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Singularities of Discriminant Loci of Laurent Polynomial Systems

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Abstract. We consider a system of n Laurent polynomials in n unknowns with variable complex coefficients. For the reduced discriminant locus of such a system, we study the set of critical points of the Horn–Kapranov parametrization. In a special instance ($n = 3$), the set of critical values of the parametrization is investigated. It is proved that the multiple root of the corresponding system is degenerate.

Keywords: Laurent polynomial, discriminant locus, $\mathcal{A}$–discriminant, mixed discriminant, degenerate multiple root

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Научная статья

Особенности дискриминантных множеств систем полиномов Лорана

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Аннотация. Рассматривается система n полиномов Лорана от n неизвестных с переменными комплексными коэффициентами. Для приведённого дискриминантного множества системы найдено множество критических точек параметризации Горна –

Капранова. В специальном случае ($n = 3$) исследовано множество критических значений параметризации и доказано, что кратный корень соответствующей системы является вырожденным.

Ключевые слова: полином Лорана, дискриминантное множество, $\mathcal{A}$–дискриминант, смешанный дискриминант, вырожденный кратный корень

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1. Introduction

Key objects of investigation in the theory by I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, which is called to be the $\mathcal{A}$–philosophy, are $\mathcal{A}$–discriminants and $\mathcal{A}$–hypergeometric functions [5]. Their approach, based on ideas of the toric geometry, has been reflected in the multidimensional hypergeometric theory (see [11]) and in the quantum field theory (see [4] and [8]). Following the $\mathcal{A}$-philosophy, we consider the system of Laurent polynomials

$$f_i(y) := \sum_{\lambda \in \mathcal{A}^{(i)}} a_\lambda^{(i)} y^\lambda = 0, \qquad i = 1, \dots, n, \tag{1.1}$$

where coefficients $a_\lambda^{(i)}$ vary in the vector space $\mathbb{C}_a^N$, sets of exponents $\mathcal{A}^{(i)} \subset \mathbb{Z}^n$ are fixed and generate the lattice as an additive group. Solutions $y := (y_1, \dots, y_n)$ are assumed to be found in the complex algebraic torus $(\mathbb{C} \setminus 0)^n$, so without loss of generality we assume that all sets $\mathcal{A}^{(i)}$ contain the zero element $\overline{0}$. We identify $\mathcal{A}^{(i)}$ with sets of monomials $y^\lambda := y_1^{\lambda_1} \cdot \ldots \cdot y_n^{\lambda_n}$, $\lambda \in \mathcal{A}^{(i)}$.

We denote by ∇^0 the set of all coefficients for which the mapping $f = (f_1, \dots, f_n) \colon (\mathbb{C} \setminus 0)^n \to \mathbb{C}^n$ associated with the system (1.1) has multiple zeros, that is, zeros where the Jacobian of f vanishes. *The discriminant locus* ∇ of the mapping f is defined to be the closure of the set ∇^0 in the space of coefficients $\mathbb{C}_a^N$. The set ∇ is appropriately called the $(\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(n)})$–discriminant locus, by analogy with the $\mathcal{A}$–discriminant locus considered in [5].

By virtue of the polyhomogeneity property of the algebraic vector-function $y(a) := (y_1(a), \dots, y_n(a))$ the system (1.1) admits a dehomogenization resulting in the reduced system and the corresponding *reduced discriminant locus* ∇'. A universal parametrization defining the irreducible components of the reduced discriminant locus is comprehensively studied in [2]. It is proved that if ∇' is an irreducible hypersurface depending on coefficients of all equations of the system, then the parametrization is the inversion of the logarithmic Gauss mapping for ∇'. We note that the Gauss mapping for ∇' is not always birational, so $(\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(n)})$–discriminant loci can not generally be reduced to $\mathcal{A}$–discriminant loci.

If the $\mathcal{A}$-discriminant locus is a hypersurface, then the unique (up to sign) irreducible polynomial with integer coefficients defining it is said to be the $\mathcal{A}$-*discriminant* [5]. The discriminant of the (1.1) in the context of the theory of $\mathcal{A}$-discriminants is studied in [3]. The authors call it the mixed discriminant and prove its equivalence to the $\mathcal{A}$-discriminant of the Cayley configuration $\mathcal{A} = Cay(\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(n)})$. The discriminant locus in [3] is defined to be the closure of the locus of coefficients for which the system (1.1) has a *non-degenerate multiple root.* In this case, an isolated solution $u \in (\mathbb{C} \setminus 0)^n$ is a *non-degenerate multiple root* if n gradient vectors $\nabla_y f_i(u)$ of polynomials of the system are linearly dependent, but any $n-1$ of them are linearly independent. This means that u is a regular point of the curve defined by any set of $(n-1)$ equations of the system (1.1).

Our objects of research are Laurent polynomial systems which have degenerate multiple roots. The specified class of systems is defined in the space of coefficients as a set of critical values of the parameterization for the discriminant hypersurface. In case of one algebraic equation, singular strata of the reduced discriminant locus responsible for the presence of roots of a certain multiplicity were studied in [9]. It turns out that they coincide with critical strata of the Horn-Kapranov parametrization, which, in turn, are restrictions of the parametrization onto a chain of embedded linear subspaces of the projective space (see also [10]).

The plan of the paper is as follows. In Section 2, we briefly characterize parameterizations for the reduced discriminant locus (Theorem 1) and multiple roots (Theorem 2). Our main result is stated and proved in Section 3. It contains a description of the set of critical points for the parametrization of the reduced discriminant hypersurface (Theorem 3). In Section 4, we deal with a special case ($n = 3$) and distinguish a class of polynomial systems with degenerate multiple roots.

2. Horn-Kapranov Parametrization

The system (1.1) admits a monomial (with rational exponents) transformation $G\colon a \mapsto x$ of coefficients. As a result, in each equation two coefficients are fixed, and the rest are variable. Polynomial supports $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(n)}$ remain unchanged. We recall that the support of a polynomial is defined to be the set of exponents of its monomials with nonzero coefficients.

We fix an element $\omega^{(i)}$ in each set $\mathcal{A}^{(i)}$ and form a matrix

$$\omega := (\omega^{(1)}, \dots, \omega^{(n)})$$

with columns $\omega^{(i)}$. The matrix ω is assumed to be non-degenerate. As a result of the transformation, we get *the reduced (dehomogenized) system* of the form

$$g_i(y) := y^{\omega^{(i)}} + \sum_{\lambda \in \Lambda^{(i)}} x_\lambda^{(i)} y^\lambda - 1 = 0, \qquad i = 1, \dots, n, \tag{2.1}$$

with new unknowns $y := (y_1, \dots, y_n)$, variable complex coefficients $x := (x_\lambda^{(i)})$ and $\Lambda^{(i)} := \mathcal{A}^{(i)} \setminus \{\omega^{(i)}, \bar{0}\}$. The dehomogenization procedure is based on the polyhomogeneity property, which can be expressed by the formula:

$$y(\dots r^{(i)} l^\lambda a_\lambda^{(i)} \dots) = (l_1^{-1} y_1(\dots a_\lambda^{(i)} \dots), \dots, l_n^{-1} y_n(\dots a_\lambda^{(i)} \dots)),$$

where $r := (r^{(1)}, \dots, r^{(n)})$, $l := (l_1, \dots, l_n) \in (\mathbb{C} \setminus 0)^n$. If $\{D_j(x) = 0\}$ is a system of equations for the discriminant locus of the system (2.1), then the discriminant locus of the system (1.1) is defined by equations $\{D_j\,(G(a)) = 0\}$. The solution of the system (1.1) can be recovered from the solution of the reduced system. A detailed description of the dehomogenization procedure is given in [2].

Next, we consider the parametrization of the discriminant locus of the system (2.1), see [2]. We call it *the reduced discriminant locus* and denote by ∇'. Let Λ be the disjoint union of sets $\Lambda^{(i)}$ and N' the cardinality of Λ. The set of coefficients of the system (2.1) is the vector space $\mathbb{C}^\Lambda \cong \mathbb{C}_x^{N'}$, where the coordinates of points $x = (x_\lambda)$ are indexed by elements $\lambda \in \Lambda$. We interpret the set Λ as a matrix $\Lambda = \left(\Lambda^{(1)} | \dots | \Lambda^{(n)}\right)$.

Let us define a multivalued algebraic mapping

$$h \colon \mathbb{CP}_s^{N'-1} \to \mathbb{C}_x^{N'},$$

by putting

$$x_\lambda^{(i)} = -\frac{s_\lambda^{(i)}}{\langle \tilde{\varphi}_i, s \rangle} \prod_{k=1}^{n} \left(\frac{\langle \tilde{\varphi}_k, s \rangle}{\langle \varphi_k, s \rangle} \right)^{\varphi_{k\lambda}}, \; \lambda \in \Lambda^{(i)}, \; i = 1, \dots, n, \tag{2.2}$$

where $\varphi_k, \tilde{\varphi}_k$ are the rows of matrices $\Phi := \omega^{-1}\Lambda$ and $\tilde{\Phi} := \Phi - \chi$ respectively, χ is a block matrix consisting of 0 and 1, the i-th row of which represents the indicator of the subset $\Lambda^{(i)} \subset \Lambda$, and $\varphi_{k\lambda}$ is the coordinate indexed by $\lambda \in \Lambda^{(i)} \subset \Lambda$ in the row φ_k.

Following [2], we assume that the discriminant locus ∇' depends on all variable coefficient groups $x^{(i)}$, that is, it can not be factorized in the form $\tilde{\nabla}' \times \mathbb{C}^{\Lambda^{(i)}}$, where $\tilde{\nabla}'$ is an algebraic subset in $\mathbb{C}^{\Lambda^{(1)}} \times \dots [i] \dots \times \mathbb{C}^{\Lambda^{(n)}}$.

We say that a system of k Laurent polynomials $g_i(y_1, \dots, y_n)$ satisfies the condition $(*)$ if the set of all exponents of the system does not lie in a k-dimensional subspace.

Theorem 1. [2] *If each subsystem of the system (2.1) satisfies the condition* $(*)$ *(in particular, if Newton polytopes of all equations in (2.1) are* n*-dimensional), then the discriminant locus* ∇' *is parametrized by the mapping (2.2).*

Recall that the Newton polytope of a Laurent polynomial is defined to be the convex hull of its support in $\mathbb{R}^n$. The full dimension of the Newton polytope of each equation in the (2.1) means that exponents of its monomials do not lie in a hyperplane. The fulfillment of this condition automatically implies the condition $(*)$ for all subsystems of the system (2.1).

Next, we need the notion of the *logarithmic Gauss mapping* of a hypersurface $\nabla' = \{D(x) = 0\} \subset (\mathbb{C} \setminus 0)^{N'}$. This is the mapping $\gamma\colon \nabla'_{reg} \to \mathbb{CP}^{N'-1}$ defined in coordinates $(x_1, \ldots, x_{N'}) \in (\mathbb{C} \setminus 0)^{N'}$ by the formula

$$(x_1, \ldots, x_{N'}) \mapsto \left(x_1 D'_{x_1} \colon \ \ldots : x_{N'} D'_{x_{N'}}\right).$$

Here ∇'_{reg} is the set of regular points of ∇'.

Theorem 2. [2] *If the discriminant locus* ∇' *of the system (2.1) is an irreducible hypersurface depending on all groups of variables, then the parametrization (2.2) is the inversion of the logarithmic Gauss mapping:* $h(s) = \gamma^{-1}(s)$.

Moreover, if matrices Φ and $\tilde{\Phi}$ do not contain zero entries, then the set ∇' is a hypersurface, see Propositon 2 in [2].

Theorem 2 is an analogue of Kapranov's theorem in [7], where he proved that the Horn uniformization parametrizes the $\mathcal{A}$-discriminantal hypersurface being the inversion of the logarithmic Gauss mapping. Thus, the parameterization of ∇' in the form $\gamma^{-1}(s)$ is appropriately said to be the *Horn-Kapranov parametrization.* For the classical discriminant the analogous fact is proven in [10]. We note that in our case the Gauss mapping for $\nabla^{'}$ is not always birational (see Example 3 in [2]).

Let the set ∇' be a hypersurface and the defining polynomial $D(x)$ depends on all groups of variable coefficients. We consider $x \in \nabla'_{reg}$ and a mapping $\tau(x) = (\tau_1(x), \ldots, \tau_n(x))$ with components

$$\tau_j(x) = \prod_{i=1}^{n} \left(\frac{\langle \varphi_i, \gamma_{\nabla'}(x) \rangle}{\langle \tilde{\varphi}_i, \gamma_{\nabla'}(x) \rangle} \right)^{\varepsilon_i^{(j)}}, \ j = 1, \ldots, n, \tag{2.3}$$

where $\varepsilon^{(j)}$ are columns of the matrix ω^{-1}. According to Lemma 5 in [1], the values of the mapping $\tau(x)$ are multiple roots of the system (2.1).

3. The Critical Set of the Parametrization

Let us study critical points of the mapping (2.2) for a system of the form

$$y^{\omega^{(i)}} + x^{(i)} y^{\lambda} - 1 = 0, \quad i = 1, \dots, n, \tag{3.1}$$

where the number of variable coefficients $x^{(i)}$ equals the number of equations. The matrix ω with columns $\omega^{(i)}$ is assumed to be non-degenerate and the system (3.1) to satisfy the condition $(*)$. The monomial substitution $y = t^{\omega^{-1}}$ transforms it as follows

$$t_i + x^{(i)} t^{\varphi^{(i)}} - 1 = 0, \quad i = 1, \dots, n, \tag{3.2}$$

where exponents $\varphi^{(i)}$ are columns of the matrix Φ.

Let us start with some notations:

$$\varphi_i[i] := (\varphi_i^{(1)}, \dots, \varphi_i^{(i-1)}, \varphi_i^{(i+1)}, \dots, \varphi_i^{(n)}),$$

$$s[i] := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n),$$

$$\langle \varphi_i[i], s[i] \rangle := \sum_{j \neq i} \varphi_i^{(j)} s_j.$$

Here and throughout the text, the symbol $[i]$ denotes the omission of the i'th element.

Consider the matrix

$$M^{(n)}(s) := \begin{pmatrix} \langle \varphi_1[1], s[1] \rangle & -\varphi_1^{(2)} s_1 & \cdots & -\varphi_1^{(n)} s_1 \\ -\varphi_2^{(1)} s_2 & \langle \varphi_2[2], s[2] \rangle & \cdots & -\varphi_2^{(n)} s_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_n^{(1)} s_n & -\varphi_n^{(2)} s_n & \cdots & \langle \varphi_n[n], s[n] \rangle \end{pmatrix}. \tag{3.3}$$

Lemma 1. *The minor $M_{j,j}^{(n)}(s)$ of the matrix $M^{(n)}(s)$ obtained by excluding the j'th row and the j'th column admits the factorization:*

$$M_{j,j}^{(n)}(s) = s_j \cdot l_j(s),$$

here $l_j(s)$ is a homogeneous polynomial of the degree $(n-2)$ of variables $s = (s_1, \dots, s_n)$.

Proof. We first note that $\det M^{(n)}(s) \equiv 0$. Indeed, the transposition of a matrix does not change its determinant, so we write it as follows

$$\det M^{(n)}(s) = \begin{vmatrix} \langle \varphi_1[1], s[1] \rangle & -\varphi_1^{(2)} s_2 & \cdots & -\varphi_1^{(n)} s_n \\ -\varphi_2^{(1)} s_1 & \langle \varphi_2[2], s[2] \rangle & \cdots & -\varphi_2^{(n)} s_n \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_n^{(1)} s_1 & -\varphi_n^{(2)} s_2 & \cdots & \langle \varphi_n[n], s[n] \rangle \end{vmatrix}.$$

Here we see that columns are linearly dependent, hence the determinant vanishes.

Next, we prove the statement of the lemma for the minor

$$M_{n,n}^{(n)}(s) := \begin{vmatrix} \langle \varphi_1[1], s[1] \rangle & -\varphi_1^{(2)} s_1 & \cdots & -\varphi_1^{(n-1)} s_1 \\ -\varphi_2^{(1)} s_2 & \langle \varphi_2[2], s[2] \rangle & \cdots & -\varphi_2^{(n-1)} s_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_{n-1}^{(1)} s_{n-1} & -\varphi_{n-1}^{(2)} s_{n-1} & \cdots & \langle \varphi_{n-1}[n-1], s[n-1] \rangle \end{vmatrix}. \quad (3.4)$$

Let us decompose the (3.4) into a sum of determinants using the following representations of its diagonal entries:

$$\langle \varphi_i[i], s[i] \rangle = \langle \varphi_i[i,n], s[i,n] \rangle + \varphi_i^{(n)} s_n, \quad i = 1, \ldots, n-1.$$

As a result, we get

$$M_{n,n}^{(n)}(s) = \det M^{(n-1)}(s) + s_n l_n(s),$$

where $l_n(s)$ is a homogeneous polynomial of the degree $(n-2)$. The first summand vanishes, so we obtain $M_{n,n}^{(n)}(s) = s_n l_n(s)$. For minors $M_{j,j}^{(n)}(s)$, $j = 1, \ldots, n-1$, the proof is similar. □

Next, we use homogeneous polynomials $l_j(s), j = 1, \ldots, n$, which are defined as follows

$$l_j(s) = \frac{1}{s_j} M_{j,j}^{(n)}(s). \quad (3.5)$$

Theorem 3. *The critical set of the parameterization (2.2) of the discriminant hypersurface ∇' for the system (3.2) is given in the form:*

$$\mathcal{L} = \{l_1(s) = \ldots = l_n(s) = 0\} \subset \mathbb{CP}^{n-1}, \quad (3.6)$$

where $l_1(s), \ldots, l_n(s)$ are defined by formulae (3.5).

Proof. We consider a homogeneous mapping $h\colon \mathbb{CP}_s^{n-1} \to \mathbb{C}_x^n$ with coordinates $x^{(i)}(s)$ defined on $\mathbb{C}^n \setminus \{0\}$ and prove that the rank of the Jacobi matrix $(\partial x^{(i)} / \partial s_j)$ drops below $(n-1)$ on the set $\mathcal{L}$. It is equivalent to study the rank property of the matrix

$$J := (\partial \ln x^{(i)} / \partial s_j), \; i, j = 1, \ldots, n.$$

Note that due to the homogeneity of the mapping $\ln x^{(i)}(s)$, the Euler identity holds:

$$\sum_{j=1}^{n} s_j \frac{\partial \ln x^{(i)}}{\partial s_j} \equiv 0,$$

whence it follows that

$$\det J = 0.$$

It remains to show that the set of common zeros of all $(n-1)$-minors of the matrix J is the set $\mathcal{L}$.

Let us consider differential forms

$$\varkappa^{(i)} := d\ln x^{(i)} = \tfrac{Q_i(s)ds_i - s_i dQ_i(s)}{s_iQ_i(s)} + \\ + \sum_{j=1}^{n} \varphi_j^{(i)} \frac{P_j(s)dQ_j(s) - Q_j(s)dP_j(s)}{P_j(s)Q_j(s)},\ i=1,\dots,n,$$

where $P_i(s)$ and $Q_i(s)$ denote linear forms $\langle\varphi_i, s\rangle$ and $\langle\tilde{\varphi}_i, s\rangle$ respectively.

We note that

$$P_j(s)dQ_j(s) - Q_j(s)dP_j(s) = s_j dP_j(s) - P_j(s)ds_j,$$

and

$$Q_i(s)ds_i - s_i dQ_i(s) = P_i(s)ds_i - s_i dP_i(s),$$

so $\varkappa^{(i)}$ can be written as follows

$$\varkappa^{(i)} = -\frac{\langle\varphi_i[i], s[i]\rangle}{s_iP_i(s)Q_i(s)} A_i + \sum_{j\neq i} \frac{\varphi_j^{(i)}}{P_j(s)Q_j(s)} A_j,$$

where

$$A_i := s_i dP_i(s) - P_i(s)ds_i.$$

Let us fix $i \in \{1,\dots,n\}$. The minor $J_{i,k}$ of the matrix J obtained by excluding the i'th row and the k'th column is equal to the coefficient at $ds_1 \wedge \dots [k] \dots \wedge ds_n$ in the form

$$\varkappa[i] := \varkappa^{(1)} \wedge \dots [i] \dots \wedge \varkappa^{(n)}.$$

The form $\varkappa[i]$ admits the following representation

$$\varkappa[i] = \sum_j \tilde{M}_{i,j}^{(n)}(s) A[j],$$

where $A[j] := A_1 \wedge \dots [j] \dots \wedge A_n$, and $\tilde{M}_{i,j}^{(n)}(s)$ is a minor of the matrix

$$\tilde{M}^{(n)}(s) = \begin{pmatrix} -\dfrac{\langle\varphi_1[1], s[1]\rangle}{s_1P_1(s)Q_1(s)} & \dfrac{\varphi_2^{(1)}}{P_2(s)Q_2(s)} & \cdots & \dfrac{\varphi_n^{(1)}}{P_n(s)Q_n(s)} \\ \dfrac{\varphi_1^{(2)}}{P_1(s)Q_1(s)} & -\dfrac{\langle\varphi_2[2], s[2]\rangle}{s_2P_2(s)Q_2(s)} & \cdots & \dfrac{\varphi_n^{(2)}}{P_n(s)Q_n(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\varphi_1^{(n)}}{P_1(s)Q_1(s)} & \dfrac{\varphi_2^{(n)}}{P_2(s)Q_2(s)} & \cdots & -\dfrac{\langle\varphi_n[n], s[n]\rangle}{s_nP_n(s)Q_n(s)} \end{pmatrix}$$

An easy computation now leads to the following expression:

$$A[j] = A_1 \wedge \ldots [j] \ldots \wedge A_n = (-1)^j l_j(s) \sum_{k=1}^{n} (-1)^k s_k \, ds[k].$$

Thus we get

$$\varkappa[i] = \left(\sum_{j=1}^{n} (-1)^j l_j(s) \tilde{M}_{i,j}^{(n)}(s) \right) \sum_{k=1}^{n} (-1)^k s_k \, ds[k].$$

It remains to do the following calculations:

$$\tilde{M}_{i,j}^{(n)} = \frac{1}{\prod\limits_{k \neq j} s_k P_k(s) Q_k(s)} \begin{vmatrix} -\langle \varphi_1[1], s[1] \rangle & \ldots & [j] & \ldots & \varphi_n^{(1)} s_n \\ \vdots & & \vdots & & \vdots \\ [i] & \ldots & & \ldots & [i] \\ \vdots & & \vdots & & \vdots \\ \varphi_1^{(n)} s_1 & \ldots & [j] & \ldots & -\langle \varphi_n[n], s[n] \rangle \end{vmatrix} =$$

$$= \frac{(-1)^{i+j+n-1} s_i l_j(s)}{\prod\limits_{k \neq j} s_k P_k(s) Q_k(s)}.$$

We end up the proof with the representation:

$$\varkappa[i] = (-1)^{n+i} s_i \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{j=1}^{n} \frac{s_j l_j^2(s)}{\prod\limits_{m \neq j} P_m(s) Q_m(s)} \right) \frac{ds_1}{s_1} \wedge \ldots [k] \ldots \wedge \frac{ds_n}{s_n}. \quad (3.7)$$

According to (3.7), we conclude that the set of common zeros of coefficients of the form $\varkappa[i]$ is the set (3.6). □

4. Degenerate Multiple Roots ($n = 3$)

In this section, we deal with the special instance of (3.2). It is the system of the form

$$t_i + x^{(i)} t^{\varphi^{(i)}} - 1 = 0, \quad i = 1, 2, 3, \quad (4.1)$$

with unknowns $t = (t_1, t_2, t_3)$ and variable coefficients $x = (x^{(1)}, x^{(2)}, x^{(3)})$. We assume that matrices Φ and $\tilde{\Phi}$ do not contain zero entries. According to Theorem 3, the critical set of the parametrization for the discriminant hypersurface ∇' of the system (4.1) is given by a system of linear equations:

$$\begin{cases} l_1(s) = \varphi_2^{(1)}\varphi_3^{(1)}s_1 + \varphi_2^{(1)}\varphi_3^{(2)}s_2 + \varphi_2^{(3)}\varphi_3^{(1)}s_3 = 0, \\ l_2(s) = \varphi_1^{(2)}\varphi_3^{(1)}s_1 + \varphi_1^{(2)}\varphi_3^{(2)}s_2 + \varphi_1^{(3)}\varphi_3^{(2)}s_3 = 0, \\ l_3(s) = \varphi_1^{(3)}\varphi_2^{(1)}s_1 + \varphi_1^{(2)}\varphi_2^{(3)}s_2 + \varphi_1^{(3)}\varphi_2^{(3)}s_3 = 0, \end{cases} \tag{4.2}$$

where $s = (s_1 \colon s_2 \colon s_3)$ are homogeneous coordinates in $\mathbb{CP}^2$. The system (4.2) has a nontrivial solution if and only if, the condition

$$\varphi_1^{(2)}\varphi_2^{(3)}\varphi_3^{(1)} = \varphi_1^{(3)}\varphi_2^{(1)}\varphi_3^{(2)} \tag{4.3}$$

holds. In this case, the rank of the matrix of (4.2) equals one.

Let $\mathcal{C} \subset \nabla' \subset \mathbb{C}^3_x$ be the subset of critical values of the parameterization (2.2) for the system (4.1). It is defined by the restriction of the mapping $x = x(s)$ to the set $\mathcal{L} \subset \mathbb{CP}^2$ given by (4.2). This is made precise in the following proposition.

Proposition 1. *Suppose that exponents $\varphi^{(i)}$ of the system (4.1) satisfy the condition (4.3).*

1. *The set $\mathcal{C}$ consists of a single point $x_0 = (x_0^{(1)}, x_0^{(2)}, x_0^{(3)})$ with coordinates*

$$x_0^{(1)} = -\frac{\varphi_2^{(3)}u^{\varphi^{(1)}}}{\tilde{\Phi}_{[3,2]}}, \quad x_0^{(2)} = \frac{\varphi_3^{(1)}u^{\varphi^{(2)}}}{\tilde{\Phi}_{[1,3]}}, \quad x_0^{(3)} = -\frac{\varphi_1^{(2)}u^{\varphi^{(3)}}}{\tilde{\Phi}_{[2,1]}}, \tag{4.4}$$

where $u^{\varphi^{(i)}} := u_1^{\varphi_1^{(i)}} \cdot u_2^{\varphi_2^{(i)}} \cdot u_3^{\varphi_3^{(i)}}$,

$$u := (u_1, u_2, u_3) = \left(\frac{\tilde{\Phi}_{[3,2]}}{\Phi_{[3,2]}}, \frac{\tilde{\Phi}_{[1,3]}}{\Phi_{[1,3]}}, \frac{\tilde{\Phi}_{[2,1]}}{\Phi_{[2,1]}} \right), \tag{4.5}$$

and $\Phi_{[i,j]}, \tilde{\Phi}_{[i,j]}$ are minors of matrices Φ and $\tilde{\Phi}$, respectively, formed by excluding the i'th row and the j'th column.

2. *The u given by (4.5) is a root of the system (4.1) when coefficients x are equal x_0.*

Proof. As mentioned above, critical values of the parametrization can be obtained by restricting the mapping $x = x(s)$ to the set $\mathcal{L}$. The second statement can be proved by the direct substitution x_0 and u into the system (4.1). □

Next, we will verify that the multiple root u is degenerate. By definition, an isolated solution $u \in (\mathbb{C} \setminus 0)^3$ of the system (4.1) is said to be the non-degenerate multiple root if three vectors $\nabla_t g_i(u)$ are linearly dependent and any two of them are linearly independent.

The Jacobi matrix of the system (4.1) is as follows:

$$J(t,x) = \left(\delta_i^j + \varphi_j^{(i)} x^{(i)} t^{\varphi^{(i)} - e_j}\right)_{i,j}, \tag{4.6}$$

where e_j form the standard basis in $\mathbb{Z}^3$ and δ_i^j is the Kronecker symbol. Let us substitute the Horn-Kapranov parameterization (2.2), and the mapping (2.3) written in terms of parameters s for the system (4.1) into (4.6). As a result, we obtain the following matrix:

$$J(\tau(x(s)), x(s)) = \left(\delta_i^j - \frac{\varphi_j^{(i)} s^{(i)}}{\langle \tilde{\varphi}_i, s\rangle} \prod_{l=1}^{3} \left(\frac{\langle \tilde{\varphi}_l, s\rangle}{\langle \varphi_l, s\rangle}\right)^{\delta_l^j}\right)_{i,j}. \tag{4.7}$$

Calculations yield that the dependence condition for any subset of two rows of the matrix (4.7) is equivalent to the condition (4.2). Let s^0 be a nontrivial solution to (4.2). The existence of such a solution is provided by the condition (4.3). Note that

$$\langle \tilde{\varphi}_1, s^0\rangle = \frac{\tilde{\Phi}_{[3,2]}}{\varphi_2^{(3)}} s_1^0, \ \langle \tilde{\varphi}_2, s^0\rangle = -\frac{\tilde{\Phi}_{[1,3]}}{\varphi_3^{(1)}} s_2^0, \ \langle \tilde{\varphi}_3, s^0\rangle = \frac{\tilde{\Phi}_{[2,1]}}{\varphi_1^{(2)}} s_3^0. \tag{4.8}$$

Finally, we take $s = s^0$ in parameterizations $x = x(s)$ and $\tau = \tau(x(s))$ and use relations (4.8). As a result, we get $x = x_0, \tau = u$. Thus, we proved

Proposition 2. *If exponents of the system (4.1) satisfy the condition (4.3), and coefficients $x = x_0$, then the multiple root $u = (u_1, u_2, u_3)$ is degenerate.*

Example 1. We consider a system of the form

$$\begin{cases} t_1 + x^{(1)} t_1^3 t_2 t_3 - 1 = 0, \\ t_2 + x^{(2)} t_1 t_2^3 t_3 - 1 = 0, \\ t_3 + x^{(3)} t_1 t_2 t_3^3 - 1 = 0, \end{cases} \tag{4.9}$$

with exponents $\varphi^{(1)} = (3,1,1)$, $\varphi^{(2)} = (1,3,1)$, $\varphi^{(3)} = (1,1,3)$ satisfying the condition (4.3).

The Horn-Kapranov parametrization for the system (4.9) is as follows

$$\begin{aligned} x^{(1)}(s) &= \frac{-s_1}{\langle \tilde{\varphi}_1, s\rangle} \left(\frac{\langle \tilde{\varphi}_1, s\rangle}{\langle \varphi_1, s\rangle}\right)^3 \left(\frac{\langle \tilde{\varphi}_2, s\rangle}{\langle \varphi_2, s\rangle}\right) \left(\frac{\langle \tilde{\varphi}_3, s\rangle}{\langle \varphi_3, s\rangle}\right), \\ x^{(2)}(s) &= \frac{-s_2}{\langle \tilde{\varphi}_2, s\rangle} \left(\frac{\langle \tilde{\varphi}_1, s\rangle}{\langle \varphi_1, s\rangle}\right) \left(\frac{\langle \tilde{\varphi}_2, s\rangle}{\langle \varphi_2, s\rangle}\right)^3 \left(\frac{\langle \tilde{\varphi}_3, s\rangle}{\langle \varphi_3, s\rangle}\right), \\ x^{(3)}(s) &= \frac{-s_3}{\langle \tilde{\varphi}_3, s\rangle} \left(\frac{\langle \tilde{\varphi}_1, s\rangle}{\langle \varphi_1, s\rangle}\right) \left(\frac{\langle \tilde{\varphi}_2, s\rangle}{\langle \varphi_2, s\rangle}\right) \left(\frac{\langle \tilde{\varphi}_3, s\rangle}{\langle \varphi_3, s\rangle}\right)^3, \end{aligned} \tag{4.10}$$

where $s = (s_1\colon s_2\colon s_3)$ are homogeneous coordinates in $\mathbb{CP}^2$, and

$$\begin{aligned} \langle\varphi_1, s\rangle &= 3s_1 + s_2 + s_3, & \langle\tilde\varphi_1, s\rangle &= 2s_1 + s_2 + s_3,\\ \langle\varphi_2, s\rangle &= s_1 + 3s_2 + s_3, & \langle\tilde\varphi_2, s\rangle &= s_1 + 2s_2 + s_3,\\ \langle\varphi_3, s\rangle &= s_1 + s_2 + 3s_3, & \langle\tilde\varphi_3, s\rangle &= s_1 + s_2 + 2s_3. \end{aligned}$$

The discriminant locus ∇' given by the mapping (4.10) is the hypersurface since matrices Φ and $\tilde\Phi$ do not contain zero entries.

The critical set of the parameterization (4.10) is the plane $\mathcal{L} = \{s_1 + s_2 + s_3 = 0\} \subset \mathbb{CP}^2$. The set of critical values $\mathcal{C} := \{x(s)|_{\mathcal{L}}\}$ consists of one point $x_0 = (-1/2^5, -1/2^5, -1/2^5)$, which is a singular point of the discriminant hypersurface ∇', and the logarithmic Gauss mapping is not defined in it.

The mapping $\tau(s)\colon \mathbb{CP}^2 \to (\mathbb{C} \setminus 0)^3$ with coordinates

$$\tau_1(s) = \frac{\langle\varphi_1, s\rangle}{\langle\tilde\varphi_1, s\rangle},\ \tau_2(s) = \frac{\langle\varphi_2, s\rangle}{\langle\tilde\varphi_2, s\rangle},\ \tau_3(s) = \frac{\langle\varphi_3, s\rangle}{\langle\tilde\varphi_3, s\rangle}$$

defines multiple roots of (4.9). The multiple root $u = \tau(s)|_{\mathcal{L}} = (2, 2, 2)$ is degenerate.

Calculations performed in the computer algebra system SageMath show that the system of equations

$$\begin{cases} 2^5 t_1 - t_1^3 t_2 t_3 - 2^5 = 0,\\ 2^5 t_2 - t_1 t_2^3 t_3 - 2^5 = 0,\\ 2^5 t_3 - t_1 t_2 t_3^3 - 2^5 = 0 \end{cases}$$

has 20 isolated roots, including 16 simple roots and the root $u = (2, 2, 2)$ of multiplicity 4.

5. Conclusion

In this paper the critical points of the parameterization for the discriminant locus of the Laurent polynomial system have been studied. In the special case ($n = 3$) it is proved that systems with degenerate multiple roots belong to the set of critical values of the parameterization in the space of coefficients. Unlike the case $n = 1$, where geometry of the discriminant strata has been studied from different points of view, starting from the paper by D. Hilbert [6], in the multidimensional case the stratification of the discriminant locus has not been investigated. The obtained results confirm efficiency of the parameterization as a tool for studiyng singularities of discriminants of polynomial systems.

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