Серия «Математика»
2025. Т. 52. С. 21—33

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского
государственного
университета

Research article

УДК 517.988.3
MSC 49J52, 49J53
DOI https://doi.org/10.26516/1997-7670.2025.52.21

On Fréchet Subdifferential of Supremum for Arbitrary Family of Continuous Functions

Dmitry V. Khlopin[1]✉

[1] Krasovskii Institute of Mathematics and Mechanics UB RAS,
Yekaterinburg, Russian Federation
✉ khlopin@imm.uran.ru

Abstract. The paper focuses on the Fréchet subdifferential of the pointwise supremum of the family of functions taken over arbitrary index sets. The all functions in the corresponding family are defined on a Fréchet smooth space; this class of Banach spaces includes all reflexive spaces and all separable Asplund spaces. The new upper estimates, including non-convex ones, establish for the Fréchet subdifferentials of the suprema of continuous functions and lower semicontinuous functions. In these estimates, an additional requirement is imposed on every ε-active index that corresponds continuous function: the ε-closeness of the considered point of the graph of the supremum to the graph of this continuous function. The key two-sided inequalities with respect to the graph of continuous function, corresponding to ε-active index, are based on the two-sided unidirectional mean value inequality. The method of proving upper estimates combines the approaches of works of J. S. Treiman, Y. S. Ledyaev, B. S. Mordukhovich, T. Nghia, and P. Pérez–Aros

Keywords: supremum of continuous functions, Fréchet smooth space, Fréchet subdifferential

Acknowledgements: The study was performed as a part of research carried out in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number 075-02-2024-1377).

For citation: Khlopin D. V. On Fréchet Subdifferential of Supremum for Arbitrary Family of Continuous Functions. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 52, pp. 21–33.
https://doi.org/10.26516/1997-7670.2025.52.21

Научная статья

О субдифференциале Фреше супремума непрерывных функций

Д. В. Хлопин[1✉]

[1] Институт математики и механики им. Н. Н. Красовского УрО РАН,
Екатеринбург, Российская Федерация
✉ khlopin@imm.uran.ru

Аннотация. Исследуется субдифференциал Фреше поточечного супремума семейства функций, индексируемых произвольным множеством. Все рассматриваемые при этом функции заданы на гладком по Фреше пространстве; этот класс банаховых пространств включает в себя как рефлексивные пространства, так и сепарабельные пространства Асплунда. Новые оценки сверху, в том числе невыпуклые, установлены для субдифференциала по Фреше супремума непрерывных и полунепрерывных снизу функций. В этих оценках к каждому ε-активному индексу, соответствующему непрерывной функции, предъявляется дополнительное требование: ε-близость графика этой непрерывной функции к рассматриваемой точке графика супремума. Ключевые двусторонние неравенства для точки графика непрерывной функции, соответствующей ε-активному индексу, основаны на двустороннем однонаправленном варианте теоремы Лагранжа. Метод доказательства верхних оценок сочетает в себе подходы из работ Дж. С. Треймана, Ю. С. Ледяева, М. Ш. Мордуховича, Т. Нгиа и П. Перез-Ароса.

Ключевые слова: супремум непрерывных функций, гладкое по Фреше пространство, субдифференциал Фреше

Благодарности: Работа выполнена в рамках исследований, проводимых в Уральском математическом центре при финансовой поддержке Министерства науки и высшего образования Российской Федерации (№ 075-02-2024-1377).

Ссылка для цитирования: Khlopin D. V. On Fréchet Subdifferential of Supremum for Arbitrary Family of Continuous Functions // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 52. С. 21–33.
https://doi.org/10.26516/1997-7670.2025.52.21

Introduction

Consider a family of lower semicontinuous functions $f_\gamma : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ $(\gamma \in \Gamma)$ defined on a Fréchet smooth space $\mathbb{X}$. Define the following supremum function f

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x).$$

In this article we estimate the Fréchet subdifferential of this function f. There are many works devoted to this topic. For example, for the family of convex functions, it is necessary to mention the classical Ioffe–Tikhomirov theorem [4] and the very recent monograph [3]. The detailed history of the issue can be found in [9], [11]. As well as in [9], [11], we will not impose any assumptions on the index set Γ. At the same time, in this article the main attention will be paid to the case when all the f_γ are continuous.

The method of proofs follows the key ideas from [6, Theorem 6.1] and [11]. We will only apply some two-sided version of the mean value inequality (see Proposition 1) in the continuous case instead of the classical one-sided mean value inequality [6, Theorem 2.2] and use the famous estimates for subgradients of the supremum function of linear function in the convex case with [11, Proposition 3.7] instead of the nonsymmetrical minimax theorem [1, Theorem 3.6.10].

The rest of the paper is organized as follows. In Section 2, we will formulate two estimates for the Fréchet subdifferential of f (Theorem 1) and, then, discuss and compare these estimates with those previously known. The proof of Theorem 1 is contained in Section 3. But first, we will recall several definitions and two results of variational analysis.

1. Basic Definitions and Preliminaries

We will use basic notions from the set-valued and variational analyses [1;7].

For every subset $\mathcal{X}$ of Banach space $\mathbb{X}$, by $\operatorname{co}\mathcal{X}$, $\operatorname{cl}\mathcal{X}$ denote its convex hull and the closure (in the norm topology). Similarly, by $\operatorname{cl}^{w^*}\mathcal{Y}$ denote the closure in the weak* topology of a subset $\mathcal{Y}$ of $\mathbb{X}^*$. Denote also by B and B^* the closed unit balls in $\mathbb{X}$ and $\mathbb{X}^*$, respectively.

Recall that a Banach space is Asplund if each of its separable subspaces has a separable dual. We say a real Banach space is Fréchet smooth if this space has an equivalent norm that is C^1-smooth off the origin. Each Fréchet smooth space is Asplund. Observe that any reflexive Banach space and any separable Asplund space are Fréchet smooth [1, Theorem 6.1.6]. It is worth mentioning that B^* is sequentially compact in the weak* topology [10, Theorem 3.113] if $\mathbb{X}$ is Asplund.

Given an extended real-valued function $g : \mathbb{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$, define its graph $\operatorname{gph} g \stackrel{\triangle}{=} \{(x, g(x)) \in \mathbb{X} \times \mathbb{R} \mid |g(x)| < +\infty\}$. For a point $x \in \mathbb{X}$ with $|g(x)| < +\infty$, define the Fréchet subdifferential [1, (3.1.1)] of g at x as

$$\hat{\partial} g(x) \stackrel{\triangle}{=} \{\zeta \in \mathbb{X}^* \mid \liminf_{y \to x} \frac{g(y) - g(x) - \zeta(y - x)}{\|y - x\|} \geq 0\}.$$

Observe that $\partial g(x) = \varnothing$ if $|g(x)| = +\infty$.

Next, we will essentially need the two recent results of the variational analysis. First, we will use the following two-sided one-directed mean value inequality [5]:

Proposition 1. *Let $\mathbb{X}$ be a Fréchet smooth space. Let a continuous function $g : \mathbb{X} \to \mathbb{R}$ and some closed interval $[u; v]$ in $\mathbb{X}$ be given.*

Then, for a real number $s < g(v) - g(u)$ and positive ε there exist some point $\hat{z} \in [u; v] + \varepsilon B$ and Fréchet subgradient $\hat{\zeta} \in \hat{\partial} g(\hat{z})$ that satisfy

$$s < \hat{\zeta}(v - u) \quad \text{and} \quad |g(\hat{z}) - g(u)| \leq |s| + \varepsilon. \tag{1.1}$$

Second, we will also apply the following excellent estimate in the case of the direct index set $\mathbb{T}$ [11, Proposition 3.7]:

Proposition 2. *Let $\mathbb{X}$ be an Asplund space. Assume that some non-decreasing family of defined on $\mathbb{X}$ lower semicontinuous functions h_t $(t \in \mathbb{T})$ is given. Let $h \equiv \sup\{h_t \,|\, t \in \mathbb{T}\}$.*

Then, for all points $x' \in \mathbb{X}$,

$$\hat{\partial} h(x') \subset \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \bigcup_{\substack{x'' \in x' + \varkappa B,\ t \in \mathbb{T}, \\ h(x') < h_t(x'') + \varkappa,\ |h_t(x') - h_t(x'')| < \varkappa}} \hat{\partial} h_t(x''). \tag{1.2}$$

2. The main result

Let a Fréchet smooth space $\mathbb{X}$, some non-empty index set Γ, and a family of lower semicontinuous functions $f_\gamma : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ $(\gamma \in \Gamma)$ be given. Define a function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ by the following rule:

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x) \text{ for all } x \in \mathbb{X}.$$

Due to [12, Proposition 1.26], f as a supremum of a family of lower semicontinuous function is lower semicontinuous.

By (Fin)(Γ) denote the set of all finite subset of Γ. For every $\mathcal{T} \in$ (Fin)(Γ), define $f_{\mathcal{T}}(y) = \max_{\gamma \in \mathcal{T}} f_\gamma(x)$. For each $\gamma \in \Gamma$ put $|a|_\gamma \equiv |a|$ if f_γ is continuous on $\mathbb{X}$, and $|a|_\gamma \equiv a$ otherwise. Finally, by $[1 : K]$ denote the set $\{1, 2, \ldots, K\}$ for each $K \in \mathbb{N}$.

Theorem 1. *Let ξ be a Fréchet subgradient of f at a point $\hat{x} \in \mathbb{X}$.*

Then, for an weak neighborhood U of the origin in $\mathbb{X}^*$, for every positive ε, there exist a natural K, coefficients $\alpha_i \in (0; 1]$ $(i \in [1 : K])$, and pairs*

$(x_i, \gamma_i) \in \mathbb{X} \times \Gamma$ $(i \in [1 : K])$ *that satisfy*

$$\sum_{i=1}^{K} \alpha_i = 1, \tag{2.1}$$

$$\xi \in \sum_{i=1}^{K} \alpha_i \zeta_i + U, \tag{2.2}$$

$$x_i \in \hat{x} + \varepsilon B \text{ for all } i \in [1 : K], \tag{2.3}$$

$$\zeta_i \in \hat{\partial} f_{\gamma_i}(x_i) \text{ for all } i \in [1 : K], \tag{2.4}$$

$$(\zeta_i - \xi)(x_i - \hat{x}) > -\varepsilon^2 \text{ for all } i \in [1 : K], \tag{2.5}$$

$$|f_{\gamma_i}(x_i) - f(\hat{x})| < \varepsilon \text{ if } f_{\gamma_i} \text{ is continuous and } f_{\gamma_i}(x_i) - f_{\gamma_i}(\hat{x}) < \varepsilon \text{ otherwise}; \tag{2.6}$$

in particular, $|f_{\gamma_i}(x_i) - f(\hat{x})|_{\gamma_i} < \varepsilon$ *and*

$$f_{\gamma_i}(x_i) - f(\hat{x}) < \varepsilon \tag{2.7}$$

holds true for all $i \in [1 : K]$. *In addition, if* Γ *is finite, we also can guarantee that*

$$|f_{\gamma_i}(x_i) - f_{\gamma_i}(\hat{x})| < \varepsilon \text{ for all } i \in [1 : K]. \tag{2.8}$$

Furthermore, one has

$$\hat{\partial} f(\hat{x}) \subset \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \hat{x} + \varkappa B,\ \mathcal{T} \in (Fin)(\Gamma), \\ |f_{\mathcal{T}}(x') - f(\hat{x})| < \varkappa, \\ |f_\gamma(x') - f_{\mathcal{T}}(x')|_\gamma \le 0\ \forall \gamma \in \mathcal{T}}} \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \mathcal{T},\ x \in x' + \varepsilon B, \\ |f_\gamma(x') - f_\gamma(x)| < \varepsilon}} \hat{\partial} f_\gamma(x) \tag{2.9}$$

$$\subset \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \Gamma,\ x \in \hat{x} + \varkappa B, \\ |f_\gamma(x) - f(\hat{x})|_\gamma < \varkappa}} \hat{\partial} f_\gamma(x); \tag{2.10}$$

in particular, in the case of the family of continuous f_γ,

$$\hat{\partial} f(\hat{x}) \subset \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \hat{x} + \varkappa B,\ \mathcal{T} \in (Fin)(\Gamma), \\ |f_{\mathcal{T}}(x') - f(\hat{x})| < \varkappa, \\ f_\gamma(x') = f_{\mathcal{T}}(x')\ \forall \gamma \in \mathcal{T}}} \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \mathcal{T},\ x \in x' + \varepsilon B, \\ |f_\gamma(x') - f_\gamma(x)| < \varepsilon}} \hat{\partial} f_\gamma(x). \tag{2.11}$$

The proof of this theorem will be presented in Section 3. Now, the discussion of the results is presented in the form of a series of comments.

Remark 1. If $\mathbb{X} \overset{\triangle}{=} \mathbb{R}^d$, by the famous Carathéodory theorem [12, Theorem 2.29], any finite convex sum of a (co)vectors can be represented by some finite convex sum of no more than $d+1$ of them. So, this theorem holds true with additional requirement that $K \leq d+1$.

Remark 2. If each f_γ is C^1-smooth on $\mathbb{X}$, we obtain that each $\hat{\partial} f_\gamma(x)$ is a singleton, therefore $\zeta_i = f'_{\gamma_i}(x_i)$ whenever i.

Remark 3. Following [9] in the case of the family of continuous f_γ assume also that for all z close to $\hat{x}$ each set-valued map $x \mapsto \hat{\partial} f_\gamma(x)$ weak* outer stable at point z [9]: for all sequences of z_i and $\zeta_i \in \hat{\partial} f_\gamma(z_i)$, converging and w^*-converging to z and a $\zeta \in \mathbb{X}^*$ respectively, ζ lies in $\mathrm{cl}^{w^*} \hat{\partial} f_\gamma(z)$. Then, (2.11) is simplified to

$$\hat{\partial} f(\hat{x}) \subset \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \hat{x} + \varkappa B,\ \mathcal{T} \in (\mathrm{Fin})(\Gamma), \\ |f_{\mathcal{T}}(x') - f(\hat{x})| < \varkappa, \\ f_\gamma(x') = f_{\mathcal{T}}(x')\ \forall \gamma \in \mathcal{T}}} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\gamma \in \mathcal{T}} \mathrm{cl}^{w^*} \hat{\partial} f_\gamma(x').$$

Remark 4. The first inequality in (2.6) can be replaced by

$$|f_{\gamma_i}(\hat{x}) - f(\hat{x})| < \varepsilon \text{ for all } i \in [1 : K]; \tag{2.12}$$

if either all the f_γ are equicontinuous near to $\hat{x}$, or f and all the f_γ are continuous. These conditions are typical, for instance, [8, (8.33)]. In particular, (2.10) yields [9, (3.4)].

Remark 5. The condition (2.7) can not be replaced by the first inequality in (2.6) or by

$$\mathrm{dist}(x_i, f_{\gamma_i}(x_i), \mathrm{gph}\, f) < \varepsilon \text{ for all } i \in [1 : K] \tag{2.13}$$

even if f is continuous on $\mathbb{X}$ and each f_γ is continuous near $\hat{x}$ and lower semicontinuous on $\mathbb{X}$. Indeed, consider $\mathbb{X} \overset{\triangle}{=} \mathbb{R}$ and $\Gamma \overset{\triangle}{=} \mathbb{Q}$; for all $p/q \in \Gamma$ set $f_{p/q}(x) \equiv 0$ if $x \leq p/q$ and $f_{p/q}(x) \overset{\triangle}{=} 1$ otherwise. Now, $f(x) = x$ and $\hat{\partial} f(x) = \{1\}$ on $\mathbb{X}$. Note that also $\hat{\partial} f_r(x) = \{0\}$ if $r \neq x$. Consider the unique subgradient $\xi = 1$ at point $\hat{x} = e^{-1}$ and a positive $\varepsilon < e^{-1}$. Assume that we could pick finite families of α_i, γ_i, x_i, and ζ_i that satisfies (2.2). Since ξ is nonzero, at least one of ζ_i must be nonzero, therefore the corresponding γ_i coincides with x_i. This yields $f_{\gamma_i}(x_i) = 0$ for some i, and (2.13) does not hold if $\varepsilon < 1/e$. So, in this example the condition (2.13), as well as (2.6), is not consistent with (2.2).

Remark 6. The requirement of the finiteness of Γ for the condition (2.8) as well as $f_{\gamma_i}(x_i) > f_{\gamma_i}(\hat{x}) - \varepsilon$ cannot be omitted. Indeed, consider $\mathbb{X} \overset{\triangle}{=} \mathbb{R}$

and $\Gamma \triangleq \mathbb{N}$, set $f_n(x) \triangleq 0$ if $x < 0$, $f_n(x) \triangleq -nx$ if $x \in [0; 1/n]$, and $f_n(x) \triangleq 1$ otherwise. Now, $f(x) = 0$ if $x < 0$ and $f(x) = 1$ otherwise. Assume that for this subgradient $\xi = 1$ at point $\hat{x} = 0$ and a positive $\varepsilon < 1$ one finds finite families of α_i, γ_i, x_i, and ζ_i that satisfies (2.2). Since ξ is positive, at least one of ζ_i must be positive, therefore the corresponding γ_i coincides with $x_i = 1/\gamma_i$. This entails $f_{\gamma_i}(x_i) = -1$ and $|f_{\gamma_i}(x_i) - f_{\gamma_i}(0)| \geq 1$. So, in this example the condition (2.8) is not consistent with (2.2).

To the best of our knowledge, the convex estimates (2.2) and (2.10) of the Fréchet subdifferential of supremum function f in Theorem 1 are similar to the following results:

- for lower semicontinuous functions f_γ on a reflexive $\mathbb{X}$, the inclusion (2.2) with (2.3), with
$$\Big|\sum_{\gamma\in\Gamma,\ \alpha_\gamma>0} \alpha_\gamma - 1\Big| < \varepsilon \tag{2.14}$$
instead of (2.1), and without any condition on $f(x')$ [2, Theorem 3.18];
- for the lower semicontinuous functions f_γ on a reflexive $\mathbb{X}$, the inclusion (2.2) with (2.1), (2.3), and (2.7) [6, Theorem 3.2];
- for the uniformly locally Lipschitz continuous (or, merely equicontinuously subdifferentiable) family of f_γ on an Asplund space, the equivalent to (2.2) inclusion [9, Theorem 3.1(ii)]:
- for the finite family of lower semicontinuous functions f_γ on an Asplund space, the inclusion (2.2) with (2.8), (2.14),
$$(|f(x') - f_{\gamma_i}(x')| > \varepsilon) \Rightarrow (\alpha_i < \varepsilon), \tag{2.15}$$
with $x' = \hat{x}$ for each $\gamma \in \Gamma$ instead of (2.1) and (2.7) [11, Proposition 3.2(v)].

Finally, regarding the non-convex estimate (2.9), we notice a similar result in Asplund space for lower semicontinuous functions f_γ [11, Theorem 3.8]:

$$\hat{\partial} f(\hat{x}) \subset \bigcap_{\varkappa>0} \mathrm{cl}^{w^*} \bigcup_{\substack{\mathcal{T}\in(\mathrm{Fin})(\Gamma), x'\in\hat{x}+\varkappa B,\\ f_{\mathcal{T}}(x')\geq f(\hat{x})-\varkappa,\, |f_{\mathcal{T}}(x')-f_{\mathcal{T}}(\hat{x})|<\varkappa}} \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \bigcup \Big\{ \sum_{\gamma\in\mathcal{T}} \alpha_\gamma \hat{\partial} f_\gamma(x_\gamma) \,\Big|\, \alpha_\gamma \geq 0, x_\gamma \in x' + \varepsilon B \text{ satisfying (2.14), (2.15), and } |f_\gamma(x_\gamma) - f_\gamma(x')| < \varepsilon \Big\} \tag{2.16}$$

and for uniformly Lipschitz continuous functions [11, Theorem 3.1(i)]:

$$\hat{\partial} f(\hat{x}) \subset \bigcap_{\varkappa>0} \mathrm{cl}^{w^*} \bigcup_{\substack{\mathcal{T}\in(\mathrm{Fin})(\Gamma), x'\in\hat{x}+\varkappa B,\\ f_\gamma(x')=f_{\mathcal{T}}(x')\geq f(\hat{x})-\varkappa\ \forall\gamma\in\mathcal{T}}} \mathrm{co} \bigcup_{\gamma\in\mathcal{T}} \partial f_\gamma(x'). \tag{2.17}$$

3. The proof of Theorem 1

Step 0. At the beginning, we can assume, without loss of generality, that $f(\hat{x}) = 0$ and $\hat{x} = 0$.

Fix some positive $\varepsilon < 1/4$ and an weak* balanced neighborhood U of the origin in $\mathbb{X}^*$. Decreasing ε if necessary, we can assume that $4\varepsilon B^* \subset U$.

Recall that, for each $\gamma \in \Gamma$, $|a|_\gamma \equiv |a|$ if f_γ is continuous, and $|a|_\gamma \equiv a$ otherwise. Introduce the set $\Xi_\varepsilon \subset \mathbb{X}^*$ by

$$\Xi_\varepsilon \stackrel{\triangle}{=} \{\zeta \in \mathbb{X}^* \,|\, \exists \gamma \in \Gamma,\ \hat{z} \in \varepsilon B/2,\ \zeta \in \hat{\partial} f_\gamma(\hat{z}) \text{ satisfying (2.5), (2.6)}\}.$$

So, since we can choose positive ε arbitrarily, for every Fréchet subgradient ξ of f at $\hat{x} = 0$ and corresponding relations (2.1)–(2.7), we only need to prove that $\xi \in \operatorname{co} \Xi_\varepsilon + U$.

To this, introduce the scalar function $g : \mathbb{X} \to \mathbb{R}$ by the rule:

$$g(x) = -\xi x + \varepsilon^3 \|x\| \quad \text{for all } x \in \mathbb{X}.$$

Now, by the definition of Fréchet subgradient, one find a positive $\tau < \varepsilon/4 < 1$ such that

$$-g(x) < f(x) \ \text{ on } \ 2\tau B \setminus \{0\}. \tag{3.1}$$

Decreasing τ if necessary, we can ensure the inequalities

$$4|g(x)| < \varepsilon, \qquad \hat{\partial} g(x) \subset -\xi + \varepsilon^2 B^*, \qquad 4t(\|\xi\| + \varepsilon) < \varepsilon \text{ on } \tau B.$$

Step 1. We claim that, for every $v \in B$, there exists a $\zeta \in \Xi_\varepsilon$ that satisfies

$$\sup_{\zeta \in \Xi_\varepsilon} (\zeta - \xi) v \geq -\varepsilon^2 \ \text{ for all } v \in B. \tag{3.2}$$

In the case $v = 0$ this inequality is trivial. Therefore, let $v \in B$ be non-zero. By (3.1), we obtain

$$-\varepsilon/4 < -g(tv) < f(tv) \text{ for all } 0 < t \leq \tau.$$

Hence, for each $0 < t \leq \tau$, one finds a $\gamma(t) \in \Gamma$ that satisfies

$$(f_{\gamma(t)} + g)(tv) > 0.$$

By the choice of τ, one has

$$f_{\gamma(t)}(0) \leq f(0) = g(0) \leq -g(tv) + \varepsilon/4 < f_{\gamma(t)}(tv) + \varepsilon/4. \tag{3.3}$$

Assume also that there exists a positive $t \leq \tau$ with continuous $f_{\gamma(t)}$. Fix this t and the corresponding $\gamma = \gamma(t)$. The continuity of f_γ leads us to a root $t' \in [0; t)$ to the equation $-g(t'v) = f_\gamma(t'v)$. This root exists due to the intermediate value theorem by $f_\gamma(0) + g(0) \le 0 < f_\gamma(tv) + g(tv)$. We obtain the inequality

$$f_\gamma(tv) - f_\gamma(t'v) > g(tv) - g(t'v) \ge (t - t')(\xi v - \varepsilon^2).$$

Now, let's apply Proposition 1 to this inequality with data $u \triangleq t'v$, $v \triangleq tv$, $s \triangleq (t - t')(\xi v - \varepsilon^2)$, $\varepsilon \triangleq t/2$. Then, we pick a point $\hat{z} \in \mathbb{X}$ and a Fréchet subgradient $\zeta \in \hat{\partial} f_\gamma(\hat{z})$ that satisfies

$$(t - t')(\xi v - \varepsilon^2) < (t - t')\zeta v, \ \|\hat{z}\| < t + t < \varepsilon/2,$$

$$|f_\gamma(\hat{z}) - f_\gamma(t'v)| \le |(t - t')(\xi v - \varepsilon^2)| + t.$$

By the choice of $t < \varepsilon/4$, in the account to $g(t'v) = f_\gamma(t'v)$, this yields that $\hat{z} \in \varepsilon B/2$ and

$$|f_\gamma(\hat{z})|_\gamma = |f_\gamma(\hat{z})| \le |f_\gamma(t'v)| + \varepsilon/2 + t < \varepsilon;$$

in particular, $\zeta \in \Xi_\varepsilon$ and $\xi v - \varepsilon^2 < \zeta v$. In this case, (3.2) has been proved.

Assume also that, for all positive $t \le \tau$, from (3.3) it follows that $f_{\gamma(t)}$ is not continuous. Fix this t and the corresponding $\gamma = \gamma(t)$. Now, from $f_\gamma(tv) + g(tv) > s \ge 0 \ge f_\gamma(0) + g(0)$ it follows that

$$(f_\gamma + g)(tv) - (f_\gamma + g)(0) > s \ge 0.$$

One-sided unidirectional mean value theorem [1, Theorem 3.4.6] to $f_\gamma + g$, $u \triangleq 0$, $v \triangleq tv$, $s = 0$, $\varepsilon \triangleq t$ gives a point $z \in \mathbb{X}$ and a Fréchet subgradient $\zeta' \in \hat{\partial}(f_\gamma + g)(z)$ that satisfies

$$0 < t\zeta' v, \ \|z\| < t + t < \varepsilon/2, f_\gamma(z) + g(z) \le f_\gamma(0) + g(0) + t < f_\gamma(0) + \varepsilon/2$$

by the choice of $t < \varepsilon/4$. Applying the strong approximate sum rule [1, Theorem 3.3.1], we pick a point $\hat{z} \in \mathcal{X}$, a Fréchet subgradient $\hat{\zeta} \in \hat{\partial} f_\gamma(\hat{z})$, and a Fréchet subgradient $\hat{\xi} \in \cup_{z' \in \varepsilon B/2} \hat{\partial} g(z') \subset (-\xi + \varepsilon^2 B^*)$ that satisfies

$$0 \le s < t(\hat{\zeta} + \xi')v, \ \|\hat{z}\| < \varepsilon/2, \ f_\gamma(\hat{z}) < f_\gamma(0) + \varepsilon.$$

Further, from $0 = s < t(\hat{\zeta} + \xi')v$ it follows $-\xi v < \hat{\zeta} v$. Since $v \in B$ and $\xi' \in -\xi + \varepsilon^2 B^*$, we obtain $\xi v - \varepsilon^2 < \hat{\zeta} v$. So, we have verified that $\hat{\zeta}$ lies in Ξ_ε.

Thus, in all the cases for each $v \in B$ there exists a $\zeta \in \Xi_\varepsilon$ that satisfies (3.2).

Step 2. Define the sequence of functions $h_n : \mathbb{X} \to \mathbb{R}$ by the following rule:

$$h_n(x) \triangleq \sup_{\zeta \in \Xi_\varepsilon \cap nB^*} (\zeta - \xi)x \text{ for all } x \in \mathbb{X}, n \in \mathbb{N}.$$

Notice that, for each n all the functions of family $\Xi_\varepsilon \cap nB^*$ are uniformly locally Lipschitz continuous, therefore, by [9, Theorem 3.1(ii)], one has

$$\hat{\partial} h_n(x) \subset \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\zeta \in \Xi_\varepsilon \cap nB^*} \hat{\partial}(\zeta - \xi)(x) \\ = \mathrm{cl}^{w^*} \mathrm{co} \Big(-\xi + \Xi_\varepsilon \cap nB^* \Big) \subset -\xi + \mathrm{cl}^{w^*} \mathrm{co}\, \Xi_\varepsilon \quad (3.4)$$

whenever x and n.

Further, the sequence of h_n allows us to rewrite (3.2) as

$$h(x) \stackrel{\triangle}{=} \sup_{n \in \mathbb{N}} h_n(x) \geq -\varepsilon^2 \ \text{ for all } x \in B.$$

Since $h_n(0) = 0$ whenever n, we establish that 0 is an ε^2-minimum of h on B. Invoking the lower subdifferential variational principle [7, Theorem 2.28], there exists a point $\check{x} \in \varepsilon B$ and $\check{\zeta} \in \hat{\partial} h(\check{x})$ such that $\|\check{\zeta}\|_* \leq \varepsilon$. In particular, $0 \in \hat{\partial} h(\check{x}) + \varepsilon B^*$. Since sequence of h_n is non-decreasing, Lemma 2 ensures

$$0 \in \bigcap_{\varkappa > 0} \mathrm{cl}^{w^*} \bigcup_{x'' \in x' + \varkappa B,\ n \in \mathbb{N}} \hat{\partial} h_n(x'') + \varepsilon B^*.$$

In account to $\varepsilon B^* \subset U/4$, substituting $\varkappa \stackrel{\triangle}{=} \varepsilon$, we find a $x'' \in 2\varepsilon B$ and $n \in \mathbb{N}$ that satisfies $0 \in \hat{\partial} h_n(x'') + U/2$. Now, (3.4) entails

$$0 \in \mathrm{cl}^{w^*} \mathrm{co}(\Xi_\varepsilon \cap nB^*) - \xi + U/2 \subset \mathrm{co}\, \Xi_\varepsilon - \xi + U$$

and $\xi \in \mathrm{cl}^{w^*} \mathrm{co}\, \Xi_\varepsilon + U$ for sufficiently small ε. Since we can choose a neighborhood U arbitrary, the relations (2.1)–(2.6) has been proved.

In the case of finite subset Γ, we can pick a positive $\varepsilon' < \varepsilon$ that satisfies $f_\gamma < f_\gamma(\hat{x}) + \varepsilon$ on $\varepsilon' B$ for continuous f and $f_\gamma > f_\gamma(\hat{x}) - \varepsilon$ on $\varepsilon' B$ for upper continuous f. Now, $\xi \in \mathrm{cl}^{w^*} \mathrm{co}\, \Xi_{\varepsilon'} + U$ yields (2.8). In particular, we obtain

$$\xi \in \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \Gamma,\ x \in \hat{x} + \varepsilon B, \\ |f_\gamma(x) - f(\hat{x})|_\gamma < \varepsilon,\ |f_\gamma(x) - f_\gamma(\hat{x})| < \varepsilon}} \hat{\partial} f_\gamma(x). \quad (3.5)$$

Step 4. Fix a positive $\varkappa$. Recall that $f(\hat{x}) = 0$, $\hat{x} = 0$. Following the proof of [11, Theorem 3.8], consider the set $\mathbb{T} = (\mathrm{Fin})(\Gamma)$ of all finite subsets $\mathcal{T}$ of Γ. This set is ordered by the relation $\subset$, the family of $f_{\mathcal{T}}$ with $\mathcal{T} \in (\mathrm{Fin})(\Gamma)$ is increasing and its supremum coincides with f. Applying Proposition 2 to this family, substituting (3.5) with x' instead of $\hat{x}$ in (1.2), we get

$$\xi \in \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \varkappa B,\ \mathcal{T} \in \mathbb{T}, \\ f_{\mathcal{T}}(x') > -\varkappa,\ |f_{\mathcal{T}}(\hat{x}) - f_{\mathcal{T}}(x')| < \varkappa}} \hat{\partial} f_{\mathcal{T}}(x') \subset \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \varkappa B,\ \mathcal{T} \in (\mathrm{Fin})(\Gamma), \\ |f_{\mathcal{T}}(x')| < \varkappa,\ |f_{\mathcal{T}}(\hat{x})| < 2\varkappa}} \hat{\partial} f_{\mathcal{T}}(x')$$
$$\subset \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \varkappa B,\ \mathcal{T} \in (\mathrm{Fin})(\Gamma), \\ |f_{\mathcal{T}}(x')| < \varkappa,\ |f_{\mathcal{T}}(\hat{x})| < 2\varkappa}} \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \mathcal{T},\ x \in x' + \varepsilon B, \\ |f_\gamma(x) - f_{\mathcal{T}}(x')|_\gamma < \varepsilon,\ |f_\gamma(x') - f_\gamma(x)| < \varepsilon}} \hat{\partial} f_\gamma(x)$$
$$\subset \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \Gamma,\ x \in 2\varkappa B, \\ |f_\gamma(x)|_\gamma < 2\varkappa}} \hat{\partial} f_\gamma(x).$$

Furthermore, $|f_\gamma(x) - f_{\mathcal{T}}(x')|_\gamma < \varepsilon$ together $|f_\gamma(x') - f_\gamma(x)| < \varepsilon$ gives $|f_\gamma(x') - f_{\mathcal{T}}(x')|_\gamma < 2\varepsilon$. Since we can choose ε small enough, we can exclude continuous f_γ in the case $f_\gamma(x') \neq f_{\mathcal{T}}(x')$. Therefore, $f_\gamma(x')$ has to coincide with $f_{\mathcal{T}}(x')$ for each continuous f_γ. This yields

$$\xi \in \mathrm{cl}^{w^*} \bigcup_{\substack{x' \in \varkappa B,\ \mathcal{T} \in (\mathrm{Fin})(\Gamma),\ |f_{\mathcal{T}}(x')| < \varkappa, \\ |f_\gamma(x') - f_{\mathcal{T}}(x')|_\gamma \leq 0\ \forall \gamma \in \mathcal{T}}} \bigcap_{\varepsilon > 0} \mathrm{cl}^{w^*} \mathrm{co} \bigcup_{\substack{\gamma \in \mathcal{T},\ x \in x' + \varepsilon B, \\ |f_\gamma(x') - f_\gamma(x)| < \varepsilon}} \hat{\partial} f_\gamma(x).$$

Since we can choose $\varkappa$ arbitrary, we obtain (2.9) and (2.10). The theorem has been proved.

Remark 7. In the proof above, the fact that a given $\mathbb{X}$ is not only is an Asplund but also is Fréchet smooth, was applied only once: when referring to a two-sided inequality (1.1) of Proposition 1. This allows us to hope that this Theorem 1 will be fulfilled in all Asplund spaces.

4. Conclusion

In this article we refine the estimates of the Fréchet subdifferential for the supremum. In particular, we replace the requirements (2.14),(2.15) with (2.6),(2.1) for the case of continuous functions. By doing so, the obtained in Theorem 1 convex and nonconvex estimates of Fréchet subdifferential bridge the gap between the result of [9, Theorem 3.1(ii)] in the uniform Lipschitz case and the result of [11, Theorem 3.8] in the lower semicontinuous case.

References

1. Borwein J.M., Zhu Q.J. *Techniques of variational analysis.* New York, Springer-Verlag, 2005, 362 p.

2. Borwein J.M., Zhu Q.J. A survey of subdifferential calculus with applications. *Nonlinear Analysis*, 1999, vol. 38, no. 6, pp. 687–774. https://doi.org/10.1016/S0362-546X(98)00142-4
3. Correa R., Hantoute A., López M.A. *Fundamentals of convex analysis and optimization*. Cham, Springer, 2023, 444 p.
4. Ioffe A.D., Tikhomirov V.M. *Theory of Extremal Problems*. Amsterdam, North-Holland, 1979, 460 p.
5. Khlopin D. On two-sided unidirectional mean value inequality in a Fréchet smooth space. *Ural Mathematical Journal*, 2023, vol. 9, no. 2, pp. 132–140. http://doi.org/10.15826/umj.2023.2.011
6. Ledyaev Y.S., Treiman J.S. Sub-and supergradients of envelopes, semicontinuous closures, and limits of sequences of functions. *Russ. Math. Surv.,* 2012, vol. 67, pp. 345–373. https://doi.org/10.1070/rm2012v067n02abeh004789
7. Mordukhovich B.S. *Variational analysis and generalized differentiation, I: Basic theory.* Springer-Verlag: Berlin, 2006. xxii+579 p.
8. Mordukhovich B.S. *Variational analysis and applications.* Cham, Springer, 2018, 622 p.
9. Mordukhovich B.S., Nghia T. Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with lipschitzian data. *SIAM J. Control Optim*, 2013, vol. 23, pp. 406–431. https://doi.org/10.1137/110857738
10. Penot J.P. *Calculus without derivatives.* New York, Springer, 2013, 524 p.
11. Pérez-Aros P. Subdifferential formulae for the supremum of an arbitrary family of functions, *SIAM Journal on Optimization*, 2019, vol. 29, no. 2, pp. 1714–1743. https://doi.org/10.1137/17M1163141
12. Rockafellar R.T., Wets R.J.B. *Variational analysis.* Berlin, Springer-Verlag, 2009, 735 p.

Список источников

1. Borwein J. M., Zhu Q. J. Techniques of variational analysis. New York : Springer-Verlag, 2005. 362 p.
2. Borwein J.M., Zhu Q.J. A survey of subdifferential calculus with applications // Nonlinear Analysis. 1999. Vol. 38, N 6. P. 687–774. https://doi.org/10.1016/S0362-546X(98)00142-4
3. Correa R., Hantoute A., López M. A. Fundamentals of convex analysis and optimization. Cham : Springer, 2023. 444 p.
4. Ioffe A. D., Tikhomirov V. M. Theory of Extremal Problems. Amsterdam : North-Holland, 1979. 460 p.
5. Khlopin D. On two-sided unidirectional mean value inequality in a Fréchet smooth space // Ural Mathematical Journal. 2023. Vol. 9, N 2. P. 132–140. http://doi.org/10.15826/umj.2023.2.011
6. Ledyaev Y. S., Treiman J. S. Sub-and supergradients of envelopes, semicontinuous closures, and limits of sequences of functions // Russ. Math. Surv. 2012. Vol. 67. P. 345–373. https://doi.org/10.1070/rm2012v067n02abeh004789
7. Mordukhovich B. S. Variational analysis and generalized differentiation, I: Basic theory. Berlin : Springer-Verlag, 2006. 579 p.
8. Mordukhovich B. S. Variational analysis and applications. Cham : Springer, 2018. 622 p.

9. Mordukhovich B. S., Nghia T. Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with lipschitzian data // SIAM J. Control Optim. 2013. Vol. 23. P. 406–431. https://doi.org/10.1137/110857738
10. Penot J. P. Calculus without derivatives. New York : Springer, 2013. 524 p.
11. Pérez-Aros P. Subdifferential formulae for the supremum of an arbitrary family of functions // SIAM Journal on Optimization. 2019. Vol. 29, N 2. P. 1714–1743. https://doi.org/10.1137/17M1163141
12. Rockafellar R. T., Wets R. J. B. Variational analysis. Berlin : Springer-Verlag, 2009. 735 p.

Об авторах

Хлопин Дмитрий Валерьевич,
канд. физ.-мат. наук,
Институт математики и механики им. Н. Н. Красовского,
Екатеринбург, 620208, Российская Федерация,
khlopin@imm.uran.ru,
https://orcid.org/0000-0002-8942-6520

About the authors

Dmitry V. Khlopin,
Cand. Sci. (Phys.–Math.),
Krasovskii Institute of Mathematics and Mechanics UB RAS,
Yekaterinburg, 620208, Russian Federation,
khlopin@imm.uran.ru,
https://orcid.org/0000-0002-8942-6520

Поступила в редакцию / Received 28.08.2024
Поступила после рецензирования / Revised 17.10.2024
Принята к публикации / Accepted 24.10.2024