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Obstacle Problem for a Discontinuous Stieltjes String

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Abstract: In this paper, we consider a boundary value problem with a nonlinear boundary condition and discontinuous solutions. This problem models the deformation process of a discontinuous Stieltjes string (a chain of Stieltjes strings connected by springs) under the action of an external load. The shape of the string is described by an integro-differential equation with a derivative with respect to the measure and with a generalized Stieltjes integral. This representation allows us to analyze both solutions and relations at each point. We assume that there is an obstacle at the left end of the chain. Depending on the applied external force, the corresponding end of the chain either touches the boundary points of the obstacle or remains free. This creates a nonlinear boundary condition, since it is not known in advance how the solution will behave. The existence and uniqueness theorems of the solution are proved, a formula for the representation of the solution is obtained, loads at which the end of the chain touches the obstacle are found, and the dependence of the solution on the size of the obstacle is studied.

Keywords: obstacle problem, variation, measure, Stieltjes integral, nonlinear boundary condition

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Научная статья

Задача с препятствием для разрывной стилтьесовской струны

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Аннотация: Рассматривается краевая задача с нелинейным краевым условием и разрывными решениями. Эта задача моделирует процесс деформаций разрывной стилтьесовской струны (цепочки из стилтьесовских струн, соединенных между собой пружинами) под воздействием внешней нагрузки. Форма струны описывается интегро-дифференциальным уравнением с производными по мере и обобщенным интегралом Стилтьеса. Такое представление позволяет проводить поточечный анализ как решений, так и соотношений. Предполагается, что на левом конце струнной цепочки установлено препятствие на перемещение. В зависимости от приложенной внешней силы соответствующий конец цепочки либо соприкоснется с граничными точками препятствия, либо останется свободным. Это порождает нелинейное краевое условие, поскольку заранее неизвестно, как поведет себя решение. Доказаны теоремы существования и единственности решения, получена формула представления решения, найдены нагрузки, при которых происходит соприкосновение конца цепочки с граничными точками препятствия, и изучена зависимость решения от размера препятствия.

Ключевые слова: задача с препятствием, вариация, мера, интеграл Стилтьеса, нелинейное краевое условие

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1. Introduction

In recent years, special attention has been given to the study of mathematical models of string systems, because they are relevant in many areas of science and technology. Note that the term ”string” has a purely mathematical character. The string deformations equation can be used to model processes in quantum mechanics, electric circuits, acoustic tubes, nerve fibers, various waveguides, etc. (see, for example, [1; 2]).

The presence of singularities (elastic supports, concentrated external effects, discontinuities in the solutions) in such models can lead to difficult problems due to loss of smoothness. This fact excludes the possibility of using ordinary derivatives in modeling and analysis.

In this paper, we will not apply the theory of generalized functions. We will use the pointwise approach with the Stieltjes integral. This approach has been applied in works of M. G. Krein, F. R. Gantmakher, O. Kellogg (see comments in [2]), Yu. V. Pokornyi [8; 9], S. A. Shabrov [12], R. Ch. Kulaev [6], M. Tverdý [7; 15]. It allows studying qualitative properties of the solutions that are important for applications, such as the number of zeros, extrema, sign changes, etc. (see, for example, [9; 10]), since solutions are determined at each point. Following this approach, in works [4; 11; 16] problems, modeling deformations of elastic systems with localized singularities and nonlinearities have been studied. However, the boundary value problem with discontinuous solutions, which is the subject of this paper, has not been investigated until now.

Obstacle problems have been studied by many scientists (see, for example, [3; 13; 14]). In such works, mainly the questions of existence and uniqueness of solutions are discussed, and algorithms for finding approximate solutions are developed. However, the formulas for the representation of exact solutions are not given explicitly.

In this paper, a formula for the exact solution of the boundary value problem

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x u d[Q] = F(x) - F(0), \\ p(0)u'_\mu(0) - \gamma_1 u(0) + f_1 \in N_{[-m,m]}(u(0)), \\ p(l)u'_\mu(l) + \gamma_2 u(l) = f_2 \end{cases}$$

is obtained in explicit form, the uniqueness of the solution is proved, and properties of the solution are studied. Here the set $N_{[-m,m]}(u(0))$ denotes the outward normal cone to $[-m, m]$ at point $u(0)$; u'_μ is the derivative with respect to the measure generated by a strictly increasing function $\mu(x)$ on the segment $[0, l]$. The integral $\int\limits_0^x u d[Q]$ is understood in the extended sense according to Stieltjes. To emphasize that we are talking about the generalized Stieltjes integral, we will enclose the function under the differential sign in square brackets.

2. Preliminaries

In this section, we recall some notions and facts (see [5; 8; 9; 11]) that we will need in the sequel.

Outward normal cone. Let H be a Hilbert space, and $G \subset H$ be a nonempty closed convex set. For $x \in G$, the set

$$N_G(x) = \{\xi \in H : \langle \xi, c - x \rangle \leq 0, \ \forall c \in G\}$$

denotes the *outward normal cone* to G at x.

Jumps of BV functions. Let $BV[0,l]$ denote the set of functions of bounded variation on the segment $[0,l]$. By a *simple jump* of $u(x)$ at a point $x = \xi$, we mean the value $\Delta u(\xi) = u(\xi+0) - u(\xi-0)$. By the *left jump* $u(x)$ at the point $x = \xi$, we mean the value $\Delta^- u(\xi) = u(\xi) - u(\xi-0)$. By the *right jump* $u(x)$ at the point $x = \xi$, we mean the value $\Delta^+ u(\xi) = u(\xi+0) - u(\xi)$.

μ–absolute continuity. Suppose the function $\mu(x)$ is strictly increasing on the segment $[0,l]$. From the Radon-Nikodym theorem [5] it follows that the function $u(x)$ is μ–absolutely continuous if and only if $u(\beta) - u(\alpha) = \int\limits_\alpha^\beta f d\mu, \quad \alpha, \beta \in [0,l]$, where the integral is understood in the Lebesgue-Stieltjes sense. The function f is called the μ derivative of u with respect to the measure μ and is denoted by u'_μ.

Let

$$E = \{u : u \text{ is } \mu\text{–absolutely continuous on } [0,l] \text{ and } u'_\mu \in BV[0,l]\}.$$

The generalized Stieltjes integral. The generalized Stieltjes integral $\int\limits_\alpha^\beta ud[v]$ was first introduced by Yu. V. Pokornyi in [8]. According to [8], the integral $\int\limits_\alpha^\beta ud[v]$ can be written as

$$\int\limits_\alpha^\beta ud[v] = \int\limits_\alpha^\beta udv_c + \sum_{\alpha < s \leq \beta} u(s-0)\Delta^- v(s) + \sum_{\alpha \leq s < \beta} u(s+0)\Delta^+ v(s),$$

where $u(x)$ and $v(x)$ are bounded variation functions, v_c is the continuous part of v, and the integral $\int\limits_\alpha^\beta udv_c$ is understood in the Lebesgue–Stieltjes sense (see [5]). For the generalized Stieltjes integral, we have (see [8], [9])

$$\int\limits_\alpha^\beta u \, d[v] = u(\beta)v(\beta) - u(\alpha)v(\alpha) - \int\limits_\alpha^\beta v \, du,$$

where the integral $\int\limits_\alpha^\beta v \ du$ is understood in the Lebesgue–Stieltjes sense. Note that the main difference between the classical Stieltjes integral $\int\limits_\alpha^\beta udv$

and the generalized integral $\int_{\alpha}^{\beta} ud[v]$ is the following. In the classical integral, to each point s of discontinuity of the function $v(x)$ there corresponds a term of the form $u(s)\Delta v(s)$, where the jump $\Delta v(s)$ determines the measure of the point s. In this case, the value of the function u at the point s is important. In the generalized Stieltjes integral, the measure of any point s of the discontinuity of the function v is split into two components determined by the left $\Delta^{-}v(s)$ and right $\Delta^{+}v(s)$ jumps, respectively. In this case, only the limits $u(s-0)$ and $u(s+0)$ of the function $u(x)$ are considered at the point s.

Definition 1. *By a* zero point *of the function $u(x)$, we mean a point $s \in [0,l]$ such that $u(s-0)u(s+0) \le 0$.*

Suppose the function $p(x)$ has bounded variation on $[0,l]$ and $\inf\limits_{[0,l]} p(x) > 0$; the function $Q(x)$ is non-decreasing on the segment $[0,l]$ and continuous at the points $x = 0$ and $x = l$; the function $\mu(x)$ is strictly increasing and continuous at the points $x = 0$ and $x = l$. Consider the homogeneous equation

$$-(pu'_{\mu})(x) + \int_{0}^{x} ud[Q] = -(pu'_{\mu})(0), \tag{2.1}$$

where $u \in E$.

Definition 2. *We say that Equation (2.1) is* non-oscillating *on $[0,l]$ if every non-trivial solution of (2.1) has no more than one zero point on $[0,l]$.*

For the non-oscillation of Equation (2.1) on $[0,l]$, it is sufficient that the function $Q(x)$ is monotonically non-decreasing on the segment $[0,l]$(Theorem 2.4 in [11]).

Lemma 1. *Let $\varphi_1(x)$ be a non-trivial solution of the homogeneous equation (2.1) and satisfy the condition $p(l){\varphi_1}'_{\mu}(l)+\gamma_2\varphi_1(l) = 0$, where $\gamma_2 > 0$. Then the function $\varphi_1(x)$ does not have zero points on the segment $[0,l]$.*

Proof. Assume $\varphi_1(l) = 0$. Thus, we have ${\varphi_1}'_{\mu}(l) = 0$, and according to Theorem 1 in [9], we obtain $\varphi_1(x) \equiv 0$. Hence, $\varphi_1(l) \ne 0$. For definiteness, let $\varphi_1(l) > 0$. Let ξ be the zero point of the function $\varphi_1(x)$ closest to $x = l$. Thus, $\varphi_1(x) > 0$ for all $x \in (\xi, l]$. Let us consider the case where at point ξ at least one of the functions p, Q, F, μ has the discontinuity. If $x > \xi$ we have

$$-(p{\varphi_1}'_{\mu})(x) + (p{\varphi_1}'_{\mu})(\xi+0) + \int_{\xi+0}^{x} \varphi_1 d[Q] = 0. \tag{2.2}$$

If the function $\varphi_1(x)$ is continuous at the point ξ, then $\varphi_1(\xi) = 0$, ${\varphi_1}'_\mu(\xi + 0) > 0$. From (2.2) it follows that ${\varphi_1}'_\mu(x) > 0$ for all $x > \xi$, which contradicts $p(l){\varphi_1}'_\mu(l) = -\gamma_2\varphi_1(l) < 0$. If the function $\varphi_1(x)$ is discontinuous at the point ξ, then $\Delta\varphi_1(\xi) > 0$. From Equation (2.1) we have

$$p(\xi)\frac{\Delta\varphi_1(\xi)}{\Delta\mu(\xi)} - p(\xi+0){\varphi_1}'_\mu(\xi+0) + \varphi_1(\xi+0)\Delta^+Q(\xi) = 0.$$

Hence, ${\varphi_1}'_\mu(\xi + 0) > 0$ and ${\varphi_1}'_\mu(x) > 0$, where $x > \xi$, but this contradicts ${\varphi_1}'_\mu(l) < 0$. Other cases can be considered similarly. □

The following lemma can be proved in a similar way.

Lemma 2. *Let $\varphi_2(x)$ be a non-trivial solution of the homogeneous equation (2.1) and satisfy the condition $-p(0){\varphi_2}'_\mu(0) + \gamma_1\varphi_2(0) = 0$, where $\gamma_1 > 0$. Then the function $\varphi_2(x)$ does not have zero points on the segment $[0, l]$.*

3. The statement of the problem

By discontinuous Stieltjes string, we mean a chain of strings connected by springs. Further, we assume that elastic supports can be attached to the string at an arbitrary set of points (but not more than countable). Let the discontinuous Stieltjes string be stretched along the segment $[0, l]$ of the Ox axis. This position is called the equilibrium position. Under the action of an external force, the string deviates from its equilibrium position and takes on the shape determined by the function $u(x)$, where $x \in [0, l]$. We assume that there exists a strictly increasing function $\mu(x)$ such that the function $u(x)$ is μ - absolutely continuous, and $u'_\mu(x)$ is a function of bounded variation on $[0, l]$. Suppose the function of bounded variation $p(x)$, where $\inf\limits_{[0,l]} p(x) > 0$, characterizes the local tension of our physical system; the non-decreasing function $Q(x)$ describes the elasticity of the external environment, where $x \in (0, l)$; the function of bounded variation $F(x)$ characterizes the external force, where $x \in (0, l)$. Let $\gamma_1 > 0$ and $\gamma_2 > 0$ denote the elasticities of the springs attached at the points $x = 0$ and $x = l$. Suppose f_1 and f_2 are the forces concentrated at points $x = 0$ and $x = l$, respectively. Moreover, at the point $x = 0$ we have the obstacle $[-m, m]$ on the deviation of the elastically fixed end of the string. The condition of the obstacle is $|u(0)| \leq m$. Using [9], the potential energy functional for our physical system has the form

$$\Phi(u) = \int_0^l \frac{p{u'_\mu}^2}{2} d\mu + \int_0^l \frac{u^2}{2} d[Q] + \frac{u^2(0)}{2}\gamma_1 + \frac{u^2(l)}{2}\gamma_2 - \int_0^l u d[F] - u(0)f_1 - u(l)f_2. \tag{3.1}$$

In (3.1), the first integral is understood in the classical sense and defines the work of the string tension force. The second and third integrals are understood in the generalized sense and define the work of the elastic force of the external environment and the work of the external force, respectively. Generalized integrals make it possible to take into account the forces that are concentrated at the ends of the strings attached to a spring and the presence of additional springs attached to these ends. The functions $u \in E$ and satisfy the condition

$$|u(0)| \leq m, \tag{3.2}$$

the functions $\mu(x)$, $Q(x)$, $F(x)$ are continuous at the points $x = 0$ and $x = l$. Note that the function $u(x)$ can only be discontinuous at points where $\mu(x)$ is discontinuous. The functions Q, F can be singular with respect to the measure μ.

According to the Hamilton-Lagrange principle, the real form $u_0(x)$ of the string minimizes the functional Φ with condition (3.2). Consider functions $u \in E$ such that $u(x) = u_0(x) + \lambda h(x)$, where $\lambda \in \mathbb{R}$, $h \in E$ and $h(0) = h(l) = 0$. Since the function $u_0(x)$ is a minimum point of the functional Φ, we have $\Phi(u_0) \leq \Phi(u_0 + \lambda h)$. Having fixed h, consider the function $\psi_h(\lambda) = \Phi(u_0 + \lambda h)$, where $\lambda \in \mathbb{R}$. So we have $\psi_h(0) \leq \psi_h(\lambda)$, and we get $\dfrac{d}{d\lambda}\psi_h(\lambda)|_{\lambda=0} = 0$. Considering the conditions on $h(x)$, the last equality has the form

$$\int_0^l p u'_{0\mu} h'_\mu d\mu + \int_0^l u_0 h d[Q] - \int_0^l h d[F] = 0.$$

Integrating the second and third integrals by parts, and considering the properties of the function $h(x)$, we obtain

$$\int_0^l (p u'_{0\mu}(x) - \int_0^x u_0 d[Q] + F(x)) h'_\mu d\mu = 0. \tag{3.3}$$

Applying Lemma 3.1 in [11], we have

$$(p u'_{0\mu})(x) - \int_0^x u_0 d[Q] + F(x) = \text{const}, \tag{3.4}$$

i.e., $-(p u'_{0\mu})(x) + \int_0^x u d[Q] = F(x) - F(0) - (p u'_{0\mu})(0)$.

Consider functions $h \in E$ such that $h(0) = 0$. Similar to the previous case, we obtain $\frac{d}{d\lambda}\psi_h(\lambda)|_{\lambda=0} = 0$, i.e.,

$$\int_0^l ((p u'_{0_\mu})(x) - \int_0^x u_0 d[Q] + F(x)) dh + h(l) \int_0^l u_0 d[Q] - h(l)F(l) + \tag{3.5}$$

$$\gamma_2 u_0(l)h(l) - f_2 h(l) = 0.$$

Equality (3.4) can be represented as

$$(pu'_{0_\mu})(x) - \int\limits_0^x u_0 d[Q] + F(x) = p(l)u'_{0\mu}(l) - \int\limits_0^l u_0 d[Q] + F(l).$$

Substituting this representation into (3.5), we obtain

$$\left(p(l)u'_{0\mu}(l) + \gamma_2 u_0(l) - f_2\right) h(l) = 0.$$

Since $h(l)$ is arbitrary, we have $p(l)u'_{0\mu}(l) + \gamma_2 u_0(l) = f_2$.

Fix any $c \in [-m, m]$ and consider functions $h \in E$ satisfying the conditions $h(0) = c - u_0(0)$, $h(l) = 0$. Let $u(x) = u_0(x) + \lambda h(x)$, where $\lambda \in [0, 1]$. Note that $u \in E$, $|u(0)| \le m$. With h fixed, consider the function $\psi_h(\lambda) = \Phi(u_0 + \lambda h)$. Then $\psi_h(0) \le \psi_h(\lambda)$. Hence, the right derivative at $\lambda = 0$ satisfies $\frac{d^+}{d\lambda}\psi_h(\lambda)|_{\lambda=0} \ge 0$, i.e.,

$$\int\limits_0^l ((pu'_{0_\mu})(x) - \int\limits_0^x u_0 d[Q] + F(x))dh + h(0)F(0)+ \tag{3.6}$$

$$\gamma_1 u_0(0)h(0) - f_1 h(0) \ge 0.$$

Thus, we obtain $\left(-p(0)u'_{0\mu}(0) + \gamma_1 u_0(0) - f_1\right) h(0) \ge 0$ and for all $c \in [-m, m]$, with respect to $h(0) = c - u_0(0)$, we have

$$(p(0)u'_{0\mu}(0) - \gamma_1 u_0(0) + f_1)(c - u_0(0)) \le 0,$$

i.e.,

$$p(0)u'_{0\mu}(0) - \gamma_1 u_0(0) + f_1 \in N_{[-m,m]}(u_0(0)).$$

We have just proved the following theorem.

Theorem 1. *Assume the function $u_0(x)$ is a minimum point of the functional $\Phi(u)$, where $u \in E$ and $|u(0)| \le m$. Then $u_0(x)$ is a solution to the problem*

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x ud[Q] = F(x) - F(0), \\ p(0)u'_\mu(0) - \gamma_1 u(0) + f_1 \in N_{[-m,m]}(u(0)), \\ p(l)u'_\mu(l) + \gamma_2 u(l) = f_2. \end{cases} \tag{3.7}$$

The equation in (3.7) is defined at each point of the special extension of the segment $[0, l]$, denoted by $\overline{[0, l]}_S$. This set contains together with each discontinuity point ξ of the function $\mu(x)$ the pair of points denoted as $\{\xi - 0, \xi + 0\}$, and each point s, where the function μ is continuous, but at least one of the functions p, Q, F is discontinuous is replaced by the pair of points denoted as $\{s-0, s+0\}$(see [8], [9], [11]). The values of the functions at the points $\xi \pm 0$, $s \pm 0$ coincide with the limit values. From the equation in (3.7) it follows that at discontinuity points ξ of the function $\mu(x)$ the equalities

$$-p(\xi)\frac{\Delta u(\xi)}{\Delta\mu(\xi)} + p(\xi-0)u'_\mu(\xi-0) + u(\xi-0)\Delta^-Q(\xi) = \Delta^-F(\xi), \quad (3.8)$$

$$p(\xi)\frac{\Delta u(\xi)}{\Delta\mu(\xi)} - p(\xi+0)u'_\mu(\xi+0) + u(\xi+0)\Delta^+Q(\xi) = \Delta^+F(\xi), \quad (3.9)$$

hold, and at the points s, where the function μ is continuous, but at least one of the functions p, Q, F is discontinuous, the equality

$$-p(s+0)u'_\mu(s+0) + p(s-0)u'_\mu(s-0) + u(s)\Delta Q(s) = \Delta F(s)$$

holds.

4. Main results

Theorem 2. *If a solution to Problem (3.7) exists, it is unique.*

Proof. Suppose $u_1(x)$ and $u_2(x)$ are solutions to Problem (3.7). Then $u(x) = u_2(x) - u_1(x)$ is a solution to Equation (2.1). Assume $u(x) \neq 0$. Since $u_1(x)$ and $u_2(x)$ are solutions to Problem (3.7), we have $|u_1(0)| \leq m$, $|u_2(0)| \leq m$ and

$$(p(0)u'_{1\mu}(0) - \gamma_1 u_1(0) + f_1)(c - u_1(0)) \leq 0, \quad (4.1)$$

$$(p(0)u'_{2\mu}(0) - \gamma_1 u_2(0) + f_1)(c - u_2(0)) \leq 0 \quad (4.2)$$

for all $c \in [-m, m]$. Putting in (4.1) $c = u_2(0)$ and in (4.2) $c = u_1(0)$, we obtain

$$(p(0)u'_{1\mu}(0) - \gamma_1 u_1(0) + f_1)(u_2(0) - u_1(0)) \leq 0,$$
$$(p(0)u'_{2\mu}(0) - \gamma_1 u_2(0) + f_1)(u_1(0) - u_2(0)) \leq 0.$$

Thus, we have

$$(p(0)u'_\mu(0) - \gamma_1 u(0))u(0) \geq 0. \quad (4.3)$$

Since $u_1(x)$ and $u_2(x)$ are solutions to Problem (3.7), we have

$$p(l)u'_\mu(l) + \gamma_2 u(l) = 0. \quad (4.4)$$

According to Lemma 1, the function $u(x)$ does not have zero points on the segment $[0, l]$. Assume $u(x) > 0$ for all $x \in [0, l]$. Thus, we have $u(0) > 0$ and according to (4.3), $p(0)u'_\mu(0) - \gamma_1 u(0) \geq 0$, i.e., $p(0)u'_\mu(0) \geq \gamma_1 u(0) > 0$. From Equation (2.1) we have $p(x)u'_\mu(x) > 0$. Hence, $p(l)u'_\mu(l) + \gamma_2 u(l) > 0$, but this contradicts (4.4). Also the case $u(x) < 0$ is not possible. So $u(x) \equiv 0$. The theorem is proved. □

Theorem 3. *Let the functions $\varphi_1(x)$ and $\varphi_2(x)$ be solutions of Equation (2.1) and satisfy the conditions*

$$-p(0)\varphi'_{1\mu}(0)+\gamma_1\varphi_1(0)=1, p(l)\varphi'_{1\mu}(l)+\gamma_2\varphi_1(l)=0,$$

$$-p(0)\varphi'_{2\mu}(0)+\gamma_1\varphi_2(0)=0, p(l)\varphi'_{2\mu}(l)+\gamma_2\varphi_2(l)=1.$$

If $\left|\int_0^l \varphi_1 d[F]+\varphi_1(0)f_1+\varphi_2(0)f_2\right| < m$ *then the solution to Problem (3.7) is*

$$u(x)=\frac{\varphi_1(x)}{\varphi_2(0)}\int\limits_0^x \varphi_2(s)d[F(s)]+\frac{\varphi_2(x)}{\varphi_2(0)}\int\limits_x^l \varphi_1(s)d[F(s)]+\varphi_1(x)f_1+\varphi_2(x)f_2. \tag{4.5}$$

If $\int_0^l \varphi_1 d[F]+\varphi_1(0)f_1+\varphi_2(0)f_2 \geq m$ *then the solution to Problem (3.7) is*

$$u(x)=\frac{\varphi_1(x)m}{\varphi_1(0)}+\frac{\varphi_1(x)}{\varphi_2(0)}\int\limits_0^x \varphi_2(s)d[F(s)]+\frac{\varphi_2(x)}{\varphi_2(0)}\int\limits_x^l \varphi_1(s)d[F(s)]+\varphi_2(x)f_2-$$

$$-\frac{\varphi_1(x)}{\varphi_1(0)}\int\limits_0^l \varphi_1(s)d[F(s)]-\frac{\varphi_1(x)\varphi_1(l)f_2}{\varphi_1(0)}. \tag{4.6}$$

If $\int_0^l \varphi_1 d[F]+\varphi_1(0)f_1+\varphi_2(0)f_2 \leq -m$ *then the solution to Problem (3.7) is*

$$u(x)=-\frac{\varphi_1(x)m}{\varphi_1(0)}+\frac{\varphi_1(x)}{\varphi_2(0)}\int\limits_0^x \varphi_2(s)d[F(s)]+\frac{\varphi_2(x)}{\varphi_2(0)}\int\limits_x^l \varphi_1(s)d[F(s)]+\varphi_2(x)f_2-$$

$$-\frac{\varphi_1(x)}{\varphi_1(0)}\int\limits_0^l \varphi_1(s)d[F(s)]-\frac{\varphi_1(x)\varphi_1(l)f_2}{\varphi_1(0)}. \tag{4.7}$$

Proof. Note that the problem

$$\begin{cases} -(p\varphi'_{1\mu})(x)+(p\varphi'_{1\mu})(0)+\int\limits_0^x \varphi_1 d[Q]=0, \\ \quad -p(0)\varphi'_{1\mu}(0)+\gamma_1\varphi_1(0)=1, \\ p(l)\varphi'_{1\mu}(l)+\gamma_2\varphi_1(l)=0 \end{cases} \tag{4.8}$$

has a unique solution. Let us represent $\varphi_1(x)$ as $\varphi_1(x)=c_1u_1(x)+c_2u_2(x)$, where $u_1(x)$ and $u_2(x)$ are solutions of Equation (2.1), satisfying the initial conditions $u_1(0)=0$, $u'_{1\mu}(0)=1$ and $u_2(0)=1$, $u'_{2\mu}(0)=0$, respectively.

Since the function $Q(x)$ does not decrease on $[0,l]$, Equation (2.1) is nonoscillating on $[0,l]$ and hence, the function $u_1(x)$ does not have any other zero points, except $x=0$. From the condition $u'_{1\mu}(0)=1$ it follows $u_1(x)>0$ for all $x\in(0,l]$. Since $(pu'_{1\mu})(x)=(pu'_{1\mu})(0)+\int\limits_0^x u_1d[Q]$, we obtain $p(x)u'_{1\mu}(x)>0$. In particular, $p(l)u'_{1\mu}(l)>0$ and $p(l)u'_{1\mu}(l)+\gamma_2u_1(l)>0$. Let us show that the function $u_2(x)$ does not have zero points on $[0,l]$. Assume that $\xi\in(0,l)$ is the zero point of the function $u_2(x)$ closest to $x=0$. Since $u_2(0)>0$, we obtain $u_2(x)>0$ and $p(x)u'_{2\mu}(x)\geq 0$ for all $x<\xi$. Thus, $u_2(x)$ is not decreasing function for $x<\xi$. Since ξ is the zero point of the function $u_2(x)$ and $u_2(\xi-0)>0$, we have $u_2(\xi+0)\leq 0$. Hence, $\Delta u_2(\xi)<0$. But

$$p(\xi)\frac{\Delta u_2(\xi)}{\Delta\mu(\xi)}=p(\xi-0)u'_{2\mu}(\xi-0)+u_2(\xi-0)\Delta^-Q(\xi)\geq 0.$$

Hence, $u_2(x)>0$ for all $x\in[0,l]$ and $p(x)u'_{2\mu}(x)\geq 0$. In particular, we obtain

$$p(l)u'_{2\mu}(l)+\gamma_2u_2(l)>0. \tag{4.9}$$

If $\xi=l$, we obtain $u_2(x)>0$, where $x\in[0,l)$, and we also have (4.9). We denote by $l_1u_1=p(l)u'_{1\mu}(l)+\gamma_2u_1(l)>0$, $l_1u_2=p(l)u'_{2\mu}(l)+\gamma_2u_2(l)>0$. Substituting the representation for $\varphi_1(x)$ into the boundary conditions of Problem (4.8), we obtain

$$c_2=\frac{l_1u_1}{\gamma_1l_1u_1+p(0)l_1u_2}\quad\text{and}\quad c_1=\frac{-l_1u_2}{\gamma_1l_1u_1+p(0)l_1u_2}.$$

Similarly, there is a solution to the problem

$$\begin{cases}-(p\varphi'_{2\mu})(x)+(p\varphi'_{2\mu})(0)+\int\limits_0^x\varphi_2d[Q]=0,\\ -p(0)\varphi'_{2\mu}(0)+\gamma_1\varphi_2(0)=0,\\ p(l)\varphi'_{2\mu}(l)+\gamma_2\varphi_2(l)=1.\end{cases}\tag{4.10}$$

According to Lemma 1 and Lemma 2, the functions $\varphi_1(x)$ and $\varphi_2(x)$ do not have zero points on $[0,l]$. Let us show that $\varphi_1(x)>0$ and $\varphi_2(x)>0$ for all $x\in[0,l]$. Assume $\varphi_1(x)<0$. Hence, $p(l)\varphi'_{1\mu}(l)=-\gamma_2\varphi_1(l)>0$ and

$$(p\varphi'_{1\mu})(x)=(p\varphi'_{1\mu})(l)-\int_x^l\varphi_1d[Q]>0.$$

So $-p(0)\varphi'_{1\mu}(0)+\gamma_1\varphi_1(0)<0$, but this contradicts $-p(0)\varphi'_{1\mu}(0)+\gamma_1\varphi_1(0)=1$. Thus, $\varphi_1(x)>0$. Similarly, $\varphi_2(x)>0$.

Since $p(0)\varphi'_{2\mu}(0)=\gamma_1\varphi_2(0)$, we have $p(0)\varphi'_{2\mu}(0)>0$ and

$$(p\varphi'_{2\mu})(x)=(p\varphi'_{2\mu})(0)+\int_0^x\varphi_2d[Q]>0,$$

i.e., the function φ_2 increases on $[0,l]$. Similarly, the function φ_1 decreases on $[0,l]$.

Denote by $W(x)=\varphi_1(x){\varphi_2}'_\mu(x)-\varphi_2(x){\varphi_1}'_\mu(x)$ the Wronskian for $\varphi_1(x)$ and $\varphi_2(x)$ (see [11]). According to Lemma 2.1 in [11],

$$p(0)W(0)=p(l)W(l)=const,$$

and we obtain $\varphi_1(l)=\varphi_2(0)$.

Assume $\left|\int_0^l \varphi_1 d[F]+\varphi_1(0)f_1+\varphi_2(0)f_2\right|<m$. Let us show that the function (4.5) is the solution to Problem (3.7). We need to verify that $u\in E$. Suppose $\alpha<\beta$. Since the functions $\varphi_1(x)$ and $\varphi_2(x)$ are μ-absolutely continuous and

$$\begin{aligned}&u(\beta)-u(\alpha)=\\&=\frac{1}{\varphi_2(0)}\left((\varphi_1(\beta)-\varphi_1(\alpha))\int\limits_0^\beta\varphi_2 d[F]+(\varphi_2(\beta)-\varphi_2(\alpha))\int\limits_\beta^l\varphi_1 d[F]\right)+\\&+\frac{1}{\varphi_2(0)}\int\limits_\alpha^\beta((\varphi_1(\alpha)-\varphi_1(s))\varphi_2(s)+(\varphi_2(s)-\varphi_2(\alpha))\varphi_1(s))d[F(s)]+\\&+(\varphi_1(\beta)-\varphi_1(\alpha))f_1+(\varphi_2(\beta)-\varphi_2(\alpha))f_2,\end{aligned}$$

we obtain that the function $u(x)$ is μ-absolutely continuous.

Note that

$$\begin{aligned}&u'_\mu(x)=\\&=\frac{{\varphi_1}'_\mu(x)}{\varphi_2(0)}\int\limits_0^x\varphi_2(s)d[F(s)]+\frac{{\varphi_2}'_\mu(x)}{\varphi_2(0)}\int\limits_x^l\varphi_1(s)d[F(s)]+{\varphi_1}'_\mu(x)f_1+{\varphi_2}'_\mu(x)f_2.\end{aligned}\tag{4.11}$$

Thus, $u'_\mu\in BV[0,l]$, and we have $u\in E$.

Let us show that Function (4.5) is a solution to the equation in (3.7). Note that

$$\begin{aligned}&\int\limits_0^x u(s)d[Q(s)]=\frac{1}{\varphi_2(0)}\int\limits_0^x\varphi_1(s)\int\limits_0^s\varphi_2(t)d[F(t)]d[Q(s)]+\\&+\frac{1}{\varphi_2(0)}\int\limits_0^x\varphi_2(s)\int\limits_s^l\varphi_1(t)d[F(t)]d[Q(s)]+f_1\int\limits_0^x\varphi_1(s)d[Q(s)]+f_2\int\limits_0^x\varphi_2(s)d[Q(s)].\end{aligned}$$

Changing the integration limits in the first term, we obtain with (2.1) that

$$\frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_1(s)\int\limits_0^s \varphi_2(t)d[F(t)]d[Q(s)] =$$

$$= \frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_2(t)((p\varphi_1{}'_\mu)(x) - (p\varphi_1{}'_\mu)(t))d[F(t)].$$

Changing the integration limits in the second term, we obtain with (2.1) that

$$\frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_2(s)\int\limits_s^l \varphi_1(t)d[F(t)]d[Q(s)] =$$

$$= \frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_1(t)(p(t)\varphi_2{}'_\mu(t) - p(0)\varphi_2{}'_\mu(0))d[F(t)]$$

$$+ \frac{1}{\varphi_2(0)}\int\limits_x^l \varphi_1(t)(p(x)\varphi_2{}'_\mu(x) - p(0)\varphi'_{2\mu}(0))d[F(t)].$$

Substituting the resulting representation for $\int\limits_0^x ud[Q]$ into the equation from (3.7), and taking into account that $p(t)W(t) = \varphi_2(0)$, we obtain the required equality. Note that

$$u(0) = \int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2.$$

Thus, $|u(0)| < m$, and we have $p(0)u'_\mu(0) - \gamma_1 u(0) + f_1 = 0$, $p(l)u'_\mu(l) + \gamma_2 u(l) = f_2$.

Assume $\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2 \geq m$. Note that Function (4.6) is the solution to the equation from Problem (3.7), and $p(l)u'_\mu(l) + \gamma_2 u(l) = f_2$, $u(0) = m$. Let us prove that $p(0)u'_\mu(0) - \gamma_1 u(0) + f_1 \geq 0$. Using (4.11) and the conditions on the functions φ_1 and φ_2, we obtain

$$p(0)u'_\mu(0) - \gamma_1 u(0) + f_1 = \frac{\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2 - m}{\varphi_1(0)} \geq 0.$$

The last case can be considered similarly. The theorem is proved. □

Corollary 1. *Let the function $F(x)$ be non-decreasing on the segment $[0, l]$ and different from a constant, $f_1 \geq 0$, $f_2 \geq 0$. Then the solution to Problem (3.7) $u(x) > 0$ for all $x \in [0, l]$.*

Proof. If $\left|\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2\right| < m$, then the solution to Problem (3.7) is

$$u(x) = \frac{\varphi_1(x)}{\varphi_2(0)} \int_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{\varphi_2(0)} \int_x^l \varphi_1(s)d[F(s)] + \varphi_1(x)f_1 + \varphi_2(x)f_2.$$

Since $\varphi_1(x) > 0$, $\varphi_2(x) > 0$, we obtain $u(x) > 0$ for all $x \in [0, l]$.

If $\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2 \geq m$ then the solution to Problem (3.7) is

$$u(x) = \frac{\varphi_1(x)m}{\varphi_1(0)} + \frac{\varphi_1(x)}{\varphi_2(0)} \int_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{\varphi_2(0)} \int_x^l \varphi_1(s)d[F(s)] + \varphi_2(x)f_2 -$$

$$-\frac{\varphi_1(x)}{\varphi_1(0)} \int_0^l \varphi_1(s)d[F(s)] - \frac{\varphi_1(x)\varphi_1(l)f_2}{\varphi_1(0)}$$

$$= \frac{\varphi_1(x)m}{\varphi_1(0)} + \frac{\varphi_1(x)}{\varphi_1(0)\varphi_2(0)} \int_0^x \left(\varphi_1(0)\varphi_2(s) - \varphi_1(s)\varphi_2(0)\right) d[F(s)] +$$

$$+ \frac{\varphi_1(0)\varphi_2(x) - \varphi_1(x)\varphi_2(0)}{\varphi_1(0)\varphi_2(0)} \int_x^l \varphi_1(s)d[F(s)] +$$

$$+ \frac{(\varphi_1(0)\varphi_2(x) - \varphi_1(x)\varphi_2(0))f_2}{\varphi_1(0)}.$$

Since the function φ_2 increases on $[0, l]$ and the function φ_1 decreases on $[0, l]$, we have $\varphi_1(0)\varphi_2(x) - \varphi_1(x)\varphi_2(0) > 0$ for all $x \in [0, l]$. Hence, $u(x) > 0$ for all $x \in [0, l]$. The case $\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2 \leq -m$ is not possible, because $\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2 > 0$. □

Corollary 2. *If $m \to 0$, then the solution of Problem (3.7) tends to a solution of the problem*

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x u d[Q] = F(x) - F(0), \\ u(0) = 0, \quad p(l)u'_\mu(l) + \gamma_2 u(l) = f_2 \end{cases} \tag{4.12}$$

uniformly on $\overline{[0, l]}_\mu$. If $m \to \infty$, then the solution of Problem (3.7) tends to a solution of the problem

$$\begin{cases} -(pu'_\mu)(x) + (pu'_\mu)(0) + \int\limits_0^x u d[Q] = F(x) - F(0), \\ -p(0)u'_\mu(0) + \gamma_1 u(0) = f_1, \quad p(l)u'_\mu(l) + \gamma_2 u(l) = f_2 \end{cases}$$

uniformly on the set $\overline{[0,l]}_\mu$*. The set* $\overline{[0,l]}_\mu$ *contains a pair of elements* $\xi-0$ *and* $\xi+0$ *instead of any discontinuity point* ξ *of the function* $\mu(x)$ *(see [9]).*

Proof. Since $m \to 0$, we have $|\int_0^l \varphi_1 d[F] + \varphi_1(0)f_1 + \varphi_2(0)f_2| \geq m$. Let us denote by

$$z(x) = \frac{\varphi_1(x)}{\varphi_2(0)} \int\limits_0^x \varphi_2(s)d[F(s)] + \frac{\varphi_2(x)}{\varphi_2(0)} \int\limits_x^l \varphi_1(s)d[F(s)] + \varphi_2(x)f_2 -$$

$$-\frac{\varphi_1(x)}{\varphi_1(0)} \int\limits_0^l \varphi_1(s)d[F(s)] - \frac{\varphi_1(x)\varphi_1(l)f_2}{\varphi_1(0)}.$$

Taking into account that $\varphi_1 \in E$, we have $|u_m(x) - z(x)| \leq c|m| \to 0$. Thus, $u_m(x)$ converges uniformly to $z(x)$. Similar to Theorem 3, we can verify that the function $z(x)$ is a solution to Problem (4.12). The second statement can be proved in a similar way. □

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