Серия «Математика»
2025. Т. 52. С. 71—87

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 517.95
MSC 35L20, 35R30
DOI https://doi.org/10.26516/1997-7670.2025.52.71

Inverse Problems of Recovering Parameters in the Lin–Reissner–Tsien Equation

Alexandr I. Kozhanov[1,2], **Lyubov A. Telesheva**[2]✉

[1] Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russian Federation
[2] Banzarov Buryat State University, Ulan-Ude, Russian Federation
✉ love_20_09@mail.ru

Abstract. The article is devoted to the study of the solvability of new nonlinear inverse problems of finding an unknown constant together with solving the linearized Lin–Reissner–Tsien equation. For the problems under consideration, theorems of solvability are proved in classes of regular solutions, i.e., of solutions having all weak derivatives in the sense of S.L. Sobolev that occur in the corresponding equation.

The peculiarity of the problems under study is, firstly, that the unknown coefficient is a constant (which corresponds, for example, to a homogeneous medium). Secondly, a new overdetermination condition, not previously used by the predecessors, is introduced: an integral condition with respect to the time variable.

Keywords: hyperbolic equations, inverse problems, unknown parameters, integral overdetermination condition, regular solutions

Acknowledgements: The work was financially supported by the Russian Science Foundation (Project No. 23-21-00269).

For citation: Kozhanov A. I.,Telesheva L. A. Inverse Problems of Recovering Parameters in the Lin–Reissner–Tsien Equation. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 52, pp. 71–87.
https://doi.org/10.26516/1997-7670.2025.52.71

Научная статья

Обратные задачи восстановления параметров в уравнении Линя – Рейснера – Цзяня

А. И. Кожанов[1,2], **Л. А. Телешева**[2]✉

[1] Институт математики им. С. Л. Соболева СО РАН, Новосибирск, Российская Федерация,
[2] Бурятский государственный университет им. Доржи Банзарова, Улан-Удэ, Российская Федерация
✉ love_20_09@mail.ru

Аннотация. Исследуется разрешимость новых нелинейных обратных задач нахождения вместе с решением линеаризованного уравнения Линя – Рейснера – Цзяня также неизвестной константы. Для изучаемых задач доказываются теоремы разрешимости в классах регулярных решений, имеющих все обобщенные по С. Л. Соболеву производные, входящие в соответствующее уравнение. Особенностью изучаемых задач является, во-первых, то, что неизвестный коэффициент есть величина постоянная (что соответствует, например, однородной среде), во-вторых, новое условие переопределения, ранее не используемое у предшественников, а именно интегральное условие по временной переменной.

Ключевые слова: гиперболические уравнения, обратные задачи, неизвестные параметры, интегральное условие переопределения, регулярные решения

Благодарности: Работа выполнена при поддержке Российского научного фонда (проект 23-21-00269).

Ссылка для цитирования: Kozhanov A. I.,Telesheva L. A. Inverse Problems of Recovering Parameters in the Lin–Reissner–Tsien Equation // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 52. С. 71–87.
https://doi.org/10.26516/1997-7670.2025.52.71

1. Introduction

In recent years, the theory of coefficient inverse problems has been actively developing, and there are a lot of works related to this topic (see [2; 12; 15–19; 23; 32] and the references therein). In the present article, by an inverse problem for partial differential equations we mean a problem in which, together with the solution, the right-hand side of the equation (the external impact) or (and) a coefficient (coefficients) of the equation itself are unknown. If in an inverse problem, the solution and the right-hand side are unknown then the problem is linear; if the solution and at least one of the coefficients and the right-hand side are unknown then the inverse problem is nonlinear.

Problems of finding a coefficient in a differential equation that essentially depends on certain independent variables have been studied in many works. Significantly fewer papers have been devoted to problems in which the unknown coefficient is a constant (which occurs, for example, in problems of mathematical modeling of processes occurring in a homogeneous medium).

It is precisely nonlinear inverse problems with an unknown constant coefficient that is a constant value that is the subject of research in the present article.

Nonlinear inverse problems of finding a constant coefficient of an equation together with the solution were earlier studied in [3–7; 9; 20; 21; 24–31; 34]. More exactly, such problems were studied for first-order operator-differential equations [24;31], for second-order elliptic, parabolic, and hyperbolic equations [3–7;20;25–28;30;34], for pseudoparabolic equations [9;29], and for equations with multiple characteristics [21]. In the above works, various overdetermination conditions, mainly of integral form, were used.

The present article is devoted to the study of the solvability of some new inverse problems of finding, together with the solution, an unknown constant coefficient — namely, of problems of finding the lowest coefficient in the linear analog of the Lin–Reissner–Tsien equation [1; 10; 22]. Such problems have not been studied before.

2. Inverse Problem I

Let Q be the quadrangle $(0,l) \times (0,T)$. Furthermore, let $f(x,t)$, $N(t)$ be given functions defined for $x \in [0,l]$, $t \in [0,T]$, and A be a given number.

Inverse Problem I. Find a function $u(x,t)$ and a number a such that u satisfies the equation

$$u_{xt} + u_{xx} + au = f(x,t) \tag{2.1}$$

in Q and the conditions

$$u(x,0) = 0, \quad x \in (0,l), \tag{2.2}$$

$$u(0,t) = u_x(0,t) = 0, \quad t \in (0,T) \tag{2.3}$$

$$\int\limits_0^T N(t)u(l,t)dt = A. \tag{2.4}$$

Equation (2.1) is a model of the linearized Lin–Reissner–Tsien equation [1; 10; 22] which models nonstationary small perturbations in a transonic spatial gas flow. The function $u(x,t)$ is the potential of the velocity field, whereas the function $f(x,t)$ describes the external mass forces. Conditions (2.2) and (2.3) are conditions of a well-posed boundary value problem for such equations. Let us clarify that (2.4) is a condition of integral-linear overdetermination; this condition is new for this class of problems.

Let us introduce some notations for the brevity of the statements and proofs. Let L be the differential operator defined by the equality

$$Lw = w_{xt} + w_{xx}.$$

For a fixed function $w(x,t) \in W_2^2(Q)$, denote by $\phi(w)$ the number

$$\phi(w) = \int_0^T N(t)w(l,t)dt.$$

We put

$$b = \int_0^T N(t)f(l,t)dt, \widetilde{f}(x,t) = Lf(x,t), N_0 = \int_0^T N^2(t)dt.$$

Throughout the sequel, we assume that b is positive. For a fixed number $m \in (l, +\infty)$, we set

$$C_1(m) = 4\int_Q (m-x)^2 \widetilde{f}^2(x,t)dxdt,$$

$$C_2(m) = \max\left\{ \left(\int_Q (m-x)^2 \widetilde{f}_t^2(x,t)dxdt \right)^{\frac{1}{2}}, \left(\int_0^l (m-x)\widetilde{f}^2(x,0)dx \right)^{\frac{1}{2}} \right\}$$

$$K_0(m) = \sqrt{lN_0C_1(m)}.$$

Let V and V_0 be the linear spaces

$$V = \{v(x,t) \in L_2(0,T; W_2^2(\Omega)), v_t(x,t) \in L_2(0,T; W_2^1(\Omega))\},$$

$$V_0 = \{v(x,t) \in L_2(0,T; W_2^2(\Omega)), v_t(x,t) \in L_2(0,T; W_2^1(\Omega)), \\ v_{tt}(x,t) \in L_2(0,T; W_2^1(\Omega))\}$$

(here Ω is the interval $(0,l)$). Define norms in these spaces as follows:

$$||v||_V^2 = ||v||^2_{L_2(0,T;W_2^2(\Omega))} + ||v_t||^2_{L_2(0,T;W_2^1(\Omega))},$$

$$||v||_{V_0}^2 = ||v||^2_{L_2(0,T;W_2^2(\Omega))} + ||v_t||^2_{L_2(0,T;W_2^1(\Omega))} + ||v_{tt}||^2_{L_2(0,T;W_2^1(\Omega))}.$$

Theorem 1. *Suppose that*

$$f(x,t) \in W_2^2(Q), \widetilde{f}_t(x,t) \in L_2(Q), N(t) \in L_2([0,T]).$$

Moreover, assume that

$$A > 0, \quad K_0(l) < b,$$
$$f(0,t) = f_x(0,t) = 0 \quad for \quad t \in [0,T],$$
$$f(x,0) = 0 \quad for \quad x \in \overline{\Omega}.$$

Then Inverse Problem I has a solution $\{u(x,t), a\}$ such that $u(x,t) \in V$, $a > 0$.

Proof. Consider the auxiliary boundary value problem: Find a function $v(x,t)$ that satisfies the equation

$$v_{xt} + v_{xx} + \frac{1}{A}(b - \int\limits_0^T N(t)v(l,t)dt)v = \widetilde{f}(x,t), \qquad (2.1_v)$$

in Q and the conditions

$$v(x,0) = 0, \quad x \in \Omega, \qquad (2.2_v)$$

$$v(0,t) = v_x(0,t) = 0, t \in [0,T]. \qquad (2.3_v)$$

Solving this initial boundary value problem for a nonlinear "loaded" (see [11]) integro-differential equation (2.1_v) will make it possible to solve to the initial Inverse Problem I.

From a number $M \in (0,b)$, define the function $G_M(\theta)$:

$$G_M(\theta) = \begin{cases} \theta \text{ if } |\theta| \le M, \\ M \text{ if } \theta > M, \\ -M \text{ if } \theta < -M. \end{cases}$$

Consider the following problem: Find a function $v(x,t)$ that is a solution in Q to the equation

$$v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(v)))v = \widetilde{f}(x,t) \qquad (2.5)$$

and satisfies conditions (2.2_v) and (2.3_v). Let us show that problem (2.5), (2.2_v), (2.3_v) has a solution in V_0.

Make use of the regularization method and the fixed point method. Let ε be a fixed positive number. Consider the problem: Find a function $v(x,t)$ that is a solution in Q to the equation

$$v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(v)))v - \varepsilon v_{xtt} = \widetilde{f}(x,t) \qquad (2.6)$$

and satisfies conditions (2.2_v) and (2.3_v) and also the condition

$$v_t(x,T) = 0 \quad x \in \Omega. \qquad (2.7)$$

Using the fixed point method, let us show that the boundary value problem (2.6), (2.2_v),(2.3_v),(2.7) is solvable in V_0 for fixed ε and $\widetilde{f}(x,t) \in L_2(Q)$. Let $w(x,t) \in V_0)$. Consider the following problem: Find a function $v(x,t)$ satisfying the equation

$$v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(w)))v - \varepsilon v_{xtt} = \widetilde{f}(x,t) \qquad (2.8)$$

and conditions (2.2_v),(2.3_v), (2.7).

This problem is a mixed initial boundary value problem for a pseudohyperbolic equation with time variable x. Under the hypotheses of the theorem and for fixed ε, the problem is solvable in V_0 [19]. It follows that the boundary value problem (2.8),(2.2_v),(2.3_v), (2.7) generates an operator $\Phi(w)$ from V into itself. Let us show that Φ satisfies the hypotheses of Schauder's Theorem.

Due to the condition $K_0(l) < b$, there exists a number $m_0 \in (l, +\infty)$ such that $K_0(m) < b$ for $m \in (l, m_0)$. Consider the equality

$$\int_0^l \int_0^T (v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(w)))v - \varepsilon v_{xtt})(m - x)v_x dxdt = \int_0^l \int_0^T \widetilde{f}(x,t)(m-x)v_x dxdt,$$

where m is a fixed number in (l, m_0).

Integrating by parts and applying Young's inequality, we obtain two estimates as a consequence:

$$\int_Q v_x^2 dxdt \le C_1(m); \tag{2.9}$$

$$\int_0^l (m-x)v_x^2(x,T)dx + (m-l)\int_0^T v_x^2(l,t)dt + \varepsilon \int_Q (m-x)v_{xt}^2 dxdt \le C_1(m). \tag{2.10}$$

Next, consider the equality

$$\int_0^l \int_0^T (v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(w)))v - \varepsilon v_{xtt})(x - m)v_{xtt} dxdt = \int_0^l \int_0^T \widetilde{f}(x,t)(x-m)v_{xtt} dxdt.$$

Using calculations analogous to those used above, we obtain the inequality

$$\int_0^l (m-x)v_{xt}^2(x,0)dx + (m-l)\int_0^T v_{xt}^2(l,t)dt + \int_Q v_{xt}^2 dxdt + \frac{\varepsilon}{2}\int_Q (m-x)v_{xtt}^2 dxdt \le \frac{1}{\varepsilon}\int_Q (m-x)\widetilde{f}^2(x,t)dxdt.$$

The obtained inequalities imply that solutions to the boundary value problem (2.8),(2.2_v),(2.3_v), (2.7) admit the a priori estimate

$$||v||_{V_0} \leq R_1||\widetilde{f}||_{L_2(Q)} = R_0$$

with the constants R_0 and R_1 defined only by the input data and the number ε.

Let $\mathfrak{B}_{R_0}$ be the closed ball of V of radius R_0 and let $\mathfrak{M}$ be the set

$$\mathfrak{M} = \{w(x,t) : w(x,t) \in \mathfrak{B}_{R_0}, w(x,0) = 0, w_t(x,T) = 0\}.$$

Obviously, $\mathfrak{M}$ is closed, convex, and bounded. The above-obtained estimates imply that the operator Φ takes this set into itself. Let us show that Φ is continuous on the set $\mathfrak{B}_{R_0}$.

Let $\{w_m(x,t)\}_{m=1}^{\infty}$ be a sequence of function in $\mathfrak{B}_{R_0}$ such that

$$w_m(x,t) \to w_0(x,t)$$

in $W_2^{2,2}(Q)$. Put $v_m = \Phi(w_m)$, $v_0 = \Phi(w_0)$, $\bar{v}_m = v_m - v_0$. We obtain the problem

$$\bar{v}_{mxt} + \bar{v}_{mxx} + \frac{1}{A}(b - G_M(\phi(w_m)))\bar{v}_m - \varepsilon\bar{v}_{mxtt} = \frac{1}{A}(G_M(\phi(w_m)) - G_M(\phi(w_0)))v_0 \quad (2.11)$$

$$\bar{v}_m(x,0) = 0, \quad \bar{v}_{mt}(x,T) = 0 \quad x \in (0,1), \quad (2.12)$$

$$\bar{v}_m(0,t) = \bar{v}_{mx}(0,t) = 0, t \in (0,T). \quad (2.13)$$

The estimate for $\bar{v}_m(x,t)$ looks as

$$\|\bar{v}_m\|_V^2 \leq R_1\|\frac{1}{A}(G_M(\phi(w_m)) - G_M(\phi(w_0)))v_0\|_{L_2(Q)}.$$

The function $G_M(\theta)$ is Lipschitz continuous, $v(x,t) \in \mathfrak{B}_{R_0}$. Therefore,

$$\|\bar{v}_m\|_V^2 \leq R_0\|w_m - w_0\|_{L_2(Q)}$$

Since the sequence $\{w_m(x,t)\}_{m=1}^{\infty}$ converges in V to $w_0(x,t)$, the right-hand side of the last inequality vanishes as $m \to \infty$. This means that the operator Φ is continuous on $\mathfrak{B}_{R_0}$.

The compactness of Φ stems from the embedding theorems. Consider an arbitrary bounded sequence $\{w_m(x,t)\}_{m=1}^{\infty}$ in V_0. Obviously, the sequences $\{w_{mx}(x,t)\}_{m=1}^{\infty}$, $\{w_{mt}(x,t)\}_{m=1}^{\infty}$ are bounded in $W_2^1(Q)$. Since the embeddings $W_2^1(Q) \subset L_2(Q)$ and $W_2^1(Q) \subset L_2(\partial Q)$ are compact (see [8; 13]), one can extract a subsequence $\{w_{m_k}(x,t)\}_{m_k=1}^{\infty}$ converging weakly to $w_0(x,t) \in W_2^2(Q)$. Moreover, we have the following convergences as $m_k \to \infty$:

$$w_{m_kt}(x,t) \to w_{0t}(x,t) \quad \text{weakly in } L_2(Q),$$

$$w_{m_k x}(x,t) \to w_{0x}(x,t) \quad \text{weakly in } L_2(Q),$$

$$w_{m_k xx}(x,t) \to w_{0xx}(x,t) \quad \text{weakly in } L_2(Q),$$

$$w_{m_k xt}(x,t) \to w_{0xt}(x,t) \quad \text{weakly in } L_2(Q),$$

$$w_{m_k}(l,t) \to w_0(l,t) \quad \text{strongly in } L_2(0,T).$$

Now, define the functions $v_m(x,t)$, $v_0(x,t)$ as the images of the functions $w_m(x,t)$, $w_0(x,t)$ under Φ. Repeating the proof of the continuity of Φ, we conclude that $\|\bar{v}_{m_k}\|_V \to 0$ as $m_k \to \infty$. This means that Φ is compact.

What has been proved implies that Φ satisfies all the hypotheses of Schauder's Fixed Point Theorem in $\mathfrak{B}_{R_0}$. Therefore, there exists a function $v(x,t) \in \mathfrak{B}_{R_0}$ that is a solution to the boundary value problem (2.8), (2.2_v), (2.3_v), (2.7) for $v(x,t) = w(x,t)$. (In other words, the operator Φ has a fixed point.) We show that under extra conditions on $\widetilde{f}(x,t)$, the function $v(x,t)$ admits estimates uniform in ε.

Consider the equality

$$\int\limits_0^l \int\limits_0^T \left(v_{xt} + v_{xx} + \frac{1}{A}(b - G_M(\phi(v)))v - \varepsilon v_{xtt}\right)(x-m)v_{xtt}dxdt$$
$$= \int\limits_0^l \int\limits_0^T \widetilde{f}(x,t)(x-m)v_{xtt}dxdt.$$

Integrating by parts and applying Hölder's inequality, we obtain the two estimates

$$\int\limits_0^l (m-x)v_{xt}^2(x,0)dx + \int\limits_Q v_{xt}^2 dxdt \le 16C_2^2,$$

$$\frac{m-l}{2}\int\limits_0^T v_{xt}^2(l,t)dt + +\varepsilon \int\limits_Q (m-x)v_{xtt}^2 dxdt \le 4C_2^2(m).$$

Consider a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n \to 0$ as $n \to \infty$. Let $v_n(x,t)$ be a solution to the boundary value problem (2.6), (2.7), (2.2_v),(2.3_v) for $\varepsilon = \varepsilon_n$. The above-obtained estimate uniform in ε and the reflexivity of a Hilbert space imply that there exists a sequence $\{m_k\}_{k=1}^\infty$ of naturals and a function $v(x,t)$ such that the following convergences as $k \to \infty$ hold: $v_{n_k}(x,t) \to v(x,t)$ strongly in $L_2(Q)$, $v_{n_k xx}(x,t) \to v_{xx}(x,t)$ weakly in $L_2(Q)$, $v_{n_k xt}(x,t) \to v_{xt}(x,t)$ weakly in $L_2(Q)$, $\varepsilon_{n_k} v_{n_k xtt}(x,t) \to 0$ weakly in $L_2(Q)$. These convergences imply that the limit function $v(x,t)$ satisfies (2.5).

Now, consider $|\phi(v)|$:

$$|\phi(v)| = |\int_0^T N(t)v(l,t)dt| \leq \left(\int_0^T N^2(t)dt\right)^{\frac{1}{2}} \left(\int_0^T v^2(l,t)dt\right)^{\frac{1}{2}}.$$

The function $v(x,t)$ satisfies the inequality $\int_0^T v^2(l,t) \leq lC_1(m)$. Reckoning with the above-obtained estimates, $|\phi(v)| \leq K_0(m)$. By the choice of m_0, under the condition $b > K_0(l)$ of the theorem, the inequality $|\phi(v)| < b$ certainly holds. Thus, $v(x,t)$ is a solution to the auxiliary boundary value problem (2.1_v),(2.2_v),(2.3_v). We will now show that we can determine a solution to the initial boundary value problem with the use of $v(x,t)$.

Define $u(x,t)$ as a solution to the boundary value problem for the equation

$$Lu(x,t) = u_{xt} + u_{xx} = v(x,t)$$

with conditions (2.2)–(2.3). Let $a = \frac{1}{A}(b - \int_0^T N(t)(u_{xt}(l,t) + u_{xx}(l,t))dt)$. Insert the value of a in (2.1); after transformations, it takes the form

$$\left(\int_0^T N(t)u(l,t)dt - A)(\int_0^T N(t)v(l,t)dt - b\right) = 0.$$

Reckoning with the representation of a and the fact that $a > 0$, we see that the second factor is positive and hence this equality implies the fulfillment of (2.4).

Then , there is equality $L(v - au - f) = 0$. For the function $w(x,t) = v(x,t) - au(x,t) - f(x,t)$ the equation $Lw = 0$ and the conditions $w(x,0) = w(0,x) = w_x(0,x) = 0$ are satisfied. It is easy to prove that $w(x,t) = 0$. Consequently, the function $u(x,t)$ and constant a found are the solution to the inverse problem I. □

3. Inverse Problem II

Once again, let $f(x,t)$ and $N(t)$ be given functions defined for $x \in [0,l]$, $t \in [0,T]$, and let A be a given number.

Inverse Problem II. Find a function $u(x,t)$ and a number a such that the equation

$$u_{xt} + u_{xx} + au = f(x,t) \tag{3.1}$$

holds in Q and $u(x,t)$ satisfies the conditions

$$u(x,0)=0, \quad x\in(0,l), \tag{3.2}$$

$$u(l,t)=u_x(0,t)=0, \quad t\in(0,T) \tag{3.3}$$

and also the condition

$$\int\limits_0^T N(t)u(0,t)dt = A. \tag{3.4}$$

We specify that (3.4) is an integral overdetermination condition; this condition is new for this class of problems.

For proving the solvability of Inverse Problem II, we introduce some additional notations.

For a fixed function $v(x,t)\in W_2^2(Q)$, denote by $\Phi(v)$ the number

$$\Phi(v)=\int\limits_0^T N(t)v(0,t)dt.$$

Denote for brevity $b_0=\int\limits_0^T N(t)f(0,t)dt, \widetilde{f}(x,t)=Lf(x,t)$, where L is the differential operator defined above;

$$f_1(x,t)=\widetilde{f}_t(x,t)(T-t)-\widetilde{f}(x,t),$$

$$N_0=\left(\int\limits_0^T N^2(t)dt\right)^{\frac{1}{2}}, \; R_0=N_0 l^2\left(\int\limits_Q f_1^2(x,t)dxdt\right)^{\frac{1}{2}}.$$

Theorem 2. *Suppose that*

$$f(x,t)\in W_2^2(Q), N(t)\in C^1([0,T]).$$

Moreover, assume the fulfillment of the conditions

$$f_x(0,t)=0, f(l,t)=0, f(x,0)=0$$

and the inequality $-b_0>R_0$.

Then Inverse Problem II has a solution $\{u(x,t),a\}$ such that $u(x,t)\in V$, $a<0$.

Proof. Consider the auxiliary direct problem: Find a function $v(x,t)$ that is a solution in Q to the equation

$$v_{xt}+v_{xx}+\frac{1}{A}(b_0-\int\limits_0^T N(t)v(0,t)dt)v=\widetilde{f}(x,t) \tag{3.5}$$

and satisfies the conditions

$$v(x,0)=0,\quad x\in(0,l), \tag{3.6}$$

$$v(l,t)=v_x(0,t)=0,\quad t\in(0,T). \tag{3.7}$$

The solvability of this problem will make it possible to establish the solvability of Inverse Problem II.

Define the function $G_M(\theta)$ as above; but now M is a fixed number such that $b_0<-M$. Let ε be a positive number.

Consider the following problem: Find a function $v(x,t)$ that is a solution in Q to the equation

$$v_{xt}+v_{xx}+\frac{1}{A}(b_0-G_M(\Phi(v)))v+\varepsilon v_{xxt}=\widetilde{f}(x,t) \tag{3.8}$$

and satisfies conditions (3.6), (3.7). We will now prove the presence of "good" a priori estimates.

Note that under conditions (3.6), (3.7), the function $v(x,t)$ satisfies the inequalities

$$\int_0^T v^2(0,t)dt\le l^2\int_Q v_x^2dxdt,\ \int_Q v^2dxdt\le\frac{l^2}{2}\int_Q v_x^2dxdt,\ \int_Q v_x^2dxdt\le\frac{l^2}{2}\int_Q v_{xx}^2dxdt.$$

Consider the equality

$$\int_Q(v_{xt}+v_{xx}+\frac{1}{A}(b_0-G_M(\Phi(v)))v+\varepsilon v_{xxt})(T-t)v_{xxt}dxdt$$
$$=\int_Q\widetilde{f}(x,t)(T-t)v_{xxt}dxdt. \tag{3.9}$$

Using integration by parts and Young's inequality, we obtain the estimate

$$\frac{1}{2}\int_0^T(T-t)v_{xt}^2(l,t)dt+\frac{1}{2}\int_Q v_{xx}^2dxdt-\frac{1}{2A}(b_0-G_M(\Phi(v)))\int_Q v_x^2dxdt$$
$$+\frac{\varepsilon}{2}\int_Q(T-t)v_{xxt}^2dxdt\le\frac{1}{2\varepsilon}\int_Q(T-t)\widetilde{f}^2(x,t)dxdt. \tag{3.10}$$

Consider the equality

$$\int_Q(v_{xt}+v_{xx}+\frac{1}{A}(b_0-G_M(\Phi(v)))v+\varepsilon v_{xxt})v_{xt}dxdt=\int_Q\widetilde{f}(x,t)v_{xt}dxdt. \tag{3.11}$$

Once again using the technique connected with integration by parts, we infer that solutions $v(x,t)$ to the boundary value problem (3.8), (3.6), (3.7) satisfy the inequality

$$\int\limits_Q v_{xt}^2 dxdt + \varepsilon \int\limits_0^l v_{xx}^2(x,T)dx \leq 3\left(\frac{T}{\varepsilon} + \frac{Tl^4}{4A^2\varepsilon}(b_0 - M)^2 + 1\right)\int\limits_Q \widetilde{f}^2(x,t)dxdt \quad (3.12)$$

Relations (3.10) and (3.12) obviously imply the estimate

$$||v||_{V_0} \leq K_1 ||\widetilde{f}||_{L_2(Q)}, \quad (3.13)$$

where the constant K_1 is defined only by the input data and ε. This estimate implies that we can apply the technique connected with Schauder's Fixed Point Theorem similar to the technique used in proving Theorem 1. As a consequence, we have the solvability of Problem (3.8),(3.6), (3.7) for fixed ε. Now, we obtain estimates uniform in ε for $v(x,t)$.

Carry out integration by parts on the left and on the right in (3). We obtain the estimate

$$\frac{1}{2}\int\limits_0^T (T-t)v_{xt}^2(l,t)dt + \frac{1}{4}\int\limits_Q v_{xx}^2 dxdt - \frac{1}{2A}(b_0 - G_M(\Phi(v)))\int\limits_Q v_x^2 dxdt$$
$$+ \varepsilon \int\limits_Q (T-t)v_{xxt}^2 dxdt \leq \int\limits_Q f_1^2(x,t)dxdt, \quad (3.14)$$

Reckoning with (3.14), we again consider (3.11). Acting by analogy with what was done above, we infer

$$\int\limits_Q v_{xt}^2 dxdt + 2\varepsilon \int\limits_0^l v_{xx}^2(x,T)dx$$
$$\leq 3\left(\int\limits_Q \widetilde{f}^2(x,t)dxdt + (4 + \frac{l^4}{2A^2\varepsilon}(b_0 - M)^2)\int\limits_Q f_1^2(x,t)dxdt\right) \quad (3.15)$$

Relations (3.14) and (3.15) imply the estimates

$$\int\limits_Q v_{xx}^2 dxdt \leq 4\int\limits_Q f_1^2(x,t)dxdt, \quad \int\limits_Q v_x^2 dxdt \leq l^2 \int\limits_Q f_1^2(x,t)dxdt,$$

$$||v||_{V_0} \leq R_2 ||f_1||_{L_2(Q)} \quad (3.16)$$

where the constant R_2 is defined only by the input data and does not depend on ε. An estimate uniform in ε is quite enough for passage to the limit (see the proof of Theorem 1) — it makes it possible to choose a convergent sequence $\{v_{\varepsilon_n}(x,t)\}_{n=1}^{\infty}$ such that the limit function $v(x,t)$ satisfies the equation

$$v_{xt} + v_{xx} + \frac{1}{A}(b_0 - G_M(v))v = \widetilde{f}(x,t)$$

and conditions (3.6), (3.7).

The above-established estimates imply the inequality

$$|\phi(v)| \leq R_0.$$

By the choice of M, under the condition $R_0 < -b$ of the theorem, the inequality $|\phi(v)| < -b$ certainly holds. Thus, the function $v(x,t)$ is a solution to the auxiliary boundary value problem (3.5),(3.6),(3.7).

All subsequent arguments correspond to the arguments in the proof of Theorem 1. As a result, we obtain what was required. The theorem is proved. □

4. Appendices and Comments

4.1. Inverse Problems I and II complement each other to a certain extent: in one case, a positive coefficient a is searched for, and in the other case, a negative coefficient.

4.2. The article is devoted to the case of model equations (2.1) and (3.1). It is obvious that generally analogous results can be obtained also for more general equations — for example, for equations with variable coefficients, with additional lower terms. The corresponding conditions and calculations will become substantially more cumbersome but the essence of the results will not change.

4.3. It is not hard to construct examples of input data for which the hypotheses of Theorems 1 and 2 are fulfilled.

Suppose that $N(t) = \text{const} = 1$ and $f(x,t) = x^2 t$. All conditions of Theorem 1 are fulfilled.

Let $f(x,t) = t(x^2 - l^2)$, $N(t) = \text{const} = 1$. Then the obvious condition $-b_0 > R_0$ takes the form

$$\frac{T^2}{4} > \sqrt{\frac{2}{3}}(T^2 l + l^3)^{\frac{1}{2}}.$$

Thus, all conditions of Theorem 2 are fulfilled for sufficiently small l.

Of course, there are a lot of other examples.

References

1. Glazatov S.N. On solvability of a spatial periodic problem for the Lin–Reissner–Tsien equation of transonic gas dynamics. *Mathematical Notes*, 2010, vol. 87, iss. 1, pp. 130–134. https://doi.org/10.1134/S0001434610010177
2. Kabanikhin S.I. *Inverse and Ill-Posed Problems.*Novosibirsk, Sib. Knizh. Izdat. Publ., 2009. (in Russian)
3. Kozhanov A.I. Inverse Problems of Finding the Absorption Parameter in the Diffusion Equation. *Mathematical Notes,* 2019, vol. 106, iss. 3, pp. 378–389. https://doi.org/10.1134/S0001434619090074 (in Russian)
4. Kozhanov A.I. The Heat Transfer Equation with an Unknown Heat Capacity Coefficient. *Journal of Applied and Industrial Mathematics*, 2020, vol. 14, iss. 1, pp. 104–114. https://doi.org/10.1134/S1990478920010111 (in Russian)
5. Kozhanov A.I. On the solvability of the inverse problems of parameter recovery in elliptic equations. *Mathematical notes of NEFU*, 2020, vol. 27, iss. 4, pp. 14–29. https://doi.org/10.25587/SVFU.2020.57.53.002 (in Russian)
6. Kozhanov A.I., Safiullova R.R. Determination of parameters in telegraph equation. *Ufa Mathematical Journal*, 2017, vol. 9, iss. 1, pp. 62–74. https://doi.org/10.13108/2017-9-1-62 (in Russian)
7. Kozhanov A.I., Telesheva L.A. Inverse problems of restoring parameters in parabolic and hyperbolic equations. *Mathematical Notes of NEFU*, 2022, vol. 29, No. 3 , pp. 57–69. https://doi.org/10.25587/SVFU.2022.85.24.005 (in Russian)
8. Ladyzhenskaya O.A., Solonnikov V.A., Uraltseva N.N. *Linear and Quasilinear Equations of Parabolic Type*. Moscow, Nauka Publ., 1967. (in Russian)
9. Lyubanova A.Sh. Identificacion of a coefficient in the leading term of a pseudoparabolic equation of filtration. *Sib. Math. J.*, 2013 , vol. 54, no. 6, pp. 1046–1058. https://doi.org/10.1134
10. Mamontov E.V. Ob uravneniyakh malykh vozmushcheniy v nestatsionarnom okolozvukovom potoke gaza.[On the equations of small perturbations in a nonstationary transonic flow of gas]. Nestacionarnye problemy mekhaniki [Nonstationary problems of mechanics]. Novosibirsk, Hidrodynamics Institute SB USSR Academy of Sciences, 1978, no. 37, pp. 1390–143. (in Russian)
11. Nakhushev A.M. *Nagruzhennye uravneniya i ikh primenenie*. Moscow, Nauka Publ., 2012, 232 p. (in Russian)
12. Romanov V.G.*Stability in Inverse Problems*. Moscow, Nauchny Mir Publ., 2005. (in Russian)
13. Sobolev S.L.*Some Applications of Functional Analysis in Mathematical Physics*. Moscow, Nauka Publ., 1973. (in Russian)
14. Trnogin V.A.*Finctional Analysis*. Moscow, Fizmatlit Publ., 2002. (in Russian)
15. Anikonov Yu.E. *Inverse problems for kinetic and other evolution equations.* Utrecht, VSP, 2001. https://doi.org/10.1515/9783110940909
16. Belov Yu.Ya. *Inverse problems for Partial Differential equations.* Utrecht, VSP, 2002. https://doi.org/10.1515/9783110944631
17. Isakov V. *Inverse Problems for Partial Differential Equations.* Springer Science, 2006. http://dx.doi.org/10.1007/0-387-32183-7
18. Ivanchov M. Inverse problems for equations of parabolic type. *Mathematical studies. Monograph series. Vol. 10.* https://doi.org/10.1201/9781482292985-6
19. Kozhanov A.I. *Composite type equations and inverse problems.* Utrecht, VSP, 1999. https://doi.org/10.1515/9783110943276
20. Kozhanov A.I. Hyperbolic Equations with Unknown Coefficients. *Symmetry*, 2020, vol. 12, iss. 9, pp. 1539. https://doi.org/10.3390/sym12091539.

21. Kozhanov A.I., Abylkayrov U.U., Ashurova G.R. Inverse problems of parameter recovery in differential equations with multiple characteristics. *Journal of Mathematics, Mechanics and Computer Science*, 2022. vol. 113, no. 1, pp. 3-16. https://doi.org/10.26577/JMMCS.2022.v113.i1.01
22. Lin C.C., Reissner E., Tsien H.S. On two-dimensional non-steady motion of a slender body in a compressible fluid. *Journal of Mathematical Physics*, 1948, vol. 27, no. 3, pp. 220–231. http://dx.doi.org/10.1016/B978-0-12-398277-3.50028-2
23. Lorenzi A. *An Introduction to Mathematikal Problems via Functional Analysis.* Utrecht, VSP, 2001. https://doi.org/10.1515/9783110940923
24. Lorenzi A., Mola G. Identification of real constant in linear evolution equation in a Hilbert spaces. *Inverse Problems Imaging*, 2011, vol. 5, no. 3, pp. 695-714. https://doi.org/10.3934/ipi.2011.5.695
25. Lorenzi A., Recovering two constants in a linear parabolic equaton. *Inverse problem in applied sciences*, 2007, vol. 73, pp. 1-15. https://doi.org/10.1088/1742-6596/73/1/012014
26. Lorenzi A., Mola G. Recovering the reaction and the diffusion coefficients in a linear parabolic equation. *Inverse Problems*, 2012, vol. 28, no. 7. http://dx.doi.org/10.1088/0266-5611/28/7/075006
27. Lyubanova A.Sh. Identification of a constant coefficient in an elliptic equation. *Applicable Analysis*, 2008, vol. 87, no. 10-11, pp. 1121-1128. https://doi.org/10.1080/00036810802189654
28. Lyubanova A.Sh., Velisevich A.V. An inverse problem for a quasilinear elliptic equation. *Journal of Mathematical Sciences*, 2023, vol. 270. no. 4. https://doi.org/10.1007/s10958-023-06370-9
29. Lyubanova A.Sh. Identificacion of a coefficient in the leading term of a pseudoparabolic equation of filtration. *Siberian Mathematical Journal*, 2013, vol. 54, no. 6, pp. 1046–1058. https://doi.org/10.1134
30. Mola G. Identification of the Diffussion Coefficient in Linear Evolution Equations in Hilbert Spaces. *J. Abstr. Diff. Equat. Appl.*, 2011, vol. 2, pp. 18–28. http://dx.doi.org/10.3934/ipi.2011.5.695
31. Mola G., Okazawa N., Yokota T. Reconstruction of two constant coefficients in linear anisotropic diffusion model. *Inverse Problems*, 2016 , vol. 32, no. 11. http://dx.doi.org/10.1088/0266-5611/32/11/115016
32. Prilepko A.I., Orlovsky D.C., Vasin I.A. .*Methods for solving inverse problems in mathimatical physics.* New York, Dekker, 1999. https://doi.org/10.1201/9781482292985
33. Triebel H. *Interpolation Theory, Function Spaces, Differential Operators.* Amsterdam, Noth-Holland Publ., 1978. https://doi.org/10.1002/zamm.19790591227
34. Velisevich A.V. On an Inverse Problem for the Stationary Equation with a Boundary Condition of the Third Kind. *Journal of Siberian Federal University. Mathematics & Physics*, 2021, vol. 14, no.5, pp. 659–666. https://doi.org/10.17516/1997-1397-2021-14-5-659-666

Список источников

1. Глазатов С. Н. О разрешимости пространственно- периодической задачи для уравнения Линя – Рейснера – Цзяня трансзвуковой динамики // Математические заметки. 2010. Т. 87, вып. 1. С. 137–140.
2. Кабанихин С. И. Обратные и некорректные задачи // Новосибирск : Сиб. кн. изд-во, 2009.

3. Кожанов А.И. Обратные задачи определения параметра поглощения в уравнении диффузии // Математические заметки. 2019. Т. 106, вып. 3. С. 395–408.
4. Кожанов А. И. Уравнение теплопроводности с неизвестным коэффициентом теплоемкости // Сибирский журнал индустриальной математики. 2020. Т. 23, вып. 1. С. 93–106.
5. Кожанов А.И. О разрешимости обратных задач восстановления параметров в эллиптических уравнениях// Математические заметки СВФУ. 2020. Т. 27, bып.4. С. 14–29.
6. Кожанов А. И., Сафиуллова Р. Р. Определение параметров в телеграфном уравнении // Уфимский математический журнал. 2017. Т. 9, № 1. С. 63–74.
7. Кожанов А. И., Телешева Л. А. Обратные задачи восстановления параметров в параболическом и гиперболическом уравнениях // Математические заметки СВФУ. 2022. Т. 29, № 3. С. 57–69.
8. Ладыженская О. А., Солонников В. А., Уральцева Н. Н. Линейные и квазилинейные уравнения параболического типа. М. : Наука, 1967. 736 с.
9. Любанова А. Ш. Идентификация коэффициента в старшем члене псевдопараболического уравнения типа фильтрации // Сибирский журнал индустриальной математики. 2013. Т. 54, № 6. С. 1315–1330.
10. Мамонтов Е. В. Об уравнениях малых возмущений в нестационарном околозвуковом потоке газа // Нестационарные проблемы механики : сб. науч. тр. Новосибирск : Сиб. отд-ние АН СССР, Ин-т гидродинамики, 1978. № 37. С. 139–143.
11. Нахушев А. М. Нагруженные уравнения и их применения // М. : Наука, 2012. 232с.
12. Романов В. Г. Устойчивость в обратных задачах. М. : Науч. мир, 2005.
13. Соболев С. Л. Некоторые применения функционального анализа в математической физике // М.:Наука, 1973.
14. Треногин В. А. Функциональный анализ. М. : Физмалит, 2002. 488 с.
15. Anikonov Yu. E. Inverse problems for kinetic and other evolution equations. Utrecht : VSP, 2001. 275 p.
16. Belov Yu. Ya. Inverse problems for Partial Differential equations. Utrecht : VSP, 2002. 211 p.
17. Isakov V. Inverse Problems for Partial Differential Equations. Springer Science, 2006. 343 p.
18. Ivanchov M. Inverse problems for equations of parabolic type // Mathematical studies. Monograph series. Vol. 10.
19. Kozhanov A. I. Composite type equations and inverse problems. Utrecht : VSP, 1999.
20. Kozhanov A. I. Hyperbolic Equations with Unknown Coefficients // Symmetry. 2020. Vol. 12, Iss. 9. 1539. doi.org/10.3390/sym12091539.
21. Kozhanov A. I.,Abylkayrov U. U., Ashurova G. R. Inverse problems of parameter recovery in differential equations with multiple characteristics // Journal of Mathematics, Mechanics and Computer Science. 2022. Vol. 113, N 1. P. 3-16.
22. Lin C. C., Reissner E., Tsien H. S. On two-dimensional non-steady motion of a slender body in a compressible fluid // Journal of Mathematical Physics. 1948. Vol. 27, N 3. P. 220–231.
23. Lorenzi A. An Introduction to Mathematical Problems via Functional Analysis. Utrecht : VSP, 2001.
24. Lorenzi A., Mola G. Identification of real constant in linear evolution equation in a Hilbert spaces // Inverse Problems Imaging. 2011. Vol. 5, N 3. P. 695-714.
25. Lorenzi A., Recovering two constants in a linear parabolic equatin// Inverse problem in applied sciences. 2007. Vol. 73. P. 1-15.

26. Lorenzi A., Mola G. Recovering the reaction and the diffusion coefficients in a linear parabolic equation // Inverse Problems. 2012. Vol. 28, N 7.
27. Lyubanova A. Sh. Identification of a constant coefficient in an elliptic equation// Applicable Analysis. 2008. Vol. 87, N 10-11. P. 1121-1128.
28. Lyubanova A. Sh., Velisevich A. V. An inverse problem for a quasilinear elliptic equation // Journal of Mathematical Sciences. 2023. Vol. 270, N 4.
29. Lyubanova A. Sh. Identification of a coefficient in the leading term of a pseudoparabolic equation of filtration// Siberian Mathematical Journal. 2013. Vol. 54, N 6. P. 1046–1058.
30. Mola G. Identification of the Diffusion Coefficient in Linear Evolution Equations in Hilbert Spaces // J. Abstr. Diff. Equat. Appl. 2011. Vol. 2. P. 18-28.
31. Mola G., Okazawa N., Yokota T. Reconstruction of two constant coefficients in linear anisotropic diffusion model // Inverse Problems. 2016. Vol. 32.
32. Prilepko A. I., Orlovsky D. C., Vasin I. A. Methods for solving inverse problems in mathematical physics. New York : Dekker, 1999.
33. Triebel H. Interpolation Theory, Function Spaces, Differential Operators. Amsterdam : Noth-Holland Publ., 1978.
34. Velisevich A. V. On an Inverse Problem for the Stationary Equation with a Boundary Condition of the Third Kind // Journal of Siberian Federal University. Series: Mathematics. Physics. 2021. Vol. 14, Iss. 5. P. 659–666.

Об авторах

Кожанов Александр Иванович, д-р физ.-мат. наук, проф., Институт математики им. С. Л. Соболева, Новосибирск, 630090, Российская Федерация, kozhanov@math.nsc.ru

Телешева Любовь Александровна, канд. физ.-мат. наук, Бурятский государственный университет им. Доржи Банзарова, Улан-Удэ, 670000, Российская Федерация, love_20_09@mail.ru

About the authors

Aleksandr I. Kozhanov, Dr. Sci. (Phys.-Math.), Prof., Sobolev Institute of Mathematics SB RAS, 630090, Novosibirsk, Russian Federation, kozhanov@math.nsc.ru

Lyubov A. Telesheva, Cand. Sci. (Phys.Math.), Assoc. Prof., Banzarov Buryat State University, Ulan-Ude, 670000, Russian Federation,

Поступила в редакцию / Received 17.09.2024
Поступила после рецензирования / Revised 28.01.2025
Принята к публикации / Accepted 30.01.2025