\,\varepsilon/\theta}e^{-(\varrho-r)\,t/\theta}$ if $t \leq \varepsilon$ and $\check{u}(t) \overset{\triangle}{=} \hat{u}(t)$ otherwise, here $\sigma = \varrho - (1-\theta)r > 0$. Routine calculation yield that, for small ε, this triplet is admissible and, for all $t \geq \varepsilon$. its arc and quality satisfy $\check{x}(t) = At + B = \hat{x}(t)$ and

$$\hat{J}(t) - \check{J}(t) = A\sigma\varepsilon^2\frac{B^{-\theta} - A^{-\theta}}{2\theta}(1 + o(\varepsilon)).$$

This leads to $\hat{J}(t) - \check{J}(t) > A(\varrho - (1-\theta)r)\varepsilon^2(B^{-\theta} - A^{-\theta})/3\theta > 0$ for all large t and small ε if $A < B$. Since $\hat{J}$ is optimal, we have checked that $A \geq B$.

The case 1c): $\hat{x}(0) < B$, $(1-\theta)r < \varrho$, $\theta \neq 0$, $\hat{x}(\hat{\tau}) = y(\hat{\tau})$ for some $\hat{\tau}$. We can also assume that $\hat{x}(t) < y(t)$ for all $t < \hat{\tau}$; in particular, $(\hat{w} - B)\theta \neq 0$ that ensures (4.3) on $[0;\hat{\tau})$; in addition, the arc $\hat{x}$ is smooth on $[0;\hat{\tau}]$ and we can propose that $\hat{u}$ is the same. Again, (4.6) and (4.4) entail $\hat{x}(t) = y(t)$ for all $t \geq \hat{\tau}$. This also yields $\hat{u}|_{[0;\hat{\tau}]} \neq 0$, $\hat{v}(T) \equiv 0$, and $\theta < 1$.

Let N be a natural number that more than $\hat{\tau}$. Then, $(\hat{x}, \hat{u}, \hat{\tau})$ is C-local optimal to the following auxiliary problem:

$$\begin{aligned} \text{maximize } & \frac{x^{1-\theta}(0)}{1-\theta} + \int_0^T e^{-\varrho\,t}\frac{u^{1-\theta}(t)}{1-\theta}\,dt + \int_T^N e^{-\varrho\,t}\frac{(e^{-r\,t}A)^{1-\theta}}{1-\theta}\,dt \\ \text{s.t. } & \dot{x}(t) = e^{-r\,t}u(t),\ u(t) \geq 0 \text{ a.e.}, \\ & x(0) \geq 0,\ \hat{x}(\tau) \leq A\tau + B\ \forall \tau \in [0;N]. \end{aligned}$$

Now, by the Pontryagin Maximum Principle [6, Theorem 3.4.2], there exists a non-zero pair $(\bar{p}, \bar{\lambda}) \in \mathbb{R} \times \{0,1\}$ that satisfies (4.1), (4.2), and the following transversality condition at $\hat{\tau}$:

$$\frac{\bar{\lambda}e^{-\varrho\,\hat{\tau}}(e^{r\,\hat{\tau}}A)^{1-\theta}}{1-\theta} - \frac{\bar{\lambda}e^{-\varrho\,\hat{\tau}}\hat{u}^{1-\theta}(\hat{\tau})}{1-\theta} - (e^{-r\,\hat{\tau}}\hat{u}(\hat{\tau}) - A)\bar{p} = 0. \tag{4.8}$$

Now, (4.3) with $\hat{u} \neq 0$ gives $\hat{u}^{\theta}(\hat{\tau})|\bar{p}| = \bar{\lambda}e^{(r-\varrho)\,\hat{\tau}}$, $\hat{\lambda} = 1$, $\bar{p} < 0$ and, by (4.8),

$$\frac{(e^{r\,\hat{\tau}}A)^{1-\theta} - \hat{u}^{1-\theta}(\hat{\tau})}{1-\theta} = \hat{u}^{-\theta}(\hat{\tau})(\hat{u}(\hat{\tau}) - e^{r\,\hat{\tau}}A).$$

By the strict concavity of h_0, $h_0(\upsilon')-h_0(\upsilon)=\frac{dh_0(\upsilon)}{d\upsilon}(\upsilon'-\upsilon)$ iff $\upsilon=\upsilon'$, hence the corresponding substitution gives $\hat{u}(\hat{\tau})=e^{r\,\hat{\tau}}A>0$. Further, $\hat{x}(0)=\hat{w}<B$ and (4.1) yield $\hat{u}(\hat{\tau})=e^{(r-\varrho)\,\hat{\tau}/\theta}\hat{w}$, $A=e^{-\sigma r\,\hat{\tau}/\theta}\hat{w}<\hat{w}<B$, $\hat{\tau}=\frac{\theta\ln(\hat{w}/A)}{\sigma}$, here $\sigma\overset{\triangle}{=}\varrho-(1-\theta)r>0$. Hence $\hat{x}(\hat{\tau})=y(\hat{\tau})$ entails

$$\frac{A\theta\ln(\hat{w}/A)}{\sigma}+B=A\hat{\tau}+B=\hat{w}+\int_0^{\hat{\tau}}e^{-\sigma\,t/\theta}\hat{w}\,dt=\hat{w}+\frac{(\hat{w}-A)\theta}{\sigma}.$$

Thus, $\hat{u}(t)=e^{r\,t}\max(A,e^{-(\varrho-(1-\theta)r)\,t/\theta}\hat{w})$ if $\theta>0$, $0<A<\hat{w}<B$, here $\hat{w}$ solves the equation $(A-\hat{w}+A\ln(\hat{w}/A))\theta=(\varrho-(1-\theta)r)(\hat{w}-B)$.

The case 1d): $\theta=0$ and $r<\varrho$.

We claim that in this case $\hat{x}(t)=y(t)$ for all positive t. Indeed, were this not so. Then, there exists a positive t_0 for which $\hat{x}(t_0)\neq y(t_0)$. By (4.6), there exists a minimal $\tau>0$ such that $\hat{x}(t_0)+y(t_0+2\tau)=2\hat{x}(t_0+2\tau)$. Hence $\int_{t_0+2\tau-\varepsilon}^{t_0+2\tau}\hat{u}(t)\,dt>0$ for all small $\varepsilon>0$. Therefore, one find a positive ε and Lebesgue subset $A\subset[t_0+\tau;t_0+2\tau]$ for which $\hat{u}(t)>e^{r\,t}\varepsilon$ on A and $0<\varepsilon\int_A dt<y(t_0+2\tau)-\hat{x}(t_0+2\tau)$. Define $\check{u}(t)\overset{\triangle}{=}\hat{u}(t)+e^{r\,t}\varepsilon$ if $t+\tau\in A$, $\check{u}(t)\overset{\triangle}{=}\hat{u}(t)-e^{r\,t}\varepsilon$ if $t\in A$, and $\check{u}(t)\overset{\triangle}{=}\hat{u}(t)$ otherwise. The control $(\check{w}\overset{\triangle}{=}\hat{w},\check{u},\check{v}\overset{\triangle}{=}\hat{v})$, by $\hat{x}|_{[t_0+2\tau;\infty)}=\check{x}|_{[t_0+2\tau;\infty)}$, should be admissible and its quality $\check{J}$ should satisfy

$$\check{J}(T)-\hat{J}(T)=\int_{-\tau+A}e^{(r-\varrho)\,t}\varepsilon\,dt-\int_A e^{(r-\varrho)\,t}\varepsilon\,dt=(e^{(\varrho-r)\,\tau}-1)\varepsilon\int_A e^{(r-\varrho)\,t}\,dt$$

By $r<\varrho$, this difference is unboundedly increasing. This contradicts the optimality of $\hat{J}$. So, $\hat{x}(t)=y(t)$ for all positive t. Similar the case 1b), this gives (4.7).

The case 2: $(1-\theta)r=\varrho$.

Note that $(1-\theta)r=\varrho$ means that the quality J can be written as

$$J(T)=\frac{w^{1-\theta}}{1-\theta}+\int_0^T\frac{(e^{-r\,t}u(t))^{1-\theta}}{1-\theta}\,dt+\frac{(e^{-r\,T}v(T))^{1-\theta}}{1-\theta}.\tag{4.9}$$

Now, from the optimality of $(\hat{x},\hat{w},\hat{u},\hat{v})$ it follows that

$$\liminf_{T\to\infty}\Big[\frac{(y(T)-\hat{x}(T))^{1-\theta}}{1-\theta}-\frac{(e^{-r\,T}\hat{v}(T))^{1-\theta}}{1-\theta}\Big]=0,$$

i.e. $y(T_n)-\hat{x}(T_n)-e^{-r\,T_n}\hat{v}(T_n)\to 0$ for some $T_n\uparrow\infty$. Note that, this condition is also sufficient for optimality in the case of $\theta=0$, when $J(T)=x(T)+e^{-r\,T}v(T)\leq y(T)$.

Let $\theta\neq 0$. In this case we checked above the inequality (4.5); further, the equality $(1-\theta)r=\varrho$ ensures the equality in (4.5). Hence $\dfrac{\hat{x}(t'')-\hat{x}(t')}{t''-t'}$

is constant on any interval with negative $\hat{x} - y$. Since $\hat{x} \leq y$, the graph of y is a line and any connected bounded subset of graph of $\hat{x}$ lying below the graph of y is a line segment, we obtain that the graph of $\hat{x}$ is the line and

$$\hat{x}(t) = \hat{w} + (\hat{x}(1) - \hat{w})t, \qquad \forall t \geq 0.$$

We claim that $\hat{x}(1) - \hat{w} = A$. Were this not so, let $a \overset{\triangle}{=} \hat{x}(1) - \hat{w} < A$, define $\overline{a} \overset{\triangle}{=} \frac{A+a}{2}$. Increasing $\hat{v}(T)$ if necessary, we can also assume that $e^{-r\,T_n}\hat{v}(T_n) \geq (A-a)T_n = 2(\overline{a}-a)T_n$. Now, $(\check{w} \overset{\triangle}{=} \hat{w}, \check{u}(t) \overset{\triangle}{=} e^{r\,t}\overline{a}, \check{v}(T) \overset{\triangle}{=} \hat{v}(T) - e^{r\,T}(\overline{a}-a)T)$ is admissible and, by (4.9) the difference $\check{J}(T_n) - \hat{J}(T_n)$ is no more than

$$\frac{T_n\,(\overline{a}^{1-\theta} - a^{1-\theta}) + (e^{-r\,T_n}\hat{v}(T_n) - (\overline{a}-a)T_n)^{1-\theta} - (e^{-r\,T_n}\hat{v}(T_n))^{1-\theta}}{1-\theta}$$
$$\geq \frac{T_n\,(\overline{a}^{1-\theta} - a^{1-\theta})}{1-\theta} - (\overline{a}-a)T_n\Big(e^{-r\,T_n}\hat{v}(T_n) - (\overline{a}-a)T_n\Big)^{-\theta}$$
$$\geq \frac{T_n\,(\overline{a}^{1-\theta} - a^{1-\theta})}{1-\theta} - (\overline{a}-a)^{1-\theta}T_n^{1-\theta}.$$

By $\theta > 0$, the right side is unboundedly increasing, hence the difference is the same. This contradicts the optimality of $\hat{J}$. Thus, $\hat{x}(t) = \hat{w} + At$, $\hat{u}(t) = e^{r\,t}A$ for all nonnegative t.

Since $\hat{x}(t) = \hat{w} + At$, we can assume $\hat{w} + e^{-r\,t}\hat{v}(T_n) \leq B$. By (4.9), $\hat{w}$ has to maximize $\left(\frac{w^{1-\theta}}{1-\theta} + \frac{(B-w)^{1-\theta}}{1-\theta}\right)$; this yields $\hat{w} = B/2$. Now, by $\hat{w} < B$, from (4.1) and (4.2) it follows $B/2 = \hat{w} = \arg\max_{\upsilon \geq 0}\left[p\upsilon + \frac{\upsilon^{1-\theta}}{1-\theta}\right] = \hat{u}(0) = A$.

Thus, in the case of $B \neq 2A$ no weakly overtaking process; in the case of $B = 2A$ we obtain $(A, e^{r\,t}A, e^{-r\,T}A - \delta(T))$.

The case 3: $(1-\theta)r > \varrho$**.**

We claim that $\hat{w} = 0$, $\hat{u} \equiv 0$, $\hat{x} \equiv 0$, $\theta > 1$. Suppose $\hat{w} > 0$. Consider $(\check{w} \overset{\triangle}{=} \hat{w}/2, \check{u} = \hat{u}, \check{v}(T) \overset{\triangle}{=} \hat{v}(T) + e^{r\,T}\hat{w}/2)$. This triplet is admissible and

$$\frac{\check{v}^{1-\theta}(T) - \hat{v}^{1-\theta}(T)}{1-\theta} \geq (\hat{v}(T) + e^{r\,T}\hat{w}/2)^{-\theta}e^{r\,T}\hat{w}/2 \geq \hat{y}^{-\theta}(T)e^{(1-\theta)r\,T}\hat{w}/2.$$

However, the quality $\check{J}$ of this process would be better than $\hat{J}$. Indeed, $\check{J}(T) - \hat{J}(T) \geq e^{((1-\theta)r-\varrho)\,T}y^{-\theta}(T)\hat{w}/2 - 2^{\theta-1}h_0(\hat{w})$, there the right side of this inequality is positive because $(1-\theta)r - \varrho > 0$ and y has a sublinear growth. This contradiction has shown $\hat{w} = 0$.

In the case $\theta > 0$, by (4.1) and (4.3), $\hat{w} = 0$ gives $\hat{u} \equiv 0$ and $\theta < 1$.

Let $\theta = 0$, by (4.1) and (4.2), $\hat{w} = 0$ gives $p(0) + \lambda \leq 0$ and $p(t) + e^{(r-\varrho)\,t}\lambda \leq 0$ for all $t \geq 0$. Then, from $r > \varrho$ it follows either $\lambda = 0$, $p(0) < 0$, or $\hat{x}(t) = y(t)$ for large t. Hence $\hat{u}(t) = 0$ as long as $p(t) \leq 0$ or at least $\hat{x}(t) < y(t)$. Since $\hat{u}|_{[0;\tau]} = 0$ gives $\hat{x}|_{[0;\tau]} = 0$, we obtain $\hat{u} \equiv 0$ too.

Hence $\hat J(T) = e^{-\varrho T}\hat v^{1-\theta}(T)/(1-\theta)$ and, by the definition of optimality,

$$e^{-\varrho T_n}\Big[\frac{(e^{r\,T_n}y(T_n))^{1-\theta}}{1-\theta}-\frac{\hat v^{1-\theta}(T_n)}{1-\theta}\Big] \ge e^{((1-\theta)r-\varrho)\,T_n}(y(T_n)-e^{-r\,T_n}\hat v(T_n))(e^{-r\,T_n}\hat v(T_n))^{-\theta}\to 0$$

for some $T_n\uparrow\infty$. Then, $y(T)-e^{-r\,T}\hat v(T)=e^{(\varrho-(1-\theta)r)\,T}y^\theta(T)\delta(T)$. □

5. Conclusion

In this paper, we consider a new formulation of the infinite-horizon control problem. Unlike the usual formulation [1], this formulation allows us to hope for the existence of an optimal solution for the consumer problem. However, in the simplest case that we have already analyzed, there may still not be a weakly overtaking optimal solution.

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