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One Remark on the Existence Theorems for Generalized Impulsive Control Problems

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Abstract: In this work, generalized solutions to optimal control problems are discussed. A notion of generalized impulsive control is introduced. Some extension is proposed for a constrained control problem governed by the dynamics of a general type. A corresponding existence theorem is formulated within the class of discontinuous arcs. The presented extension is smaller than those previously derived in literature for this type of problems, as it contains less generalized impulsive controls, and, correspondingly, less trajectories. This is achieved by rejecting the problem convexification. As the main tool for investigation, the generally known Lebesgue discontinuous time variable change is employed. It is important noting that the obtained existence theorem is not always applicable. Therefore, a task of finding more subtle conditions for the existence of a solution arises. In this regard, a number of classical variational calculus problems are discussed in the context of presented nonlinear impulsive extension. This article is dedicated to the memory of Vladimir Alexandrovich Dykhta.

Keywords: optimal impulsive control, generalized solutions, existence theorems

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Научная статья

Одно замечание о теоремах существования для обобщенных задач импульсного управления

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Аннотация: Рассматриваются обобщенные решения задач оптимального управления. Вводится понятие обобщенного импульсного управления. Предлагается некоторое расширение для задачи управления с ограничениями, подчиняющейся динамике общего вида. Сформулирована соответствующая теорема существования в классе разрывных дуг. Представленное расширение является более узким, чем ранее полученные в литературе для задач этого типа, поскольку содержит меньше обобщенных импульсных управлений и, соответственно, меньше траекторий. Это достигается за счет отказа от конвексификации задачи. В качестве основного инструмента исследования применяется общеизвестная разрывная замена переменной времени Лебега. Эта замена переменной реализуется за счет некоторой редукции задачи. Важно отметить, что полученная теорема существования не всегда применима. Поэтому возникает задача нахождения более тонких условий существования решения. В связи с этим обсуждается ряд классических задач вариационного исчисления в контексте представленного нелинейного импульсного расширения. Статья посвящается памяти Владимира Александровича Дыхты.

Ключевые слова: оптимальное импульсное управление, обобщенные решения, теоремы существования

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1. Introduction

Some problems of Calculus of variations do not have continuous solutions. There exists a number of classical examples illustrating this phenomenon such as Catenary, Dido's problem, etc., when the boundary conditions reach critical values. The absence of a classical solution naturally gives rise to the issue of extension or relaxation of the problem. The extension leads to the concept of a generalized solution. Regarding the topic of extensions, one can read, for example, in the books [1; 5; 10].

Passing to a broader setting, one can consider controlled dynamics

$$\dot{x}(t) = f(x(t), u(t), t), \quad u(t) \in U \ \text{ a.e. } t \in [0, 1],$$

where $u(t)$ is a measurable function in $L_p([0, 1])$, $p \geq 1$. Considering endpoint constraints and a minimizing cost, one can examine an optimal control problem over this dynamics, in which discontinuous arcs can be candidates for solutions. Such arcs may only arise when the set U is unbounded. Indeed, on the one hand, it is clear that discontinuities appear when $\dot{x}$ begins to take unbounded values. On the other hand, in the case of a bounded set U, and under fairly general assumptions, an extension into the class of absolutely continuous trajectories is implementable. It was proposed in [8] and is based on the concept of generalized control. For compact U, the set of generalized controls is weakly sequentially compact which is of decisive importance for the existence of a solution in the extended problem. However, if U is unbounded, then this is not the case. For this reason, when extending a control problem, it follows convenient to split the control parameter in two parts u and v, and to put that $u \in U$, and U is compact, while $v \in V$, and V is some closed, but necessarily unbounded set, for example, a cone.

Consider a simple type of extension in case when $V \neq \varnothing$. The measurable control function $v(t)$ can be replaced by a Borel measure μ on $[0, 1]$. Indeed, on the one hand, any absolutely continuous Borel measure generates an integrable function, namely its Radon-Nikodym derivative: $d\mu/dt = v(t)$. On the other hand, there are measures which cannot be associated with any measurable integrable function, for example, Dirac's measure. Then, it is simple to verify that this approach offers some extension for the following dynamical system affine in the control variable v:

$$\dot{x} = f(x, u, t) + G(t)v, \quad v \in K,$$

where $u = u(t)$ is an ordinary bounded control with values in U, $v = v(t)$ is an unbounded control function with values in some convex closed cone K, and G is some smooth matrix-valued function. Replacing v with μ, and minimizing the cost on the solutions to the extended control system with measure, one arrives at the optimal impulsive control problem in its simplest form. The generalized trajectories $x(t)$ are functions of bounded variation, and therefore, can exhibit discontinuities. The generalized controls are Borel vector-valued measures which are also termed as impulsive controls. The idea of such an extension was first proposed in [19; 24].

The complexity of the extension increases if one considers more general control systems, for example, when the matrix G begins to depend also on the state variable: $G = G(x, t)$, or even on the both state and control variables: $G = G(x, u, t)$. It follows that in this case, the class of Borel measures is not sufficient to describe the generalized control in the extended problem. The impulsive control turns out to be something more than just a Borel measure, but it is a pair $(\mu; \{v_\tau\})$, where μ is a Borel measure, and $\{v_\tau\}$ is a certain family of ordinary measurable and bounded controls termed attached. The attached controls act only on the breaks of the system, that is, at the moments when an impulse occurs. This case of extension was of interest to many researchers as there is an extensive literature on this topic, see, e.g., [2; 3; 5; 11; 14; 25]. This list of works is far from exhaustive.

The general case is given by the system

$$\dot{x} = g(x, v, t), \quad v \in V, \tag{1.1}$$

where g is some smooth vector-valued function, and the set V, as noted above, is closed and unbounded. The system (1.1) obviously includes the above considered cases. Various types of extensions to (1.1) have been studied, for example, in [12; 15; 18; 20; 24].

In [12], the idea of 'merging' a Borel measure μ on $[0, 1]$ with generalized Gamkrelidze controls over V was proposed by virtue of the discontinuous Lebesgue time-variable change. Such a composition of controls of different types, together with some standard compactification procedure, leads to a fairly general extension of the original problem into the class of discontinuous trajectories. In this work, an extension is proposed for a constrained control problem driven by (1.1), which is smaller than the one of [12], as it contains less generalized impulsive controls, and, correspondingly, less trajectories. This is achieved due to rejecting the problem convexification. The drawback is a weak existence theorem, which is not always applicable. Therefore, a task of finding more subtle conditions for the existence of a solution arises. In this regard, some classical variational problems are discussed in the context of a nonlinear impulsive extension.

In general, there exists an extensive literature focusing at the theory of optimal impulsive control. In addition to those already mentioned above, there are, for example, sources [4; 6; 9; 13; 16; 17; 21–23]. In conclusion, let us note profound works of Vladimir Alexandrovich Dykhta on the control theory and, in particular, the book [5], which had a great impact on the author's Ph.D. thesis, [11].

2. Statement of the problem

Below, we will study generalized solutions which may arise in the following constrained optimal control problem:

$$
\begin{array}{rl}
\text{Find the minimum of} & \varphi(x_0, x_1) \\
\text{under constraints} & \dot{x} = g(x, v, t), \\
& x_0 = x(t_0) \in A,\ x_1 = x(t_1) \in B, \\
& h(x(t), t) \leq 0 \ \ \forall\, t \in [t_0, t_1], \\
& v(t) \in V \ \ \text{a.e.}\, t \in [t_0, t_1],
\end{array}
\tag{2.1}
$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^l$ are given continuous mappings, A, B, V are given closed sets, and $[t_0, t_1]$ is a given time interval which is fixed. The measurable function $v(t)$ is a control, which is allowed to take unbounded values. The function h specifies the so-called inequality-type state constraints.

Let us associate with Problem (2.1) some a priori given scalar function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. The function ω has the following properties:

a) $\omega(0) = 1$;

b) it is continuous and increasing;
c) $\lim\limits_{y\to\infty} \omega(y) = +\infty$.[1]

Suppose that the control function $v(t)$ in Problem (2.1) is such that $\omega(|v(t)|)$ is integrable. If, for example, $\omega(y) = 1 + y$, then v is a function of class L_1, if $\omega(y) = 1 + y^2$, then v is a function of class L_2, and so on. Thus, the function ω defines the base class of measurable controls for Problem (2.1). Those functions v for which $\omega(|v|)$ is non-integrable will not participate in the construction of the extension. If it is necessary to extend the problem with bounded controls, that is, for class L_∞, then the function ω can be chosen arbitrarily taking into account its properties mentioned above, for example, $\omega(y) = 1 + y$.

It is clear that (2.1) is more general in its formulation than the previously discussed problem, in which the set of control parameters U is compact. The set V in (2.1) can be unbounded. The formulation (2.1) includes the possibility of minimizing the integral functional

$$\int_{t_0}^{t_1} f_0(x, u, t)dt.$$

This can be done by introducing an additional state variable $\chi : \dot{\chi} = f_0(x, u, t)$, $\chi_0 = 0$, with χ_1 to be minimized.

In order to perform the extension, we need a natural compactification of the space $\mathbb{R}^m$, which is obtained by adding to $\mathbb{R}^m$ the set S_∞, called the 'sphere at infinity'. The sphere at infinity is the usual $(m-1)$-dimensional unit sphere, but it is contained in the copy of $\mathbb{R}^m$. Formally, such a compactification is defined as a pair (Θ, B_1), where B_1 is a closed unit ball in the $\mathbb{R}^m$-copy, and $\Theta : \mathbb{R}^m \to B_1$ is an embedding, which is defined by the formula:

$$\Theta(v) = \frac{v}{1 + |v|}, \quad v \in \mathbb{R}^m.$$

Extending this embedding onto the sphere at infinity by the identity mapping $\Theta(l) = l$, $l \in S_\infty$, one obtains a topology on $\bar{\mathbb{R}}^m := \mathbb{R}^m \cup S_\infty$, in which the sets $\Theta^{-1}(O)$ are considered open, where O is open in the induced topology on B_1. Then, $\Theta : \bar{\mathbb{R}}^m \to B_1$ is a homeomorphism, while the space $\bar{\mathbb{R}}^m$ is topologically equivalent to the closed unit ball, and hence, is compact.

Note that the set V is closed in $\mathbb{R}^m$, but generally speaking, not in $\bar{\mathbb{R}}^m$. Denote by $\bar{V}$ its closure in the above described topology of $\bar{\mathbb{R}}^m$. Clearly, the relation $\bar{V} = \Theta^{-1}\Big(\mathrm{cl}\,\Theta(V)\Big)$ holds, which will be used below.

Consider the function

$$\bar{g}(x, v, t) := \frac{g(x, v, t)}{\omega(|v|)}.$$

[1] One can also consider the case of a finite limit. However, then, as it will become clear from the further exposition, discontinuities of trajectories do not arise. This case fits into the already known theory.

Our main assumption will be that the mapping $\bar{g}$ is *continuously extendable* from $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1$ to $\mathbb{R}^n \times \bar{\mathbb{R}}^m \times \mathbb{R}^1$. Thus, for each x, t the function $\bar{g}(x, \cdot, t)$ is defined on the compactified space $\bar{\mathbb{R}}^m$, and is continuous. In particular, it is continuous on the sphere at infinity, and one has

$$\bar{g}(x, l, t) = \lim_{v \to l} \frac{g(x, v, t)}{\omega(|v|)} \quad \forall\, x, t \in \mathbb{R}^n \times \mathbb{R}^1, \ \ \forall\, l \in S_\infty.$$

Above, the convergence of elements $v \in \mathbb{R}^m$ to an element $l \in S_\infty$ is understood in the sense of topology on $\bar{\mathbb{R}}^m$ and means that

$$\Theta(l) = \frac{v}{|v|} + \alpha(v),$$

where $\alpha(v) \to 0$ for $|v| \to \infty$, $v \neq 0$.

Consider the following definition.

Definition 1. *We will say that the control problem (2.1) admits an impulsive extension of order ω if the mapping $\bar{g}$ can be continuously extended to $\mathbb{R}^n \times \bar{\mathbb{R}}^m \times \mathbb{R}^1$ and this extension is nontrivial, that is,*

$$\bar{g}(\cdot, \cdot, \cdot)|_{\mathbb{R}^n \times S_\infty \times \mathbb{R}^1} \neq 0.$$

Thus, if there are points x, t such that the function $\bar{g}(x, \cdot, t) : S_\infty \to \mathbb{R}^n$ is not identically equal to zero, then we say that the control problem admits an impulsive extension of order ω.

Consider also the following hypothesis regarding $\bar{g}$.

H1) The mapping $\bar{g}$ satisfies the following estimate:

$$|\bar{g}(x, v, t)| \leq \kappa(t)(1 + |x|) \ \ \forall\, (x, v, t) \in \mathbb{R}^n \times \bar{\mathbb{R}}^m \times \mathbb{R}^1,$$

where κ is some integrable function on $[t_0, t_1]$.

Set $T = [t_0, t_1]$. Consider a Borel measure $\mu : \mathcal{B}(T) \to [0, +\infty)$ such that $\mu \geq \ell$. Here, $\mathcal{B}(T)$ denotes the σ-algebra of Borel subsets of T, and ℓ is the Lebesgue measure on $\mathbb{R}$. The inequality $\mu \geq \ell$ means that $\mu(B) \geq \ell(B)$ $\forall\, B \in \mathcal{B}(T)$. Below, we will identify μ with its unique completion in the Lebesgue-Stieltjes sense. Thus, one also has that $\mu(E) \geq \ell(E)$ $\forall\, E \in \mathcal{L}(T)$, where $\mathcal{L}(T)$ is the σ-algebra of Lebesgue subsets of T.

Consider the following change of variable which is, in fact, the well-known discontinuous time-variable change due to H. Lebesgue,

$$\pi(t) = \mu([t_0, t]), \ t \in (t_0, t_1], \ \ \pi(t_0) = 0. \tag{2.2}$$

It is clear that $\pi : T \to [0, \|\mu\|]$, where $\|\mu\| = \mu([t_0, t_1])$ is the total variation of the measure. It is simple to derive that, since $\mu \geq \ell$, there exists an inverse function $\theta(s)$: $[0, \|\mu\|] \to T$ such that

a) $\theta(s)$ is monotonically increasing;

b) $\theta(s)$ is Lipschitz continuous with $|\theta(s)-\theta(t)| \le |s-t|\ \forall\, s,t$;
c) $\theta(s)=\tau$, $\forall\, s\in\Gamma_\tau$, $\forall\,\tau\in T$, where $\Gamma_\tau=[\pi(\tau^-),\pi(\tau^+)]$.

Note that the function $\pi(t)$ maps μ-measurable sets to ℓ-measurable sets. Indeed, this follows directly from the definition of π and from the representation of a measurable set as union of a Borel set and a set of zero measure. Therefore, if a set E is μ-measurable, then the set $\theta^{-1}(E)$ is measurable. This implies the following important fact. The function $v(\theta(\cdot))$ is measurable if $v(\cdot)$ is μ-measurable. Thus, the following change of variable under the integral is feasible:

$$\int_{t_0}^{t_1} v(t)d\mu = \int_0^{\|\mu\|} v(\theta(s))ds.$$

By $\mathrm{Ds}(\mu)$, denote the set of atoms of μ, that is,

$$\mathrm{Ds}(\mu) := \{\tau\in T:\ \mu(\{\tau\})>0\}.$$

By $\mathcal{D}(t;\mu)$, we denote the Radon-Nikodym derivative (if exists) of the Lebesgue measure ℓ w.r.t. μ, that is, $\mathcal{D}(t;\mu):=\frac{d\ell}{d\mu}$. It is clear that $\mathcal{D}(\cdot;\mu)$ is μ-measurable and has values in $[0,1]$.

3. Extended Problem

The extension for (2.1) takes the following form:

$$\text{Find the minimum of} \qquad \varphi(x_0,x_1), \tag{3.1}$$

$$\text{subject to constraints} \qquad x(t)=\int_{t_0}^{t}\bar g(x,v,t)d\mathfrak{c},\ \ \forall\, t\in(t_0,t_1], \tag{3.2}$$

$$x_0\in A,\ x_1\in B, \tag{3.3}$$

$$h(x,t)\le 0, \tag{3.4}$$

$$\mathfrak{c}=\{\mu,v,v_\tau\},\ \operatorname{range}\mathfrak{c}\subseteq \bar V.$$

The above formulae and new notation require clarification. The symbol $\mathfrak{c}$ denotes *generalized impulsive control.* By definition, it consists of three components:
– $\mu:\mathcal{B}(T)\to[0,+\infty)$ – a non-negative Borel measure such that $\mu\ge\ell$;
– $v:T\to\bar V$ – a μ-measurable function such that[2]

$$\mathcal{D}(t;\mu)=\frac{1}{\omega(|v(t)|_{\mathbb{R}^m})}\ \ \mu\text{-a.e.}; \tag{3.5}$$

[2] In (3.5), it is naturally assumed that $|v|_{\mathbb{R}^m}=\infty\ \forall\, v\in S_\infty$, $\omega(\infty)=\infty$, and $\frac{1}{\infty}=0$. Hence, in particular, taking into account that $\mu\ge\ell$, it follows that the set of points t in which the control value $v(t)$ is on the infinitely distant sphere has zero ℓ-measure.

– $v_\tau : [0,1] \to \bar{V} \cap S_\infty$ – a family of measurable functions depending on $\tau \in \mathrm{Ds}(\mu)$ with the values from the infinitely distant sphere.

The family $\{v_\tau\}_{\tau\in\mathrm{Ds}(\mu)}$ is termed the *attached* family of controls. It is clear that all the three components of the generalized impulsive control are closely related with each other.

Let $x_\tau(s)$ denote the solution of the attached differential control system:

$$\begin{cases} \dot{x}_\tau(s) = \Delta_\tau \bar{g}(x_\tau(s), v_\tau(s), \tau), \ s \in [0,1], \\ x_\tau(0) = x(\tau^-), \end{cases}$$

where $\Delta_\tau := \mu(\{\tau\})$. A function of bounded variation $x(t)$ is called a solution of Equation (3.2) corresponding to $x_0 \in A$ provided that:

$$x(t) = x_A + \int_{t_0}^{t} \bar{g}(x(\theta), v(\theta), \theta) d\mu_{\mathrm{c}} + \sum_{\tau\in\mathrm{Ds}(\mu):\, \tau\le t} \Big(x_\tau(1) - x_\tau(0) \Big).$$

for all $t \in (t_0, t_1]$, and $x(t_0) = x_0$. Here, μ_{c} is the continuous component of μ, that is, the sum of its absolutely continuous and singular components.

The state constraints (3.4) are understood in a somewhat broader sense than the usual inequality. More precisely,

$$h(x,t) \le 0 \quad \Leftrightarrow \quad \begin{cases} h(x(t),t) \le 0, & \forall\, t \in T, \\ h(x_\tau(s),\tau) \le 0, & \forall\, s \in [0,1], \ \forall\, \tau \in \mathrm{Ds}(\mu). \end{cases}$$

A pair $(x, \mathfrak{c})$ is called a control process if condition (3.2) is valid. A control process is called admissible if the endpoint constraints (3.3) and state constraints (3.4) are satisfied. The set of all admissible processes is denoted by $\mathcal{P}$. An admissible process $(x^*, \mathfrak{c}^*)$ is called optimal, or a solution to problem (3.1), if the value of the minimizing functional is the least possible on the set $\mathcal{P}$.

Let us comment on these definitions. Problem (3.1) represents an extension of (2.1), since for any admissible control $v(t)$ in Problem (2.1) there exists a generalized impulsive control $\mathfrak{c}$ in Problem (3.1) such that the corresponding trajectories, and hence, the values of the minimizing functional, coincide. Indeed, let $v(t)$ be an ordinary control. Consider an absolutely continuous Borel measure

$$\mu(E) = \int_E \omega(|v(t)|) dt, \quad E \in \mathcal{B}(T).$$

Set $\mathfrak{c} = (\mu, v(\cdot), 0)$. It is clear that $\mathcal{D}(t;\mu) = \frac{1}{\omega(|v(t)|)}$, and hence, Condition (3.5) is valid. At the same time, by definition of $\bar{g}$, one has

$$x(t) = x_0 + \int_{t_0}^{t} \bar{g}(x, v, \theta) d\mathfrak{c} = x_0 + \int_{t_0}^{t} \frac{g(x(\theta), v(\theta), \theta)}{\omega(|v(\theta)|)} \cdot \omega(|v(\theta)|) d\theta =$$

$$= x_0 + \int_{t_0}^{t} g(x(\theta), v(\theta), \theta) d\theta.$$

Condition (3.5) is of critical importance. Indeed, without this condition, the proposed formulation is not a correct extension of the original problem (2.1), which can be easily seen in the simplest examples. Let

$$n = m = 1, \;\; x_0 = 0, \;\; g(x, v, t) = 1 + v, \;\; V = [1; \infty], \;\; \omega(y) = 1 + y.$$

It is clear that $\bar{g} \equiv 1$. Let $\mu := \ell$ and consider the time interval $[0, 1]$. The trajectory $x(t) \equiv t$ corresponds to the chosen measure and is admissible in the extended problem. However, $x(1) = 1 \notin \operatorname{cl} \mathcal{A}(1) = [2, \infty)$, where $\mathcal{A}(t)$ is the reachability set of the original system at time t. Therefore, the indicated trajectory cannot be approximated by the trajectories of the original problem, and therefore, it should not be included in the extension. This happened since Condition (3.5) is violated in the considered example.

In the case of a bounded problem (when V is compact), can any other trajectories, except for the trajectories of the original problem, be included into the extension? The answer is negative. Indeed, by virtue of (3.5), μ is absolutely continuous, while its density equals $\omega(|v(t)|)$, and therefore, is bounded. Then, by virtue of the definitions (see the above reasoning), the set of admissible trajectories does not change when passing to the extended problem. Thus, new trajectories can manifest only when the set V is unbounded.

4. Existence Theorem

One of the central questions in the theory of extensions concerns the existence of a solution in the extended problem. The following existence theorem is a version of A.F. Filippov's theorem, [7], adjusted to the case of discontinuous trajectories.

Theorem 1. *Suppose that Problem (2.1) admits an impulsive extension of order ω. Let one of the sets A or B be compact, and $\mathcal{P} \neq \varnothing$. Suppose also that there exists a constant $\kappa > 0$ such that*

$$(x, \mathfrak{c}) \in \mathcal{P} \;\; \Rightarrow \;\; \|\mu\| \leq \kappa. \tag{4.1}$$

Suppose also that the set

$$\bigcup_{v \in \bar{V}} \left(\bar{g}(x, v, t), \frac{1}{\omega(|v|_{\mathbb{R}^m})} \right) \subset \mathbb{R}^{n+1} \tag{4.2}$$

is convex for all x, t, and hypothesis H1) holds.

Then, Problem (3.1) has a solution.

Proof. The idea for the proof is to reduce the impulsive control problem (3.1) to a conventional optimal control problem using the Lebesgue discontinuous time variable change along with a compactification procedure. Consider the embedding $\Theta : \mathbb{R}^m \to B_1$ defined in Section 2. Set

$$f(x,u,t) := \bar{g}(x, \Theta^{-1}(u), t), \quad U := \Theta(\bar{V}) \subseteq B_1,$$

and consider the following conventional control problem with bounded controls.

$$\begin{array}{ll} \text{Find the minimum} & \varphi(x_0, x_1), \\ \text{subject to constraints} & \dot{x} = f(x,u,\chi), \\ & \dot{\chi} = \alpha, \ \text{a.e. } s \in [0, s_1], \\ & x(0) \in A, \ x(s_1) \in B, \\ & \chi(0) = t_0, \ \chi(s_1) = t_1, \\ & h(x,\chi) \le 0, \\ & \alpha = \frac{1}{\omega(|\Theta^{-1}(u)|_{\mathbb{R}^m})}, \\ & u(s) \in U, \ \alpha(s) \in [0,1] \ \text{a.e. } s \in [0, s_1]. \end{array} \tag{4.3}$$

The point s_1 in Problem (4.3) is not fixed, unlike Problem (3.1), which is set on a fixed time interval. The control functions in Problem (4.3) are $\alpha(s)$ and $u(s)$. Note that U is compact, as is the image of $\bar{V}$ under the continuous mapping Θ. Let us show that Problems (3.1) and (4.3) are equivalent, i.e., for any admissible process $(x, \mathfrak{c}) \in \mathcal{P}$ of Problem (3.1), there exists an admissible process $(\tilde{x}, \chi, \alpha, u, s_1)$ of (4.3) such that $\varphi(x_0, x_1) = \varphi(\tilde{x}_0, \tilde{x}_1)$, and vice versa.

Let $(x, \mathfrak{c}) \in \mathcal{P}$, where $\mathfrak{c} = \{\mu, v, v_\tau\}$. Consider the discontinuous time change $\pi(t)$. The inverse function is denoted by $\theta = \theta(s)$, $\theta : [0, s_1] \to T$, where $s_1 = \|\mu\|$. Take

$$\alpha(s) = \begin{cases} \mathcal{D}(\theta(s); \mu), & \text{if } s \in \Upsilon(\mu), \\ 0, & \text{otherwise}, \end{cases} \qquad u(s) = \begin{cases} \Theta(v(\theta(s))), & \text{if } s \in \Upsilon(\mu), \\ \Theta(v_\tau(\zeta_\tau(s))), & \text{otherwise}, \end{cases}$$

$$\tilde{x}(s) = \begin{cases} x(\theta(s)), & \text{if } s \in \Upsilon(\mu), \\ x_\tau(\zeta_\tau(s)), & \text{otherwise}, \end{cases}$$

Here,

$$\Upsilon(\mu) := [0, s_1] \setminus \bigcup_{\tau \in \mathrm{Ds}(\mu)} \Gamma_\tau, \quad \zeta_\tau(s) := \frac{s - \pi(\tau^-)}{\ell(\Gamma_\tau)} : \Gamma_\tau \to [0,1].$$

Let us demonstrate that

$$\theta(s) = t_0 + \int_0^s \alpha(\varsigma) d\varsigma. \tag{4.4}$$

Indeed, by definition one has $\ell([t_0, t]) =$

$$t - t_0 = \int_{[t_0,t]} \mathcal{D}(\sigma; \mu) d\mu = \int_{t_0}^t \mathcal{D}(\sigma; \mu) d\pi(\sigma) = \int_0^{\pi(t)} \mathcal{D}(\theta(\varsigma); \mu) d\varsigma.$$

Substituting $t = \theta(s)$, taking into account the definition of α, θ, and also the fact that $\pi(\theta(s)) = s$, as soon as π is continuous at the point $t = \theta(s)$, one arrives at (4.4). Therefore, $\chi = \theta$.

By virtue of (3.5), it is obvious that the additional control constraint $\alpha(s) = 1/\omega(|\Theta^{-1}(u(s))|)$ imposed in (4.3) is satisfied for a.e. s. It is also clear that the endpoint and state constraints in Problem (4.3) are satisfied. By changing the variable in (3.2) and taking into account the concept of extended trajectory, we obtain that the trajectory $\tilde{x}(\cdot)$ satisfies the dynamics in (4.3) on $[0, s_1]$ for (x_0, α, u). It is obvious that $\tilde{x}(s_1) = x_1$. Thus, the constructed process $(\tilde{x}, \chi, \alpha, u, s_1)$ is admissible in Problem (4.3). It is clear that $\tilde{x}_1 = x_1$, and hence, $\varphi(x_0, x_1) = \varphi(\tilde{x}_0, \tilde{x}_1)$.

Consider an arbitrary admissible process $(\tilde{x}, \chi, \alpha, u, s_1)$ of Problem (4.3). Function $\chi(s)$ is the inverse of some discontinuous time change $\pi : T \to [0, s_1]$, where $\pi(t)$ is uniquely defined as a function such that $\pi(\chi(s)) = s$, a.e. s: $\alpha(s) > 0$, $\pi(t_0) = 0$, $\pi(t_1) = s_1$, and $\pi(t)$ is continuous on the right on (t_0, t_1). Define a measure: $\mu([t_0, t]) = \pi(t)$. It is clear that $\mu \geq \ell$.

Let $v(t) = \Theta^{-1}(u(\pi(t)))$, $v_\tau(s) = \Theta^{-1}(u(\tilde{\zeta}_\tau(s)))$, where

$$\tilde{\zeta}_\tau(s) = \ell(\Gamma_\tau)s + \pi(\tau^-) : [0, 1] \to \Gamma_\tau, \quad \tau \in \mathrm{Ds}(\mu).$$

If E is measurable, then $\pi^{-1}(E)$ is μ-measurable. Therefore, v is μ-measurable. The extended trajectory $x(t) = \tilde{x}(\pi(t))$, $t \in T$, and $x_\tau(s) = \tilde{x}(\tilde{\zeta}_\tau(s))$ is by construction a solution to (3.2). This follows directly from the definitions and the change of variable under the integral. It is also clear that the endpoint and state constraints are satisfied. Therefore, the process $(x, \mathfrak{c})$, where $\mathfrak{c} = \{\mu, v, v_\tau\}$ is admissible in Problem (3.1). Moreover, $\varphi(x_0, x_1) = \varphi(\tilde{x}_0, \tilde{x}_1)$.

Thus, it has been established that Problems (3.1) and (4.3) are equivalent. Moreover, due to the one-to-one correspondence of processes,

$$s_1 = \|\mu\|. \tag{4.5}$$

Thus, if a solution exists in one of the problems, it also exists in the other. However, a solution exists in the auxiliary problem (4.3). Indeed, this is a consequence of the imposed conditions $\mathcal{P} \neq \varnothing$, H1), compactness of A or B, conditions (4.1), (4.2), (4.5) and Filippov's theorem.[3] Therefore, Problem (3.1) has a solution. □

Remark 1. It is simple to show that Condition (4.1) is certainly satisfied as soon as there are constants $\varepsilon, \kappa > 0$ such that

$$\int_{t_0}^{t_1} \omega(|v(t)|)dt \leq \kappa$$

[3] Herein, its version from [8] is employed.

for all $v(\cdot) : T \to V$, for which $\mathrm{dist}(x(t_1), B) \le \varepsilon$, $h^j(x(t), t) \le \varepsilon$, $j = 1, ..., l$, $t \in T$, where $\dot{x}(t) = g(x(t), v(t), t)$, $x(t_0) = x_0 \;\forall x_0 \in A$. Then, it is not necessary to pass to the class of extended controls in order to verify (4.1). This sufficient condition for validity of (4.1) is often simple to verify. In particular, it is fulfilled for a number of classical examples, like the Euler or Dido problems analysed in [12] with regards to their extensions.

Remark 2. Let $A = \{x_0\}$. Then, $\varphi(x_0, x_1) = \varphi(x_1)$. Condition (4.1) can be replaced by the following:

$$(x, \mathfrak{c}) \in \mathcal{P} \quad \Rightarrow \quad \|\mu\| \le r(\varphi(x_1)), \tag{4.6}$$

where $r(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous monotonically increasing function.

Should we take $\omega \equiv 1$ in (4.2), then we arrive at Filippov's theorem, or rather some of its simple generalization to the case of an unbounded set U bearing in mind the compactification of $\mathbb{R}^m$. Thus, Theorem 1 formally contains the classical result for bounded controls. However, Theorem 1 can be applied only to some narrow class of problems with discontinuous solutions. This is due to the rather strict requirement of convexity of the compound set in (4.2). In essence, this requirement says that the impulsive control is scalar and must have a sign. It is clear that, for example, System (2) in [1] satisfies the given requirement, while Example (1) of the same source does not. In this connection, a question arises of finding some more subtle conditions for the existence of a solution that would be applicable to a broader class of problems including a number of known nonlinear examples of Calculus of variations which admit discontinuous solutions, [12].

Consider candidates for such type conditions. Suppose that $A = \{a\}$, $B = \{b\} + C$, where $a, b \in \mathbb{R}^n$, and $C = \{y \in \mathbb{R}^n : y^j = 0, j = 1, ..., l\}$, where $1 \le l < n$. Let g^j be positively homogenous w.r.t. v for $j = 1, ..., l$. Then, the proposed type of extension is feasible. In case of inequalities $y^j \le 0$, one can consider the following type condition

$$g^j(x, \lambda v, t) \le \lambda g^j(x, v, t)$$

to be satisfied for all $\lambda \ge \lambda_0 > 0$. Note that a number of isoparametric problems, such as, for example, the Catenary problem, or Dido's problem, mentioned earlier in the introduction can be fitted into such a paradigm. Let us leave the verification of validity of these new hypotheses for future research.

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