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Existence of Strong Solutions for Compressible Elastic Curves in the Energy Conservation System

Chiharu Kosugi[1]✉

[1] Yamaguchi University, Yamaguchi, Japan
✉ ckosgi@yamaguchi-u.ac.jp

Abstract: In this paper, we consider initial and boundary value problems for the beam equation system accompanying by a function having a singularity point for the nonlinear strain, called a compressible stress function. This problem is constructed as the mathematical model describing motions of closed elastic curves on $\mathbb{R}^2$ in our previous work. It is known that the energy derived from the system is conserved. For this problem we have already proved existence and uniqueness of weak solutions. Also, we have obtained results for existence and uniqueness of the strong solutions to the problem with the viscosity term. Our aim of this paper is not only to establish existence and uniqueness of a strong solution to the present problem, but also convergence of solutions to the problem with the viscosity term as the viscosity coefficient tends to 0. The key to this proof is the uniform estimate for the fourth derivative with respect to the space of solutions.

Keywords: beam equation, nonlinear strain, compressible elastic curve, energy method

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Научная статья

Существование сильных решений для сжимаемых упругих кривых в системе сохранения энергии

Ч. Косуги[1]✉

[1] Университет Ямагути, Ямагути, Япония
✉ ckosgi@yamaguchi-u.ac.jp

Аннотация: Рассматриваются начальные и краевые задачи для системы уравнений балки, сопровождаемой функцией, имеющей точку сингулярности для нелинейной деформации, называемой функцией сжимаемого напряжения. Эта задача строится как математическая модель, описывающая движения замкнутых упругих кривых на $\mathbb{R}^2$ в нашей предыдущей работе. Известно, что энергия, полученная из системы, сохраняется. Для этой задачи уже доказано существование и единственность слабых решений. Также получены результаты о существовании и единственности сильных решений задачи с вязкостным членом. Цель данной работы — не только установить существование и единственность сильного решения данной задачи, но и сходимость решений задачи с вязкостным членом при стремлении коэффициента вязкости к 0. Ключ к этому доказательству — равномерная оценка четвертой производной относительно пространства решений.

Ключевые слова: уравнение балки, нелинейная деформация, сжимаемая упругая кривая, энергетический метод

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1. Introduction

In this paper, we consider the following initial and boundary value problem for the beam equation describing shrinking and stretching motions of elastic curves on $\mathbb{R}^2$. In our model, an unknown function u from the domain $Q(T) := (0,T)\times(0,1)$ to $\mathbb{R}^2$, $T>0$, is representing the position of $x\in[0,1]$ at time t (see Figure 1) and satisfies

$$\rho\frac{\partial^2 u}{\partial t^2}+\gamma\frac{\partial^4 u}{\partial x^4}-\mu\frac{\partial^3 u}{\partial t\partial x^2}=\frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\partial u}{\partial x}\right), \varepsilon=\left|\frac{\partial u}{\partial x}\right|-1 \text{ on } Q(T), \quad (1.1)$$

$$\frac{\partial^i}{\partial x^i}u(0)=\frac{\partial^i}{\partial x^i}u(1) \text{ on } (0,T) \text{ for any } i=0,1,2,3, \quad (1.2)$$

$$u(0)=u_0, \frac{\partial}{\partial t}u(0)=v_0 \text{ on } (0,1), \quad (1.3)$$

where $\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u_1}{\partial t^2}, \frac{\partial^2 u_2}{\partial t^2}\right)$ and so on, $\rho>0$ is the density, $\gamma>0$ and $\mu\geq 0$ are constants, and $|\cdot|$ is the Euclidian norm in $\mathbb{R}^2$, namely, $\left|\frac{\partial u}{\partial x}\right| = \sqrt{\left|\frac{\partial u_1}{\partial x}\right|^2+\left|\frac{\partial u_2}{\partial x}\right|^2}$, where $u=(u_1,u_2)$. We call the constant μ the viscosity coefficient, since it is concerned with the energy decay rate. Also, u_0 and v_0 are initial position and initial velocity, respectively. In our model, since we suppose that the elastic material is connected at $x=0,1$, smoothly, we choose the periodic boundary condition (1.2). Here, we note that ε and

$f(\varepsilon)u_x$ represent strain and stress, respectively, where the function $f\colon (-1, \infty) \to \mathbb{R}$ is given by

$$f(\varepsilon) = \frac{\kappa}{4}\left(1 - \frac{1}{(1+\varepsilon)^4}\right) \text{ for } \varepsilon > -1, \tag{1.4}$$

and κ is a positive constant. In our model, we call $f(\varepsilon)$ the compressible stress function. For simplicity, the system (1.1)-(1.3) is denoted by $\mathrm{P}_\mu(u_0, v_0, f)$ for $\mu \geq 0$.

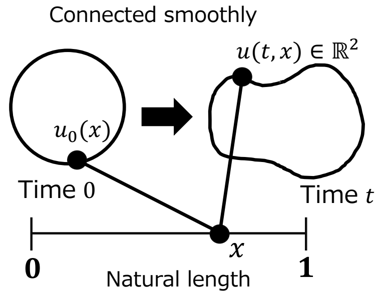

Figure 1. Unknown function u

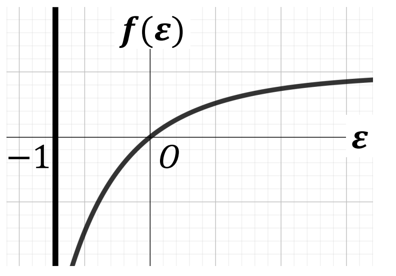

Figure 2. Compressible stress function f

Here, we list features of the present model.

(Beam equation) The equation $u_{tt} + u_{xxxx} = 0$ is often called the beam equation and is known as one of mathematical representation for elastic material motions. In [16; 18; 19] the nonlinear beam equation was already investigated. From mathematical point of view, by the fourth derivative term, we can easily deal with the equation. On the other hand, for the elastic curves its physical meaning is not clear. One of our research aims is to clarify it. Here, we note that in [15] there is an other approach on dynamics of the elastic closed curve on $\mathbb{R}^2$.

(The unknown function u) In usual cases, the unknown function for elastic material model is the displacement. However, in our model, the dimension of the domain and the range of u are 1-dimension and 2-dimension, respectively (see Figure 1). Therefore, we define the unknown function u as the position of elastic curve on $\mathbb{R}^2$. Accordingly, as mentioned in [11], the strain ε is defined by $\varepsilon = |u_x| - 1$. Clearly, the strain ε is nonlinear and not smooth with respect to u_x.

(Compressible stress function f) In general, the strain is given as the linear function of the stress by Hooke's law. However, in our model, we define the function f by (1.4). Accordingly, $f(\varepsilon)$ tends to $-\infty$ as -1 (see Figure 2). The idea for introducing this kind of functions is based on the assumption that when an elastic material is highly compressed, the magnitude of the stress should become very large. Similar functions were already studied in [5; 7; 8; 13; 14; 17] from engineering point of view and considered numerically. For example in [17], they studied the stress function whose primitive is given by

$$\frac{1}{4}\left((\varepsilon+1)^2-1-2\log(\varepsilon+1)\right).$$

Moreover, to present the validity of our function f we give some numerical results for $\mathrm{P}_0(u_0, v_0, f)$ with $u_0(x) = R_0(\cos(2\pi x), \sin(2\pi x)), v_0(x) = (0,0)$ for $x \in [0,1]$. In this case, by putting $u(t,x) = R(t)(\cos(2\pi x), \sin(2\pi x))$ for $(t,x) \in Q(T)$, u is the solution of $\mathrm{P}_0(u_0, v_0, f)$ where R is the function on $[0,T]$ and satisfies the following ordinary differential equation.

$$R'' + (2\pi)^4\gamma R + (2\pi)^2 f(2\pi R(t) - 1)R(t) = 0, R(0) = |u_0|, R'(0) = 0.$$

We note that $\varepsilon = 2\pi R - 1$. As mentioned later, we can show that $\varepsilon > -1$ on $Q(T)$, theoretically, namely, $2\pi R > 0$. Since R is the radius of the circle, the positivity of R should be expected. In Figures 3 and 5, the graphs indicate the value of R, if $f(\varepsilon) = \kappa\dfrac{\varepsilon}{|u_x|}$ where κ is a positive constant, that is, the stress is linear. As shown in Figure 5, the radius takes negative values for large initial strain. On the other hand, by adopting the our function f, in the numerical results, Figures 4 and 6, the radius are always positive, even if the initial large strain. Thus, we emphasize that our function is very useful for representing behaviors of elastic materials.

In our model, not only the numerical results but also thanks to Lemma 1, we obtain the lower boundedness of the strain ε, for instance (3.3). This guarantees that the length of the elastic material never vanishes. This is a unique feature of our model that is not obtained by any other models. We note that the generalization of the stress function is very important. However, we do not consider it in this paper, since we aim to clarify what can be shown in our present model. Of course, the generalization is our next issue to be addressed.

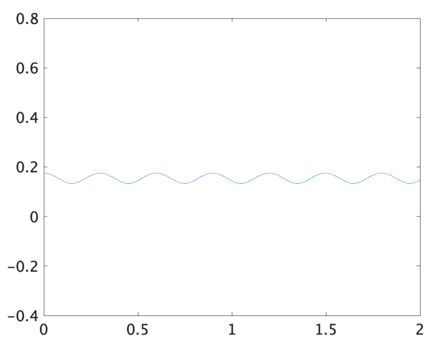

Figure 3. f: linear, small initial strain

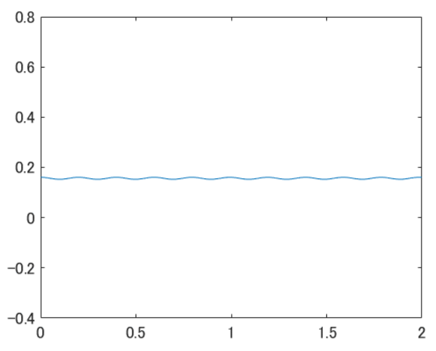

Figure 4. f: nonlinear, small initial strain

Next, we recall our previous results concerned with solvability of $\mathrm{P}_\mu(u_0, v_0, f)$.

(ODE model) In [1;11], we constructed an approximation model for elastic curves. In this model, we regarded the elastic curve as a polygon having N vertices, and derived N-dimensional second-order ordinary differential equations. In order to apply the structure preserving numerical method [6] to this model, we define the stress function similar to (1.4). For this model, we have proved existence and uniqueness of solutions in [1] and have shown existence of time-periodic solutions under appropriate initial conditions in [11]. Furthermore, in [1] we constructed a numerical scheme by the structure-preserving numerical method and showed existence and uniqueness of numerical solutions. We also have proved the sequence of the numerical solutions converges to the solution of the ODE model and compared numerical results by the structure-preserving numerical method with those by the multi-steps method in [11].

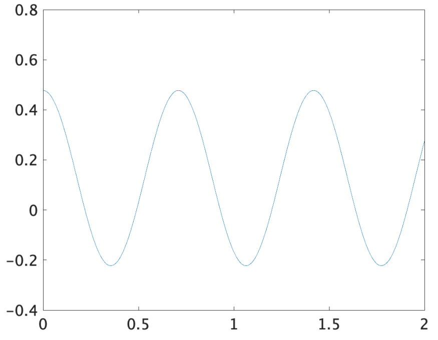

Figure 5. f: linear, large initial strain

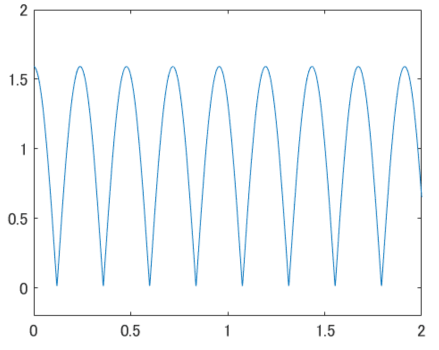

Figure 6. f: nonlinear, large initial strain

(Partial differential equation models) From the ODE model, we obtained the quasilinear wave equation which is (1.1) in case $\gamma = 0$ and $\mu = 0$. However, by the singularity of the stress function, it is not easy to deal with the differential equation, theoretically. In order to overcome this difficulty, we approximate it by adding the fourth derivative term γu_{xxxx}. Thus, we arrived at the beam equation and propose the mathematical model representing motions of the elastic curve by (1.1)-(1.3) in [11].

Our first result for the system (1.1)-(1.3) is concerned with existence and uniqueness of weak solutions to $\mathrm{P}_0(u_0, v_0, f)$, where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, monotone increasing and $f(0) = 0$. The proof of the uniqueness was based on the method applying the dual problem given in [12].

Next, in [9], for $\mu > 0$ we showed existence and uniqueness of weak solutions to $\mathrm{P}_\mu(u_0, v_0, f)$ with the compressible stress function f defined by (1.4). Moreover, we proved existence of strong solutions. In this problem, since the energy is dissipated, we could consider the stationary problem. Actually, we showed existence of a subsequence $\{t_n\}$ such that the solution $u(t_n)$ of $\mathrm{P}_\mu(u_0, v_0, f)$ tends to the stationary solution as $n \to \infty$. Also, we succeeded in improving the regularity of the strong solution to $\mathrm{P}_\mu(u_0, v_0, f)$

in [3]. However, since rotating and translating stationary solutions are also stationary solutions, the uniqueness of the stationary solutions has not been proved yet.

Furthermore, we proved existence and uniqueness of weak solutions to $\mathrm{P}_0(u_0, v_0, f)$ with the compressible stress function f in [10]. In the proof of the uniqueness, by extending f to the function on $\mathbb{R}$, we can apply the result in [2] to this problem.

The aim of this paper is to prove existence of the strong solution to $\mathrm{P}_0(u_0, v_0, f)$ with the compressible stress function f given by (1.4).

Throughout this paper, we put Hilbert spaces by $H = (L^2(0,1))^2$,
$H_1 = \{z \in (H^1(0,1))^2 | z(0) = z(1)\}$,
$H_2 = \left\{ z \in (H^2(0,1))^2 | \frac{\partial^i}{\partial x^i} z(0) = \frac{\partial^i}{\partial x^i} z(1) \text{ for } i = 0, 1 \right\}$,
$H_4 = \left\{ z \in (H^4(0,1))^2 | \frac{\partial^i}{\partial x^i} z(0) = \frac{\partial^i}{\partial x^i} z(1) \text{ for } i = 0, 1, 2, 3 \right\}$ with the standard norm denoted by $|\cdot|_H = |\cdot|_{(L^2(0,1))^2}$, $|\cdot|_{H_1} = |\cdot|_{(H^1(0,1))^2}$, $|\cdot|_{H_2} = |\cdot|_{(H^2(0,1))^2}$, $|\cdot|_{H_4} = |\cdot|_{(H^4(0,1))^2}$, and standard inner products represented as follows : $(u, v)_H = \int_0^1 u \cdot v \, dx$ for $u, v \in H$, $(u, v)_{H_i} = \sum_{j=0}^{i} (\frac{\partial^j u}{\partial x^j}, \frac{\partial^j v}{\partial x^j})_H$ for $u, v \in H_i$, $i = 1, 2, 4$, where $u \cdot v = u_1 v_1 + u_2 v_2$ for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$.

We remark definitions of the weakly$*$ convergence. If a sequence $\{u_n\} \subset L^\infty(0, T; H_4)$ satisfies the following convergence, we represent "$u_n \to u$ weakly$*$ in $L^\infty(0, T; H_4)$":

$$\int_0^T (u_n, \varphi)_{H_4} dt \to \int_0^T (u, \varphi)_{H_4} dt \text{ as } n \to \infty \quad \text{for any } \varphi \in L^1(0, T; H).$$

Also, if a sequence $\{u_n\} \subset W^\infty(0, T; H_2)$ satisfies the following convergence, we represent "$u_n \to u$ weakly$*$ in $W^{1,\infty}(0, T; H_2)$":

$$\int_0^T (u_{nt}, \varphi)_{H_2} dt \to \int_0^T (u_t, \varphi)_{H_2} dt, \quad \int_0^T (u_n, \varphi)_{H_2} dt \to \int_0^T (u, \varphi)_{H_2} dt,$$

as $n \to \infty$ for any $\varphi \in L^1(0, T; H_2)$.

First, we define a strong solution of $\mathrm{P}_\mu(u_0,\, v_0,\, f)$ for $\mu > 0$.

Definition 1. *Let $\mu > 0$. A function u from $Q(T)$ to $\mathbb{R}^2$ is called a strong solution of $P_\mu(u_0,\, v_0,\, f)$ on $[0, T]$ if u satisfies $u \in W^{2,2}(0, T, H_1) \cap W^{1,\infty}(0, T; H_2) \cap W^{1,2}(0, T, (H^3(0,1))^2) \cap L^\infty(0, T; H_4)$, $|u_x| > 0$ on $\overline{Q(T)}$ and satisfying* (1.1)-(1.3) *in a usual sense.*

The next theorem guarantees existence and uniqueness of the strong solution to $\mathrm{P}_\mu(u_0,\, v_0,\, f)$ for $\mu > 0$.

Theorem 1. *(Kosugi [9]) Let $\mu > 0$ and $T > 0$. If $u_0 \in H_4$, $|u_{0x}| > 0$ on $[0, 1]$ and $v_0 \in H_2$, then $P_\mu(u_0, v_0, f)$ has one and only one strong solution on $[0, T]$. Moreover, the following (1.5), (1.6) and (1.7) hold:*

$$\begin{aligned}
&\frac{\rho}{2}|u_t(t)|_H^2 + \frac{\gamma}{2}|u_{xx}(t)|_H^2 + \mu \int_0^t |u_{\tau x}(\tau)|_H^2 d\tau \\
&+\frac{\kappa}{8}\left(|u_x(t)|_H^2 + \int_0^1 \frac{1}{|u_x(t,x)|^2}dx\right) = \frac{\rho}{2}|v_0|_H^2 + \frac{\gamma}{2}|u_{0xx}|_H^2 \\
&\qquad +\frac{\kappa}{8}\left(|u_{0x}|_H^2 + \int_0^1 \frac{1}{|u_{0x}(x)|^2}dx\right) =: C_0 \text{ for any } t \in [0, T], \quad (1.5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \mu \int_0^t |u_{\tau xxx}(\tau)|_H^2 d\tau \\
&\le \frac{\rho}{2}|v_{0xx}|_H^2 + \frac{\gamma}{2}|u_{0xxxx}|_H^2 \\
&\qquad +\frac{1}{\mu}\int_0^t |(f(\varepsilon)u_x)_{xx}(\tau)|_H^2 d\tau \text{ for any } t \in [0, T], \quad (1.6)
\end{aligned}$$

$$|u_x| \ge \sqrt{\frac{C_0}{\gamma}} e^{-\frac{16C^2}{\kappa\gamma}} \text{ on } \overline{Q(T)}. \quad (1.7)$$

In our proof of this theorem, by using the first estimate (1.5) and the following lemma, we can obtain the third estimate (1.7).

Lemma 1. *(cf. Aiki, Kröger and Muntean, [4, Lemma 3.2]) Let $z \in (H^2(0, 1))^2$ and $K_1, K_2 > 0$. If*

$$\int_0^1 \frac{1}{|z_x|^2}dx \le K_1, \quad |z_{xx}|_H \le K_2,$$

then

$$|z_x| \ge \frac{K_2}{\sqrt{2}} e^{-K_1 K_2^2} \text{ on } [0, 1].$$

In Section 2 we give a theorem concerning existence of solutions to $\mathrm{P}_0(u_0, v_0, f)$ and prove it in Section 4. In its proof, we consider the asymptotic behavior of strong solutions u_μ to $\mathrm{P}_\mu(u_0, v_0, f)$ obtained by Theorem 1 for $\mu > 0$, when μ tends to 0. A key lemma for the proof is provided in Section 3.

2. Main result

First, we define a strong solution of $\mathrm{P}_0(u_0, v_0, f)$.

Definition 2. *A function u from $Q(T)$ to $\mathbb{R}^2$ is called a strong solution of $P_0(u_0, v_0, f)$ on $[0, T]$ if u satisfies the following properties: $u \in W^{2,2}(0,T;H) \cap W^{1,\infty}(0,T;H_2) \cap L^\infty(0,T;H_4)$, there exists $\delta > 0$ such that $|u_x| \geq \delta$ a.e. on $Q(T)$ and satisfying* (1.1)-(1.3) *in a usual sense.*

The main result of this paper is as follows:

Theorem 2. *Let $T > 0$ and $\mu > 0$. If $u_0 \in H_4$, $|u_{0x}| > 0$ on $[0,1]$, $v_0 \in H_2$ and u_μ be a strong solution of $P_\mu(u_0, v_0, f)$ on $[0,T]$, then there exists $u \in W^{2,2}(0,T;H) \cap W^{1,\infty}(0,T;H_2) \cap L^\infty(0,T;H_4)$ such that $u_\mu \to u$ weakly in $W^{2,2}(0,T;H)$, weakly$*$ in $W^{1,\infty}(0,T;H_2)$, weakly$*$ in $L^\infty(0,T;H_4)$, strongly in $C(\overline{Q(T)})$, $u_{\mu x} \to u_x$ strongly in $C(\overline{Q(T)})$, $u_{\mu xx} \to u_{xx}$ strongly in $C(\overline{Q(T)})$ as $\mu \to 0$ and u is a strong solution of $P_0(u_0, v_0, f)$ on $[0,T]$.*

Here, we note that we have already proved existence and uniqueness of weak solutions to $\mathrm{P}_0(u_0, v_0, f)$ in [10].

3. Key lemma

We note that the estimate obtained by (1.6) depends on $\mu > 0$. Hence, in order to prove Theorem 1, we show a new inequality to obtain the uniform estimate for $|u_{xxxx}(t)|_H$ as follows:

Lemma 2. *Let $T > 0$, $\mu > 0$, $u_0 \in H_4$, $|u_{0x}| > 0$ on $[0,1]$ and $v_0 \in H_2$. If u is a strong solution of $P_\mu(u_0, v_0, f)$ on $[0,T]$, then the following inequality holds:*

$$\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \mu \int_0^t |u_{\tau xxx}(\tau)|_H^2 d\tau$$
$$\leq C_1 + \left(\frac{1}{\rho} + \frac{\gamma C_2}{2}\right) C_1 T e^{\left(\frac{1}{\rho} + \frac{\gamma C_2}{2}\right)T} =: C_3 \quad \textit{for any } t \in [0,T], \tag{3.1}$$

where C_1 and C_2 are positive constants independent of $\mu > 0$.

Our proof of this lemma is rather long, since by applying time discretization to the problem we show (3.1). Thus, we omit the detail of the proof. Instead, we show (3.1) by the following formal calculation. For $\mu > 0$, let u be a strong solution of $\mathrm{P}_\mu(u_0, v_0, f)$ on $[0,T]$. By multiplying (1.1) with u_{txxxx} and integrating it on $(0,1)$ with respect to x, we have

$$\rho \int_0^1 u_{tt} \cdot u_{txxxx} dx + \gamma \int_0^1 u_{xxxx} \cdot u_{txxxx} dx$$
$$-\mu \int_0^1 u_{txx} \cdot u_{txxxx} dx - \int_0^1 (f(\varepsilon)u_x)_x \cdot u_{txxxx} dx = 0 \quad \text{a.e. on } (0,T).$$

Thanks to integration by parts, it is easy to see that

$$\frac{d}{dt}\left(\frac{\rho}{2}|u_{txx}(t)|_H^2+\frac{\gamma}{2}|u_{xxxx}(t)|_H^2+\mu\int_0^t|u_{txxx}(\tau)|_H^2d\tau\right)$$
$$=\int_0^1(f(\varepsilon)u_x)_{xxx}\cdot u_{txx}dx \text{ for a.e. } t\in(0,T),$$

and

$$\frac{\rho}{2}|u_{txx}(t)|_H^2+\frac{\gamma}{2}|u_{xxxx}(t)|_H^2+\mu\int_0^t|u_{txxx}(\tau)|_H^2d\tau$$
$$\le\frac{\rho}{2}|v_{0xx}|_H^2+\frac{\gamma}{2}|u_{0xxxx}|_H^2+\frac{1}{2}\int_0^t|\left(f(\varepsilon)u_x\right)_{xxx}(\tau)|_H^2d\tau$$
$$+\frac{1}{2}\int_0^t|u_{\tau xx}(\tau)|_H^2d\tau \text{ for any } t\in[0,T].$$

In the third term of the right hand side, we see that

$$(f(\varepsilon)u_x)_{xxx}=(f(\varepsilon))_{xxx}u_x+3\left(f(\varepsilon)\right)_x u_{xxx}+3\left(f(\varepsilon)\right)_{xx}u_{xx}+f(\varepsilon)u_{xxxx},$$

and

$$|(f(\varepsilon))_{xxx}u_x+3\left(f(\varepsilon)\right)_x u_{xxx}+3\left(f(\varepsilon)\right)_{xx}u_{xx}+f(\varepsilon)u_{xxxx}|^2$$
$$\le 4\left\{|(f(\varepsilon))_{xxx}u_x|^2+9|\left(f(\varepsilon)\right)_x u_{xxx}|^2+9\left|(f(\varepsilon))_{xx}u_{xx}\right|^2+|f(\varepsilon)u_{xxxx}|^2\right\}.$$

Accordingly, we have

$$\frac{1}{2}\int_0^t|\left(f(\varepsilon)u_x\right)_{xxx}(\tau)|_H^2d\tau$$
$$\le 2\left\{\int_0^t|(f(\varepsilon))_{xxx}u_x(\tau)|_H^2\,d\tau+9\int_0^t|(f(\varepsilon))_{xx}u_{xx}(\tau)|_H^2\,d\tau\right.$$
$$\left.+9\int_0^t|\left(f(\varepsilon)\right)_x u_{xxx}(\tau)|_H^2d\tau+\int_0^t|f(\varepsilon)u_{xxxx}(\tau)|_H^2\,d\tau\right\}$$
$$=:\sum_{i=1}^4 I_i(t) \text{ for any } t\in[0,T]. \tag{3.2}$$

For I_3, by the definition of ε, we see that $\left(f(\varepsilon)\right)_x=\kappa\dfrac{u_x\cdot u_{xx}}{|u_x|^6}$ a.e. on $Q(T)$, and

$$I_3(t)\le 9\kappa^2\int_0^t\frac{|u_{xx}|^2\,|u_{xxx}|^2}{|u_x|^{10}}dxd\tau \quad \text{for any } t\in[0,T].$$

Now, Theorem 1 (1.7) implies existence of $\delta>0$ independent of $\mu>0$ such that

$$|u_x|\ge\delta \text{ a.e. on } Q(T), \tag{3.3}$$

and by (1.5) $|u_{xx}(t)|_H^2 \le \dfrac{2C_0}{\gamma}$ for any $t \in [0,T]$ holds. Thus, we have

$$\begin{aligned} I_3(t) &\le \frac{9\kappa^2}{\delta^{10}} \int_0^t \int_0^1 |u_{xx}|^2 |u_{xxx}|^2 dx d\tau \\ &\le \frac{9\kappa^2}{\delta^{10}} \int_0^t |u_{xxx}(\tau)|^2_{(L^\infty(0,1))^2} |u_{xx}(\tau)|_H^2 d\tau \\ &\le \frac{18\kappa^2 C_0}{\gamma\delta^{10}} \int_0^t |u_{xxx}(\tau)|^2_{(L^\infty(0,1))^2} d\tau \quad \text{for any } t \in [0,T]. \end{aligned}$$

Here, we note that

$$|z|_{(L^\infty(0,1))^2} \le |z|_H + |z_x|_H \quad \text{for } z \in (H^1(0,1))^2, \tag{3.4}$$

$$|u_{xxx}|_H^2 \le |u_{xx}|_H |u_{xxxx}|_H \le \frac{1}{2}|u_{xx}|_H^2 + \frac{1}{2}|u_{xxxx}|_H^2 \text{ for any } u \in H_4. \tag{3.5}$$

Therefore, by (3.4), (1.5) and (3.5), we obtain

$$\begin{aligned} I_3(t) &\le \frac{18\kappa^2 C_0}{\gamma\delta^{10}} \left(\int_0^t |u_{xxx}(\tau)|_H^2 d\tau + \int_0^t |u_{xxxx}(\tau)|_H^2 d\tau \right) \\ &\le \frac{18\kappa^2 C_0}{2\gamma\delta^{10}} \left(\int_0^t |u_{xx}(\tau)|_H^2 d\tau + 3\int_0^t |u_{xxxx}(\tau)|_H^2 d\tau \right) \\ &\le \frac{18\kappa^2 C_0}{2\gamma\delta^{10}} \left(\frac{2C_0 T}{\gamma} + 3\int_0^t |u_{xxxx}(\tau)|_H^2 d\tau \right) \text{ for any } t \in [0,T]. \end{aligned}$$

For I_2 we see that $(f(\varepsilon))_{xx} = \kappa \left(\dfrac{|u_{xx}|^2}{|u_x|^6} - \dfrac{6(u_x \cdot u_{xx})^2}{|u_x|^8} + \dfrac{u_x \cdot u_{xxx}}{|u_x|^6} \right)$ a.e. on $Q(T)$. By (3.3), (1.5), (3.4) and (3.5), we have

$$\begin{aligned} I_2(t) &\le 9\kappa^2 \int_0^t \int_0^1 \left(\frac{|u_{xx}|^3}{|u_x|^6} + \frac{6|u_{xx}|^2}{|u_x|^6} + \frac{|u_{xx}||u_{xxx}|}{|u_x|^5} \right)^2 dx d\tau \\ &\le 27\kappa^2 \left(\int_0^t \int_0^1 \frac{|u_{xx}|^6}{|u_x|^{12}} dx d\tau + 36 \int_0^t \int_0^1 \frac{|u_{xx}|^4}{|u_x|^{12}} dx d\tau \right. \\ &\qquad \left. + \int_0^t \int_0^1 \frac{|u_{xx}|^2 |u_{xxx}|^2}{|u_x|^{10}} dx d\tau \right) \\ &\le 27\kappa^2 \left(\frac{1}{\delta^{12}} \int_0^t \int_0^1 |u_{xx}|^6 dx d\tau + \frac{36}{\delta^{12}} \int_0^t \int_0^1 |u_{xx}|^4 dx d\tau \right. \\ &\qquad \left. + \frac{1}{\delta^{10}} \int_0^t |u_{xx}|_H^2 |u_{xxx}|^2_{(L^\infty(0,1))^2} d\tau \right) \\ &\qquad \text{for any } t \in [0,T]. \end{aligned} \tag{3.6}$$

Here the following inequality holds.

$$|z(x)| \leq \sqrt{2}|z|_H^{\frac{1}{2}}|z_x|_H^{\frac{1}{2}} + |z|_H \quad \text{for any } z \in (H^1(0,1))^2. \tag{3.7}$$

By using (3.7) and (3.5) for first and second terms of the right hand side in (3.6), we have

$$\begin{aligned}
&\int_0^t \int_0^1 |u_{xx}|^6 dx d\tau \\
&\leq 32 \int_0^t \left(8|u_{xx}|_H^3 |u_{xxx}|_H^3 + |u_{xx}|_H^6\right) d\tau \\
&\leq 32 \int_0^t \left(8|u_{xx}|_H^3 |u_{xx}|_H^{\frac{3}{2}} |u_{xxxx}|_H^{\frac{3}{2}} + |u_{xx}|_H^6\right) d\tau \\
&\leq 32 \int_0^t \left(8\left(\frac{2C_0}{\gamma}\right)^{\frac{9}{2}} |u_{xxxx}|_H^{\frac{3}{2}} + \left(\frac{2C_0}{\gamma}\right)^3\right) d\tau \\
&\leq 256 \left(\left(\frac{2C_0}{\gamma}\right)^{\frac{9}{2}} \int_0^t \left(1 + |u_{xxxx}|_H^2\right) d\tau + \left(\frac{C_0}{\gamma}\right)^3 T\right) \\
&\leq 256 \left\{\left(\left(\frac{2C_0}{\gamma}\right)^{\frac{9}{2}} + \left(\frac{C_0}{\gamma}\right)^3\right) T + \left(\frac{2C_0}{\gamma}\right)^{\frac{9}{2}} \int_0^t |u_{xxxx}|_H^2 d\tau\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } t \in [0,T],
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \int_0^1 |u_{xx}|^4 dx d\tau \\
&\leq 8 \int_0^t \left(2|u_{xx}|_H^2 |u_{xxx}|_H^2 + |u_{xx}|_H^4\right) d\tau \\
&\leq 8 \int_0^t \left(\frac{4C_0}{\gamma}|u_{xxx}|_H^2 + \left(\frac{2C_0}{\gamma}\right)^2\right) d\tau \\
&\leq \frac{32C_0}{\gamma}\left(\frac{1}{2}\int_0^t \left(|u_{xx}|_H^2 + |u_{xxxx}|_H^2\right) d\tau + \frac{C_0 T}{\gamma}\right) \\
&\leq \frac{32C_0}{\gamma}\left(\frac{C_0 T}{\gamma} + \frac{1}{2}\int_0^t |u_{xxxx}|_H^2 d\tau + \frac{C_0 T}{\gamma}\right) \quad \text{for any } t \in [0,T],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t |u_{xx}|_H^2 |u_{xxx}|_{(L^\infty(0,1))^2}^2 d\tau \leq \frac{2C_0}{\gamma}\int_0^t |u_{xxx}|_{(L^\infty(0,1))^2}^2 d\tau \leq \\
&\leq \frac{2C_0}{\gamma}\int_0^t \left(|u_{xxx}|_H + |u_{xxxx}|_H\right)^2 d\tau \leq \frac{2C_0}{\gamma}\int_0^t \left(\frac{1}{2}|u_{xx}|_H^2 + \frac{3}{2}|u_{xxxx}|_H^2\right) d\tau \\
&\qquad\qquad \leq \frac{2C_0}{\gamma}\left(\frac{C_0 T}{\gamma} + \frac{3}{2}\int_0^t |u_{xxxx}|_H^2 d\tau\right) \quad \text{for any } t \in [0,T].
\end{aligned}$$

From these inequalities we infer that

$$9\int_0^t \left|(f(\varepsilon))_{xx}\,u_{xx}(\tau)\right|_H^2\,d\tau \le \tilde{C}_1 + \tilde{C}_2\int_0^t |u_{xxxx}|_H^2 d\tau \quad \text{for any } t\in[0,T],$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are positive constants independent of $\mu > 0$. Similarly, for I_1 and I_4 we have

$$f(\varepsilon) = \frac{\kappa}{4}\left(1 - \frac{1}{(1+\varepsilon)^4}\right),$$

$$\begin{aligned}
&(f(\varepsilon))_{xxx}\\
&= \kappa\Bigg(\frac{2u_{xx}\cdot u_{xxx}}{|u_x|^6} - \frac{6|u_{xx}|^2 u_x\cdot u_{xx}}{|u_x|^8} - \frac{12\left(|u_{xx}|^2 + u_x\cdot u_{xx}\right)u_x\cdot u_{xx}}{|u_x|^8}\\
&\qquad + \frac{48(u_x\cdot u_{xx})^3}{|u_x|^{10}} + \frac{u_{xx}\cdot u_{xxx} + u_x\cdot u_{xxxx}}{|u_x|^6} - \frac{6(u_x\cdot u_{xxxx})(u_x\cdot u_{xx})}{|u_x|^8}\Bigg)\\
&\hfill \text{a.e. on } Q(T),
\end{aligned}$$

$$\begin{aligned}
I_1 \;=\;& \int_0^t \left|(f(\varepsilon))_{xxx}\,u_x(\tau)\right|_H^2\,d\tau\\
\le\;& 6\kappa^2\Bigg\{\frac{4}{\delta^{12}}\int_0^t\int_0^1 |u_{xx}|^2|u_{xxx}|^2 dx\tau + \frac{36}{\delta^{14}}\int_0^t\int_0^1 |u_{xx}|^6 dxd\tau\\
&+\frac{144}{\delta^{14}}\int_0^t\int_0^1 \left(|u_{xx}|^2 + |u_x||u_{xxx}|\right)^2 |u_{xx}|^2 dxd\tau\\
&+\frac{48^2}{\delta^{14}}\int_0^t\int_0^1 |u_{xx}|^6 dxd\tau\\
&+\frac{1}{\delta^{12}}\int_0^t\int_0^1 \left(|u_{xx}||u_{xxx}| + |u_x||u_{xxxx}|\right)^2 dxd\tau\\
&+\frac{36}{\delta^{12}}\int_0^t\int_0^1 |u_{xxx}|^2|u_{xx}|^2 dxd\tau\Bigg\}\\
\le\;& 6\kappa^2\Bigg\{\frac{42}{\delta^{12}}\int_0^t |u_{xxx}|^2_{(L^\infty(0,1))^2}|u_{xx}|_H^2 d\tau\\
&+\frac{36+48^2+288}{\delta^{14}}\int_0^t\int_0^1 |u_{xx}|^6 dxd\tau\\
&+\frac{288}{\delta^{14}}\int_0^t |u_x|_{(L^\infty(0,1))^2}|u_{xxx}|^2_{(L^\infty(0,1))^2}|u_{xx}|_H^2 d\tau\\
&+\frac{2}{\delta^{12}}\int_0^t |u_{xx}|_H^2\int_0^1 |u_{xxxx}|^2 dxd\tau\Bigg\}\\
\le\;& \tilde{C}_3 + \tilde{C}_4\int_0^t |u_{xxxx}(\tau)|_H^2 d\tau,
\end{aligned}$$

$$I_4 = \int_0^t |f(\varepsilon)u_{xxxx}(\tau)|_H^2 \, d\tau \leq \tilde{C}_4 \int_0^t |u_{xxxx}(\tau)|_H^2 d\tau \quad \text{for any } t \in [0,T],$$

where $\tilde{C}_3$ and $\tilde{C}_4$ are positive constants independent of $\mu > 0$. Therefore, we can get

$$\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \mu \int_0^t |u_{\tau xxx}(\tau)|_H^2 d\tau$$
$$\leq C_1 + \left(\frac{1}{\rho} + \frac{\gamma C_2}{2}\right)\left(\frac{\rho}{2}\int_0^t |u_{\tau xx}(\tau)|_H^2 d\tau + \frac{\gamma}{2}\int_0^t |u_{xxxx}(\tau)|_H^2 d\tau\right)$$
$$\text{for any } t \in [0,T],$$

where C_1 and C_2 are positive constant independent of $\mu > 0$. By Gronwall's lemma, we obtain (3.1).

Thus, we can obtain the uniform estimate for $|u_{xxxx}|_{L^\infty(0,T;H)}$ independent of $\mu > 0$.

4. Existence of solutions

In this section, we prove existence of strong solutions of $\mathrm{P}_0(u_0, v_0, f)$ on $[0,T]$.

Proof of Existence. Let $T > 0$, $\mu > 0$, $u_0 \in H_4$, $|u_{0x}| > 0$ on $[0,1]$, $v_0 \in H_2$, and u_μ be a strong solution of $\mathrm{P}_\mu(u_0,\, v_0,\, f)$ on $[0,T]$. By (1.5) and Lemma 1 we have

$$\frac{\rho}{2}|u_{\mu t}(t)|_H^2 + \frac{\gamma}{2}|u_{\mu xx}(t)|_H^2 + \mu \int_0^t |u_{\mu\tau x}(\tau)|_H^2 d\tau) = C_0 \quad \text{for any } t \in [0,T], \tag{4.1}$$

and

$$|u_{\mu x}| \geq \sqrt{\frac{C_0}{\gamma}} e^{-\frac{16C_0^2}{\kappa\gamma}} \text{ on } \overline{Q(T)}, \tag{4.2}$$

where C_0 is a positive constant given by (1.5). We note that C_0 is independent of $\mu > 0$. In the similar way to that of [10, Section 4], we obtain

$$|u_\mu(t)|_H \leq T\sqrt{\frac{2C_0}{\rho}} + |u_0|_H \quad \text{for any } t \in [0,T].$$

Next, by Lemma 3.1, we also have

$$\frac{\rho}{2}|u_{\mu txx}(t)|_H^2 + \frac{\gamma}{2}|u_{\mu xxxx}(t)|_H^2 + \mu \int_0^t |u_{\mu\tau xxx}(\tau)|_H^2 d\tau \leq C_3 \quad \text{for any } t \in [0,T], \tag{4.3}$$

where $C_3 > 0$ independent of $\mu > 0$. By (4.1), (4.3), and (3.5), we obtain

$$|u_{\mu xxx}(t)|_H \le \sqrt{\frac{C_0 + C_3}{\gamma}}, \quad |u_{\mu tx}(t)|_H \le \sqrt{\frac{C_0 + C_3}{\rho}} \quad \text{for any } t \in [0,T].$$

From these estimates, we see that $\{u_\mu\}$ and $\{u_{\mu t}\}$ are bounded in $L^\infty(0,T;H_4)$ and $L^\infty(0,T;H_2)$, respectively. Moreover, $\{u_{\mu tt}\}$ is bounded in $L^\infty(0,T;H)$. In fact, by (1.1) and (4.3), we see that for any $\mu > 0$,

$$\begin{aligned}
&|u_{\mu tt}(t)|_H \\
&\le \frac{1}{\rho}\left\{\gamma|u_{\mu xxxx}(t)|_H + |\left(f(\varepsilon_\mu)u_{\mu x}\right)_x(t)|_H + \mu|u_{\mu txx}(t)|_H\right\} \\
&\le \frac{1}{\rho}\left\{\sqrt{2\gamma C_3} + |\left(f(\varepsilon_\mu)u_{\mu x}\right)_x(t)|_H + \mu\sqrt{\frac{2C_3}{\rho}}\right\} \text{ for a.e. } t \in (0,T),
\end{aligned}$$

where $\varepsilon_\mu = |u_{\mu x}| - 1$ a.e. on $Q(T)$. Now, $(f(\varepsilon_\mu))_x = \kappa\dfrac{u_{\mu x}\cdot u_{\mu xx}}{|u_{\mu x}|^6}$ a.e. on $Q(T)$ and by (4.2) there exists $\delta > 0$ independent of $\mu > 0$ such that $|u_{\mu x}| \ge \delta$ a.e. on $Q(T)$. Easily, we have

$$\begin{aligned}
\left(f(\varepsilon_\mu)u_{\mu x}\right)_x &= \left(f(\varepsilon_\mu)\right)_x u_{\mu x} + f(\varepsilon_\mu)u_{\mu xx} \\
&\le \kappa\frac{|u_{\mu xx}|}{|u_{\mu x}|^4} + \frac{\kappa}{4}\left(1 + \frac{1}{|u_{\mu x}|^4}\right)|u_{\mu xx}| \\
&\le \kappa\left\{\frac{1}{\delta^4} + \frac{1}{4}\left(1 + \frac{1}{\delta^4}\right)\right\}|u_{\mu xx}| \quad \text{a.e. on } Q(T).
\end{aligned}$$

By (4.1) and $|u_{\mu xx}|_H \le \sqrt{\dfrac{2C_0}{\gamma}}$ on $[0,T]$, we have

$$\begin{aligned}
|\left(f(\varepsilon_\mu)u_{\mu x}\right)_x(t)|_H &= \kappa\left\{\frac{1}{\delta^4} + \frac{1}{4}\left(1 + \frac{1}{\delta^4}\right)\right\}|u_{\mu xx}|_H \\
&\le \frac{\kappa}{4}\left(\frac{5}{\delta^4} + 1\right)\sqrt{\frac{2C_0}{\gamma}} \quad \text{for a.e. } t \in (0,T).
\end{aligned}$$

Thus, for any $\mu \in [0,1)$, we obtain

$$\begin{aligned}
|u_{\mu tt}(t)|_H &\le \frac{1}{\rho}\left\{\sqrt{2\gamma C_3} + \frac{\kappa}{4}\left(\frac{5}{\delta^4} + 1\right)\sqrt{\frac{2C_0}{\gamma}} + \mu\sqrt{\frac{2C_3}{\rho}}\right\} \\
&\le \frac{1}{\rho}\left\{\sqrt{2\gamma C_3} + \frac{\kappa}{4}\left(\frac{5}{\delta^4} + 1\right)\sqrt{\frac{2C_0}{\gamma}} + \sqrt{\frac{2C_3}{\rho}}\right\} \\
&=: C_4 \qquad \text{for a.e. } t \in (0,T).
\end{aligned}$$

Clearly, C_4 is independent of $\mu \in [0,1)$. Therefore, $\{u_{\mu tt}\}_{\mu\in(0,1]}$ is bounded in $L^\infty(0,T;H)$.

From these estimates, we can take a subsequence $\{\mu_j\} \subset \{\mu\}$ and $u \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;H_2) \cap L^\infty(0,T;H_4)$ such that $\mu_j \to 0$,

$$\left.\begin{aligned} &u_{\mu_j} \to u \text{ weakly}* \text{ in } W^{2,\infty}(0,T;H),\ \text{weakly}* \text{ in } W^{1,\infty}(0,T;H_2),\\ &\qquad\qquad \text{weakly}* \text{ in } L^\infty(0,T;H_4),\\ &u_{\mu_j} \to u, u_{\mu_j x} \to u_x, u_{\mu_j xx} \to u_{xx}, u_{\mu t} \to u_t \text{ in } C(\overline{Q(T)}) \text{ as } j \to \infty. \end{aligned}\right\} \tag{4.4}$$

Moreover, we can easily obtain $\left(f(\varepsilon_{\mu_j})u_{\mu_j x}\right)_x \to (f(\varepsilon)u_x)_x$ in $C(\overline{Q(T)})$ as $j \to \infty$. We note that for any $j \in \mathbb{Z}_{>0}$,

$$\rho u_{\mu_j tt} + \gamma u_{\mu_j xxxx} - \left(f(\varepsilon_{\mu_j})u_{\mu_j x}\right)_x - \mu u_{\mu_j txx} = 0 \quad \text{a.e. on } Q(T). \tag{4.5}$$

Let $\varphi \in C_0^\infty(Q(T))^2$ and multiply both sides of (4.5) by φ. Then, we have

$$\rho \int_{Q(T)} u_{\mu_j tt} \cdot \varphi dxdt + \gamma \int_{Q(T)} u_{\mu_j xxxx} \cdot \varphi dxdt - \int_{Q(T)} \left(f(\varepsilon_{\mu_j})u_{\mu_j x}\right)_x \cdot \varphi dxdt$$
$$-\mu_j \int_{Q(T)} u_{\mu_j txx} \cdot \varphi dxdt = 0 \quad \text{for any } \varphi \in C_0^\infty(Q(T))^2.$$

It is obvious that

$$\mu_j \int_{Q(T)} u_{\mu_j txx} \cdot \varphi dxdt \to 0 \quad \text{as } j \to \infty.$$

Hence, the convergences (4.4) imply

$$\int_{Q(T)} (\rho u_{tt} + \gamma u_{xxxx} - (f(\varepsilon)u_x)_x) \cdot \varphi dxdt = 0 \quad \text{for any } \varphi \in C_0^\infty(Q(T))^2.$$

By the fundamental lemma of the calculus of variations, we obtain

$$\rho u_{tt} + \gamma u_{xxxx} - (f(\varepsilon)u_x)_x = 0 \quad \text{a.e. on } Q(T).$$

Therefore, u satisfies the third condition for Definition 2. By using (4.4), we see that the second condition holds. Thus, the existence of strong solutions to $\mathrm{P}_0(u_0, v_0, f)$ on $[0,T]$ has been proved. Moreover, thanks to the uniqueness guaranteed by [10, Section 3], for the whole sequence $\{u_\mu\}$ the convergences (4.4) are true as μ tends to 0. □

5. Conclusion

In this paper, we have established existence and uniqueness of a strong solution to the initial and boundary value problem for the beam equation accompanying with the stress function having the singularity point. In other words, we proved the convergence of solutions to the problem with the viscosity term as the viscosity coefficient tends to 0.

References

1. Aiki T., Kosugi C. Numerical scheme for ordinary differential equations describing shrinking and stretching motion of elastic materials. *Adv. Math. Sci. Appl.*, 2020, vol. 29, pp. 459–494.
2. Aiki T., Kosugi C. Existence and uniqueness of weak solutions for the model representing motions of curves made of elastic materials. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 36, pp. 45–56. https://doi.org/10.26516/1997-7670.2021.36.44
3. Aiki T., Kosugi C. Large time behavior of solutions for a PDE model for compressible elastic curve. *Discrete Contin. Dyn. Syst. Ser. S*, 2023, vol. 16, pp. 3733–3745.
4. Aiki T., Kröger N.H., Muntean A. A macro-micro elasticity-diffusion system modeling absorption-induced swelling in rubber foarms: Proof of the strong solvability. *Quart. Appl. Math.*, 2021, vol. 3, pp. 545–579.
5. Bonet J., Lee H.C., Gil J.A., Ghavamian A. A first order hyperbolic framework for large strain computational solid dynamics. Part III: Thermo-elasticity. *Comput. Methods Appl. Mech. Engrg.*, 2021, vol. 373, Paper No. 113505, 53 pp.
6. Furihata D., Matsuo T. *Discrete variational derivative method: A structure preserving numerical method for partial differential equations.* Chapman & Hall CRC, 2010.
7. Holzapfel A.G. *Nonlinear solid mechanics: a continuum approach for engineering.* John Wiley & Sons Publ., 2000.
8. Holzapfel A.G., Simo C.J. Entropy elasticity of isotropic rubber-like solids at finite strains. *Comput. Methods Appl. Mech. Engrg.*, 1996, vol. 132, pp. 17–44.
9. Kosugi C. Solvability of a PDE model with nonlinear stress function having singularity for compressible elastic curve. *Adv. Math. Sci. Appl.*, 2023, vol. 32, pp. 155–177.
10. Kosugi C. Existence and uniqueness of weak solutions for compressible elastic curves in the energy conservation system. *Adv. Math. Sci. Appl.*, 2023, vol. 32, pp. 447–454.
11. Kosugi C., Aiki T., Anthonissen M., Okumura M. Numerical results for ordinary and partial differential equations describing motions of elastic materials. *Adv. Math. Sci. Appl.*, 2021, vol. 30, pp. 387–414.
12. Ladyzenskaya A.O., Solonnikov V.A., Uraĺceva N.N. *Linear and Quasi-Linear Equations of Parabolic Type, Transl. Math. Monograph.*, vol. 23, Amer. Math. Soc., Providence, R. I., 1968.
13. Li K., Holzapfel A.G. A multiscale viscoelastic fiber dispersion model for strain rate-dependent behavior of planar fibrous tissues. *J. Mech. Phys. Solids*, 2024, vol. 186, Paper No. 105572, 16 pp.

14. Ogden R.W. Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubber like solids. *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.*, 1972, vol. 328, pp. 567–583.
15. Okabe S. The motion of elastic planar closed curves under the area-preserving condition. *Indiana Univ. Math. J.*, 2007, vol. 56, pp. 1871–1912.
16. Racke R., Shang C. Global attractors for nonlinear beam equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 2012, vol. 142, pp. 1087–1107.
17. Simo C.J., Miehe C. Associative coupled thermoplasticity at finite strains: Formulation, numerical analysis and implementation. *Comput. Methods in App. Mech. Engi.*, 1992, vol. 98, pp. 41–104.
18. Takeda H., Yoshikawa S. On the initial value problem of the semilinear beam equations with weak damping I; Smoothing effect. *J. Math. Anal. Appl.*, 2013, vol. 401, pp. 244–258.
19. Takeda H., Yoshikawa S. On the initial value problem of the semilinear beam equations with weak damping II; Aymptotic profiles. *J. Differential Equations*, 2012, vol. 253, pp. 3061–3080.

Об авторах

Косуги Чихару, канд. физ.-мат. наук, доц., Университет Ямагути, Ямагути, 753-8511, Япония, ckosgi@yamaguchi-u.ac.jp

About the authors

Chiharu Kosugi, PhD (Phys.–Math.), Assoc. Prof., Yamaguchi University, 1677-1, Yoshida, Yamaguchi-shi, Yamaguchi, 753-8511, Japan, ckosgi@yamaguchi-u.ac.jp

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