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G-permutable Subgroups in $\mathrm{PSL}_2(q)$ and Hereditarily G-permutable Subgroups in $\mathrm{PSU}_3(q)$

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Abstract: The concept of X-permutable subgroup, introduced by A. N. Skiba, generalizes the classical concept of a permutable subgroup. Many classes of finite groups have been characterized in terms of X-permutable subgroups. In particular, W. Guo, A. N. Skiba and K. P. Shum obtained a characterization of the classes of solvable, supersolvable and nilpotent groups. Nevertheless, the further application of this concept in solving various problems in group theory is restrained by the lack of information about G-permutable and hereditarily G-permutable subgroups lying in the composition factors of groups. In this regard, the following problems were posed in the Kourovka Notebook: which finite nonabelian simple groups G have a proper G-permutable subgroup and a proper hereditarily G-permutable subgroup? In this paper, an answer is obtained to the first question for simple linear groups of dimension two and to the second question for simple unitary groups of dimension three.

Keywords: simple linear group, simple unitary group, G-permutable subgroup, hereditarily G-permutable subgroup

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Научная статья

G-перестановочные подгруппы в группе $\mathrm{PSL}_2(q)$ и наследственно G-перестановочные подгруппы в группе $\mathrm{PSU}_3(q)$

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Аннотация: Понятие X-перестановочной подгруппы, введенное А. Н. Скибой, обобщает классическое понятие перестановочной подгруппы. Многие классы конечных групп удалось охарактеризовать в терминах X-перестановочных подгрупп. В частности, В. Го, А. Н. Скиба и К. П. Шам получили характеризацию классов разрешимых, сверхразрешимых и нильпотентных групп. Тем не менее дальнейшее применение данного понятия при решении различных задач теории групп осложняется отсутствием информации о G-перестановочных и наследственно G-перестановочных подгруппах, находящихся в композиционных факторах групп. В связи с этим в «Коуровской тетради» под номером 17.112 были записаны следующие проблемы: какие конечные неабелевы простые группы G обладают собственной G-перестановочной подгруппой и собственной наследственно G-перестановочной подгруппой? В данной работе получен ответ на первый вопрос для простых линейных групп размерности два и на второй — для простых унитарных групп размерности три.

Ключевые слова: простая линейная группа, простая унитарная группа, G-перестановочная подгруппа, наследственно G-перестановочная подгруппа

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1. Introduction

A subgroup A of a group G is *permutable with a subgroup* B if $AB = BA$. If A is permutable with every subgroup of G then A is called a *permutable* [4] or *quasinormal* [16] *subgroup* of G.

The study of permutable subgroups goes back to Ore's work [16], where he proved that each permutable subgroup of a finite group is subnormal. Since then, his result has been generalized in different directions (see [11; 13; 17]).

Given two subgroups A and B of a group G, a common situation is that $AB \neq BA$ but there exists an element $x \in G$ such that $AB^x = B^xA$. A minimal example of this case is the symmetric group S_3 with two different subgroups of order 2 in S_3. The examples of this situation were presented in [8; 9] and led to the following notions.

Definition. Let A, B be subgroups of a group G and $\emptyset \neq X \subseteq G$. Then
(1) A is called *X-permutable* with B if there exists an element $x \in X$ such that $AB^x = B^xA$;
(2) A is called *hereditarily X-permutable* with B if $AB^x = B^xA$ for some $x \in X \cap \langle A, B\rangle$;
(3) A is called *(hereditarily) X-permutable in G* if A is (hereditarily) X-permutable with all subgroups of G.

The solvable, supersolvable and nilpotent groups were characterized in terms of X-permutable subgroups in [8]. In the Kourovka Notebook the following question was posed.

Problem [14, 17.112] Which finite non-abelian simple groups G possess
(a) a non-trivial G-permutable subgroup?
(b) a non-trivial hereditarily G-permutable subgroup?

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There are examples of finite non-abelian simple groups both containing and not containing proper G-permutable subgroups. However, the authors do not know of any finite non-abelian simple group with a proper hereditarily G-permutable subgroup.

Problem 1(b) was answered in the negative for the alternating groups, sporadic groups, and exceptional groups of Lie type in [19], [18], and [6] respectively. Based on these results, it was conjectured in [6] that a finite group G is simple if and only if G has no proper hereditarily G-permutable subgroups.

Problem 1(a) was solved for sporadic groups [5], Suzuki groups ${}^2\mathrm{B}_2(q)$ [6], and alternating groups of sufficiently large degree [20].

In this paper, we give an answer to Problem(a) for simple groups $\mathrm{PSL}_2(q)$ and to Problem(b) for simple groups $\mathrm{PSU}_3(q)$.

Theorem 1. *Let $G \cong \mathrm{PSL}_2(q)$, where $q = p^n \geqslant 4$. Then G has no proper G-permutable subgroups if and only if q and $(q-1)/2$ are odd. Moreover, if F is a proper G-permutable subgroup of G, then $|F| = 2$.*

Theorem 2. *Let $G \cong \mathrm{PSU}_3(q)$, where $q = p^n \geqslant 3$. Then G has no proper hereditarily G-permutable subgroups.*

2. Notation and preliminary results

Our notation is standard and mainly follows [2]. In particular, by $A{:}B$, $A{}^{\cdot}B$ and $A.B$ we denote a split, nonsplit and arbitrary extension of A by B, respectively. Symmetric, alternating, and dihedral groups are denoted by S_n, A_n, and D_{2n}, respectively. A cyclic group of order n is denoted by n. By $\mathrm{Syl}_p(G)$ we denote the set of all Sylow p-subgroups of G. The greatest common divisor of the integers $n_1, \dots, n_k$ is denoted by $(n_1, \dots, n_k)$.

In the sequel, we will need the following statements.

Lemma 1. [5, Lemma 1] *Let $T < G$ be a maximal subgroup of a group G and $G \neq TR$ for every subgroup $R < G$. If F is a G-permutable subgroup in G then $F^g \leqslant T$ for some $g \in G$. In particular, $|F|$ divides $|T|$.*

Lemma 2. [5, Lemma 2] *Let G be a finite group without factorizations and F be a G-permutable subgroup in G. Then $|F|$ divides $(|M_1|, \dots, |M_k|)$, where M_i are representatives of all classes of maximal subgroups of G.*

Lemma 3. *The following statements hold:*

(*a*) $(q-1, q+1) \in \{1, 2\}$;
(*b*) $(q-1, q^2-q+1) = 1$;
(*c*) $(q+1, q^2-q+1) \in \{1, 3\}$;
(*d*) *if* $(3, q+1) = 3$ *then* $q^2-q+1 = 3l$, *where* $(3, l) = 1$.

Proof. The statements (a)-(c) are trivial. For (d) we have $q+1 = 3m$ and $q = 3m-1$. Then $q^2-q+1 = 3(3(m^2-m)+1) = 3l$, where $(3, l) = 1$. □

Lemma 4. *Let $G = A{:}F$, where A is an elementary abelian 2-group and $3 \cong F$ is a hereditarily G-permutable subgroup. Then $G = A \times F$.*

Proof. Induction by $|A|$. If $|A| = 2$, then it is clear that $G = A \times F$. Let $|A| > 2$. Consider a subgroup $\langle a \rangle \subset A$ of order 2. By our assumption there exists $g \in G$ such that $\langle a \rangle^g F = F \langle a \rangle^g$. It is clear that $\langle a \rangle^g F = \langle a \rangle^g \times F$ and $C_A(F) \neq 1$. Since $A = [A, F] \times C_A(F)$ and $|[A, F]| < |A|$, then by inductive hypothesis we have $[[A, F], F] = 1$ and $G = A \times F$. □

We notice the following property of G-permutable subgroups, that can be checked straightforward.

Lemma 5. *Let G be a group, H, T, K are subgroups of G, and K is normal in G. If $K \subseteq T$ and H is G-permutable with T, then HK/K is G/K-permutable with T/K. In particular, if H is hereditarily G-permutable in G, then HK/K is hereditarily G/K-permutable in G/K.*

3. Proof of Theorem 1

The subgroup structure of $\mathrm{PSL}_2(q)$, where $q = p^n$, was determined by Dickson [3] and well known. Further we will use this information without any additional references. Remind that

$$|G| = \frac{1}{(2, q-1)} q(q^2-1) = \frac{1}{(2, q-1)} q(q-1)(q+1).$$

(a) Let $p = 2$. The group G has a maximal subgroup $M \cong \mathrm{D}_{2(q-1)}$. This subgroup is not a factor in any factorization of G [12]. By Lemma 1 we have $|F|$ divides $2(q-1)$. The group G has a cyclic subgroup $T \cong q+1$. Hence, there exists $g \in G$ such that FT^g is a subgroup. It is possible only if $|F| = 2$. Since all involutions of G are conjugate, for every subgroup L of even order there exists $g \in G$ such that $F^g \leqslant L$.

Let $L < G$ and $|L|$ be an odd number. Each subgroup of G of odd order is contained in a cyclic subgroup $S \cong q-1$ or in a cyclic subgroup $T \cong q+1$. Assume that $L \leqslant S$. Since $S < S : F^g \cong \mathrm{D}_{2(q-1)}$ for some $g \in G$, there exists the subgroup $S^{g^{-1}}F$. Since S is cyclic, there exists the subgroup $L^{g^{-1}}F$. The case $L \leqslant T$ can be considered in a similar way. Thus, F is G-permutable in G.

(b) Let $p \geqslant 3$. At first, consider the case $q > 11$. A subgroup $M \cong \mathrm{D}_{(q-1)}$ is maximal in G. As in the case $p = 2$ we have $|F|$ divides $|M|$ and $F \leqslant M^g$ for some $g \in G$. A subgroup $T \cong \mathrm{D}_{(q+1)}$ is maximal in G and by our assumption there exists a subgroup FT^h for some $h \in G$. It is clear that $FT^h \neq G$, therefore $|F|$ divides $|T^h|$ and $F \leqslant T^h$. Since $(q-1, q+1) = 2$, we have $|F| = 2$.

The group G has a Borel subgroup $B = U : H \cong q : (\frac{q-1}{2})$, which is maximal in G. If $\frac{q-1}{2}$ is odd, then $FB^g \neq B^gF$ for every $g \in G$. Therefore, F is not G-permutable in G.

Let $\frac{q-1}{2}$ be an even number. Without loss of generality, we can assume that $F < B$. Let L be an arbitrary subgroup of B of odd order. If $|L|$ divides $q-1$, then it was shown that there exists a subgroup FL^g for some $g \in G$.

Consider the case p divides $|L|$. At first, assume that $|L| = p^m$. Since B is the Frobenious group, for every subgroup $\langle u \rangle \subseteq U$ with $|\langle u \rangle| = p$ and for every involution $\tau \in B$ we have the equality $\langle u \rangle^\tau = \langle u \rangle$. Indeed, if $\langle u \rangle^\tau = \langle u_1 \rangle \neq \langle u \rangle$, then we can assume that $u^\tau = u_1$ and $u = u_1^\tau$. It follows from here that $(uu_1)^\tau = u^\tau u_1^\tau = u_1 u = uu_1$. The latter is impossible in the Frobenius group. Therefore, $L^\tau = L$ or $\langle \tau \rangle L = L \langle \tau \rangle$. Since all involutions of G are conjugate, there exists a subgroup FL^g for some $g \in G$. Let $L = E : \langle t \rangle$, where E is an elementary abelian group of order p^m, $m \geqslant 1$ and $\langle t \rangle$ is a cyclic subgroup of odd order dividing $q-1$. There exists a cyclic subgroup $\langle \widetilde{t} \rangle \subseteq B$ such that $\langle t \rangle \subset \langle \widetilde{t} \rangle$ and $|\langle \widetilde{t} \rangle : \langle t \rangle| = 2$. Since $\langle \widetilde{t} \rangle = \langle t \rangle \times \langle \tau \rangle$ for some involution $\tau \in B$, it follows that $\langle \tau \rangle$ normalizes E and there exists a subgroup $E : \langle \widetilde{t} \rangle = L : \langle \tau \rangle$. All involutions of G are conjugate, therefore there exists a subgroup FL^g for some $g \in G$. Hence, F is G-permutable in G.

Consider the remaining cases for odd $q \leqslant 11$. The cases $\mathrm{PSL}_2(5) \cong \mathrm{A}_5$ and $\mathrm{PSL}_2(9) \cong \mathrm{A}_6$ were mentioned in [5].

Assume that $G \cong \mathrm{PSL}_2(7)$. The group $\mathrm{PSL}_2(7)$ has a maximal subgroup isomorphic to $7 : 3$. Let $F \subseteq H \cong 7 : 3$. If $F \cong 7$, then there exists a subgroup FS^g, where S is a Sylow 2-subgroup of G and $g \in G$. This situation is impossible in G. Let $F \cong 3$. The group G has a cyclic subgroup $T \cong 4$ and therefore G has a subgroup FT^g for some $g \in G$. However, G has no such groups. Let $F = H$. In this case there exists a subgroup FU^g, where U is a subgroup of G of order 2 and $g \in G$. It is impossible because F is maximal in G.

Suppose that $F \nsubseteq H$. Since H is maximal in G, it follows that $FH^g = G$. Hence, F has a Sylow 2-subgroup S of G. If $F = S$ then there exists a subgroup FR^g for some $g \in G$, where $R \in \mathrm{Syl}_7(G)$, which is impossible. Therefore, $F \cong \mathrm{S}_4$. The group G has two conjugacy classes of subgroups isomorphic to S_4. Let T be a subgroup of G isomorphic to S_4 which is not conjugate to F. Then there exists a subgroup FT^g for some $g \in G$. The latter is impossible because F and T are maximal in G. Thus, G has no proper G-permutable subgroups.

$G \cong \mathrm{PSL}_2(11)$. The group $\mathrm{PSL}_2(11)$ has a maximal subgroup isomorphic to $11 : 5$. Suppose that $F \subseteq H \cong 11 : 5$. If $F \cong 11$, then there exists a subgroup FS^g for some $g \in G$ and a Sylow 2-subgroup S, which is impossible. Let $F \cong 5$. Then there exist $T \in \mathrm{Syl}_3(G)$ and $g \in G$ such that FT^g is a subgroup of G, which is impossible. Finally, let $F = H$. In this case, there exists a subgroup U of order 2 and $g \in G$ such that FU^g is a subgroup. Since F is maximal, the latter is also impossible.

Assume that $F \nsubseteq H$. Since H is maximal, there exists $g \in G$ such that $FH^g = G$. Therefore, $|F|$ is divisible by 12 and $F \in \{\mathrm{A}_4, \mathrm{A}_5, \mathrm{D}_{12}\}$. Let $F \cong \mathrm{A}_5$. The group G has two conjugacy classes of subgroups isomorphic to A_5. Let T be a subgroup of G isomorphic to A_5 which is not conjugate to F. Then there exists a subgroup FT^g for some $g \in G$. The latter is impossible because F and T are maximal in G.

If $F \cong \mathrm{A}_4$, then there exist $R \in \mathrm{Syl}_{11}(G)$ and $g \in G$ such that FR^g is a proper subgroup of G, which is impossible. The case $F \cong \mathrm{D}_{12}$ is considered as the previous one.

4. Proof of Theorem 2

The subgroup structure of $\mathrm{PSU}_3(q)$ is well known and can be found in [1, Table 8.5]. Further we will use this information without any additional references.

Notice that

$$|G| = \frac{1}{(3, q+1)} q^3(q^2-1)(q^3+1) = \frac{1}{(3, q+1)} q^3(q-1)(q+1)^2(q^2-q+1).$$

At first, consider the cases $G \in \{\mathrm{PSU}_3(3), \mathrm{PSU}_3(5)\}$.

Let $G \cong \mathrm{PSU}_3(3)$. Then $|\,\mathrm{PSU}_3(3)| = 2^5 {\cdot} 3^3 {\cdot} 7$ and its maximal subgroups are isomorphic to one of the following groups: $3_+^{1+2} : 8$, $\mathrm{PSL}_2(7)$, $4^{\cdot}\,\mathrm{S}_4$,

$4^2{:}\mathrm{S}_3$ [2, p. 14]. The group $\mathrm{PSU}_3(3)$ has only one factorization into maximal subgroups: $\mathrm{PSU}_3(3) = \mathrm{PSL}_2(7)(3^{1+2}_+{:}8)$ [15, Table 3].

Since $|4^{\cdot}\,\mathrm{S}_4| = 2^5 \cdot 3$, Lemma 1 implies that $|F|$ divides $2^5 \cdot 3$. We have

$$|\,\mathrm{PSU}_3(3){:}\mathrm{PSL}_2(7)| = 2^2 \cdot 3^2 \text{ and } |\,\mathrm{PSU}_3(3){:}(3^{1+2}_+{:}8)| = 2^2 \cdot 7.$$

If F is not contained in a subgroup isomorphic to $\mathrm{PSL}_2(7)$, then $|F|$ is divisible by 3^2. It is in contradiction with $|F|$ divides $2^5 \cdot 3$. Hence, F is contained in a subgroup isomorphic to $\mathrm{PSL}_2(7)$. However, $\mathrm{PSL}_2(7)$ has no proper hereditarily G-permutable subgroups [6, Proposition 1].

Let $G \cong \mathrm{PSU}_3(5)$. Then $|\,\mathrm{PSU}_3(5)| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ and its maximal subgroups are isomorphic to one of the following groups: $5^{1+2}_+ : 8$, A_7, $\mathrm{A}_6^{\cdot}\,2_3$, $2\,\mathrm{S}_5$ [2, p. 34]. The group $\mathrm{PSU}_3(5)$ has only one factorization into maximal subgroups: $\mathrm{PSU}_3(5) = \mathrm{A}_7(5^{1+2}_+{:}8)$ [15, Table 3].

Since $|2\,\mathrm{S}_5| = 2^4 \cdot 3 \cdot 5$, Lemma 1 implies that $|F|$ divides $2^4 \cdot 3 \cdot 5$. We have $|\,\mathrm{PSU}_3(5) : \mathrm{A}_7| = 2 \cdot 5^2$ and $|\,\mathrm{PSU}_3(5) : (5^{1+2}_+ : 8)| = 2 \cdot 3^2 \cdot 7$. If F is not contained in a subgroup isomorphic to A_7, then $|F|$ is divisible by 5^2. It is in contradiction with $|F|$ divides $2^4 \cdot 3 \cdot 5$. If F is not contained in a subgroup isomorphic to $5^{1+2}_+{:}8$, then $|F|$ is divisible by 7. Since $|F|$ divides $2^4 \cdot 3 \cdot 5$, we get a contradiction.

Therefore, $G = AB \cong \mathrm{A}_7(5^{1+2}_+{:}8)$ and $F \subseteq A$, $F \subseteq B$. It follows from the equality $2^4 \cdot 3^2 \cdot 5^3 \cdot 7 = \frac{3\cdot4\cdot5\cdot6\cdot7\cdot5^3\cdot2^3}{|A\cap B|}$ that $|A \cap B| = 2^2 \cdot 5$ and $|F|$ divides $2^2 \cdot 5$. Choose $T \in \mathrm{Syl}_7(G)$ and $g \in G$. The group G has no subgroups FT^g, which is a contradiction.

In the remaining cases G does not have factorizations [15, Tables 1, 3].

We will show that $|F| = 3$. Since G has no factorizations, Lemma 2 implies that $|F|$ divides $(|M_1|, \ldots, |M_k|)$, where M_i are representatives of maximal subgroups of G. The group G has a maximal subgroup of order $\frac{1}{(3,q+1)}(q^2 - q + 1) \cdot 3$. Therefore, $|F|$ is odd.

Let $(3, q + 1) = 1$. It follows from [1, Table 8.5] that G has maximal subgroups $M_1 \cong (q + 1)^2 : \mathrm{S}_3$ and $M_2 \cong (q^2 - q + 1) : 3$. It is clear that $(|M_1|, |M_2|) = 3$. By Lemma 1 we have $|F| = 3$.

Let $(3, q + 1) = 3$. By Lemma 3(d) we have $q^2 - q + 1 = 3l$, where $(3, l) = 1$. The group G has a maximal subgroup $M \cong \frac{1}{3}(q^2 - q + 1){:}3$. It is clear that $|M| = 3m$, where $(3, m) = 1$. The group G also has a maximal subgroup, isomorphic to $\frac{1}{3}{E_q}^{1+2}{:}(q^2 - 1)$. By Lemmas 1, 3 we obtain that $|F| = 3$.

Suppose that q is odd. Since $q \geqslant 7$, the group G has a maximal subgroup $M = A{:}T \cong \mathrm{PSL}_2(q){:}2$. Since $|F| = 3$, we get $F \subset A$, which contradicts to [6, Proposition 1].

Thus, q is a power of 2. The group F is contained in every maximal subgroup of G. Consider a maximal subgroup

$$M = A \times L \cong \frac{1}{(3, q+1)}(q + 1) \times \mathrm{PSL}_2(q) \cong \frac{1}{(3, q+1)}\,\mathrm{GU}_2(q).$$

By [6, Proposition 1] F is not contained in $L \cong \mathrm{PSL}_2(q)$. Then $|A|$ is divisible by 3 and $F = \langle f \rangle = \langle al \rangle$, where $a \in A$, $l \in L$. If $l \neq 1$, Lemma 5 implies that FA/A is a proper MA/A-permutable subgroup in MA/A. Since $MA/A \cong \mathrm{PSL}_2(q)$, we have a contradiction with [6, Proposition 1].

Hence, $f = a$ and $C_M(f) = M$. Notice that $C_M(f)$ has a subgroup isomorphic to $\mathrm{PSL}_2(q)$. It follows from the structure of G that $C_M(f)$ is not contained in $C_{M_i}(f)$ for all representatives M_i of maximal subgroups other than M. Therefore, $C_G(f) = M$.

The group G has a maximal subgroup $M_1 = ET$, where $E \cong E_q^{1+2}$ and $T \cong \frac{1}{(3,q+1)}(q^2-1)$. The subgroup E of order q^3 has an elementary abelian 2-subgroup U of order q. By Lemma 4 we have $U \subseteq C_{M_1}(f)$. Since E is a non-abelian 2-group, there exists an element $x \in E$ such that $|x| = 4$. Without loss of generality, we can assume that there exists a subgroup $T = \langle x \rangle : F$. According to [7, Chapter 5, Lemma 4.1] we get the equality $T = \langle x \rangle \times F$. Hence, f centralizes a 2-subgroup of order greater than q. However, the order of Sylow 2-subgroup in M is equal to q and therefore $C_{M_1}(f)$ is not contained in $C_M(f) = C_G(f)$. This contradiction completes the proof of Theorem 2.

5. Conclusion

We obtain an answer to problem 17.112(a) from the Kourovka Notebook for groups $\mathrm{PSL}_2(q)$ and to problem 17.112(b) for groups $\mathrm{PSU}_3(q)$. In particular, we confirm the conjecture posed in [6] for unitary groups $\mathrm{PSU}_3(q)$.

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