Серия «Математика»
2025. Т. 51. С. 66—81

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 517.43
MSC 45P05, 33C20, 33D05, 05A30
DOI https://doi.org/10.26516/1997-7670.2025.51.66

A Note on Extended Saigo Operators and Their q-analogues

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Abstract. Megumi Saigo derived generalized fractional operators, involving Gauss hypergeometric function, having four special cases: Riemann-Liouville, Weyl, Erdély-Kober left and right sided fractional operators. Mridula Garg and Lata Chanchalani established q-analogues of Saigo fractional integral operators. Building upon this base, the current article aims to generalize Saigo integral operators as well their q-analogues. In addition, we obtain some new results involving extended Saigo integral operators and their q-extensions.

Keywords: integral operators, generalized hypergeometric series, q-gamma functions, q-beta functions and integrals, q-calculus and related topics

Acknowledgements: Kuldipkumar K. Chaudhary is highly appreciative to the Government of India, Ministry of Science and Technology, Department of Science and Technology (DST) for the INSPIRE Fellowship Award (DST/INSPIRE Fellowship/2022/IF220147). Furthermore, the authors deeply appreciate the editor and reviewers for their insightful and comprehensive feedback on our article.

For citation: Chaudhary K. K., Rao S. B. A Note on Extended Saigo Operators and Their q-analogues. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 51, pp. 66–81.
https://doi.org/10.26516/1997-7670.2025.51.66

Научная статья

Заметка о расширенных операторах Saigo и их q-аналогах

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Аннотация. Отмечается, что Мэгуми Сайго ввел обобщенные дробные операторы, включающие гипергеометрическую функцию Гаусса, имеющие четыре частных случая: дробные операторы Римана – Лиувилля, Вейля, Эрдеи – Кобера для левой и правой частей. Мридула Гарг и Лата Чанчалани установили q-аналоги дробных интегральных операторов Сайго. Используя эти результаты, исследуются обобщения интегральных операторов Сайго, а также их q-аналоги. Кроме того, получены некоторые новые результаты, связанные с обобщенными интегральными операторами Сайго и их q-расширениями.

Ключевые слова: интегральные операторы, обобщенные гипергеометрические ряды, q-гамма функции, q-бета функции и интегралы, q-исчисление и смежные темы

Благодарности: Кулдипкумар К. Чаудхари выражает глубокую благодарность Правительству Индии, Министерству науки и технологий, Департаменту науки и технологий (DST) за премию INSPIRE Fellowship (DST/INSPIRE Fellowship/2022/ IF-y220147). Кроме того, авторы искренне благодарят редактора и рецензентов за их содержательные и всесторонние отзывы о нашей статье.

Ссылка для цитирования: Chaudhary K. K., Rao S. B. A Note on Extended Saigo Operators and Their q-analogues // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 51. С. 66–81.
https://doi.org/10.26516/1997-7670.2025.51.66

1. Introduction

The study of extending the concept of integer ordered derivatives and integrals to the concept of fractional order derivatives and integrals is called fractional calculus. Fractional calculus has important applications in the field of Physics and Engineering [10]. One important area of interest within this field is the investigation and derivation of fractional operators, including but not limited to the Riemann–Liouville, Weyl, Erdély-Kober left and right sided fractional operators, Caputo, Marchaud, and other novel operators. These operators are useful to reformulate classical models in fractional models, leading to improved outcomes in applications [4; 11; 13].

In 1978, Megumi Saigo [16] derived two (left and right sided) fractional integral operators involving the classical hypergeometric function with four distinct special cases. Since then, many mathematicians have furnished to the theory and applications in the area of fractional calculus. Rao et al. [15] derived fractional operators involving the Wright-type generalized hypergeometric function and established some new results related to fractional operators involving the generalized hypergeometric function [17]. Additionally, the emergence of q-calculus has further enriched the intersection of mathematics and physics, fostering connections with quantum theory, statistical mechanics, number theory, and combinatorics [5; 6].

The concept of fractional q-calculus, an extension of ordinary fractional calculus to include q-extensions, has found widespread utility in various domains. Beginning with Al-Salam's work on the q-analogue of Cauchy's formula, researchers such as Agarwal, Isogawa, Rajkovic, Saxena, and Yadav have made significant contributions to understanding fractional q-derivatives and q-integral operators [2]. Notably, Garg and Chanchlani extended Saigo's fractional operators to introduce their q-analogues [7], involving the infinite series closely related to q-hypergeometric series [8].

In light of this backdrop, our study delves into the realm of extended Saigo fractional operators, involving generalized hypergeometric function, defined by Virchenko et al. [17] and their q-versions, involving the infinite series closely related to generalized q-hypergeometric function [3], building upon the foundation laid by previous researchers. By exploring these operators, we aim to contribute to the evolution of fractional calculus.

2. Methodology

We obtain the boundary conditions for the newly defined integral operators and derive some interesting results using below mentioned definitions

Definition 1. *For $\mathfrak{z} \in \mathbb{C}$, the q-shifted factorial is given by [6; 8]*

$$(\mathfrak{z};q)_\ell = \begin{cases} 1, \ell = 0 \\ \prod\limits_{m=0}^{\ell-1} (1-\mathfrak{z}q^m), \ell \in \mathbb{N} \end{cases}, \quad (0<|q|<1). \tag{2.1}$$

Definition 2. *The q-extension of the power function is defined as [7]*

$$(a-b)_m = a^m (b/a;q)_m. \tag{2.2}$$

Definition 3. *For $0<|q|<1$, the q-integral is given as [6; 8]*

$$\int\limits_{\mathfrak{a}}^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} = \int\limits_0^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} - \int\limits_0^{\mathfrak{a}} f(\mathfrak{z})\, d_q\mathfrak{z}, \tag{2.3}$$

$$\textit{where} \int\limits_0^{\alpha} f(\mathfrak{z})\, d_q\mathfrak{z} = \alpha(1-q)\sum_{\ell=0}^{\infty} f\left(\alpha q^\ell\right) q^\ell, \tag{2.4}$$

$$\textit{and,} \int\limits_0^{\infty} f(z)\, d_q z = (1-q)\sum_{k=-\infty}^{\infty} f\left(q^k\right) q^k. \tag{2.5}$$

Definition 4. *The q-gamma function is defined as [6; 8]*

$$\Gamma_q(\mathfrak{z}) = \frac{(q;q)_\infty}{(q^{\mathfrak{z}};q)_\infty}(1-q)^{1-\mathfrak{z}}, \qquad (\Re(\mathfrak{z})>0, 0<|q|<1). \tag{2.6}$$

In terms of q-shifted factorial, the q-gamma function is given as [8]

$$\Gamma_q(\alpha) = \frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}}. \tag{2.7}$$

Definition 5. *The q-expression of beta function is given as [6; 8]*

$$\beta_q(z_1,z_2)=(1-q)\sum_{k=0}^{\infty}\frac{(q^{k+1};q)_\infty}{(q^{k+z_2};q)_\infty}q^{kz_1},\quad (z_1,z_2\in\mathbb{C}) \tag{2.8}$$

where $\mathrm{Re}(z_1),\mathrm{Re}(z_2)>0$ *and its integral expression is given as [8]*

$$\beta_q(z_1,z_2)=\int_0^1 x^{z_1-1}\frac{(xq;q)_\infty}{(xq^{z_2};q)_\infty}d_qx. \tag{2.9}$$

The q-beta function is described in terms of q-gamma function as [8]

$$\beta_q(z_1,z_2)=\frac{\Gamma_q(z_1)\Gamma_q(z_2)}{\Gamma_q(z_1+z_2)}. \tag{2.10}$$

Definition 6 (Refer [2])**.** *If* $\gamma\in\mathbb{R}$ *is fixed, a subset* A *of* $\mathbb{C}$ *is called* γ*-geometric if* $\gamma z\in A$ *whenever* $z\in A$*. In addition, if a subset* A *is* γ*-geometric then it contains all geometric sequences* $\{z\gamma^n\}_{n=0}^{\infty}, z\in A$*.*

Proposition 1 (Refer [2])**.** *Let* $\kappa\in\mathbb{C}$ *and let* f *be a function defined on a* γ*-geometric (with* $\gamma=\frac{1}{q}$*) set* A*. We say that* $f\in S_{q,\kappa}$ *if there exists* $\upsilon\in\mathbb{C},\Re(\upsilon)>\Re(\kappa)$ *such that* $|f(xq^{-n-\kappa})|=O(q^{n\upsilon})$ *as* $n\to\infty, x\in A$*. Which means, if* $f\in S_{q,\kappa}$ *then for each* $x>0$ *there exists a constant* $C>0$*,* C *depends on* x *and* κ *such that* $|f(xq^{-n-\kappa})|\le Cq^{n\upsilon}$*.*

3. Main Results

3.1. Extended Saigo integral operators involving generalized hypergeometric function

We define integral operators involving the generalized hypergeometric function as

$$\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi(y)=\frac{1}{y^{\mu+\eta}\Gamma(\mu)}\int_0^y\frac{{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{t}{y}\right)\right)}{(y-t)^{1-\mu}}\xi(t)\,dt, \tag{3.1}$$

$$\text{and, } \mathcal{J}_\tau^{\mu,\eta,\lambda}\xi(y)=\frac{1}{\Gamma(\mu)}\int_y^\infty\frac{{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{y}{t}\right)\right)}{(t-y)^{1-\mu}t^{\mu+\eta}}\xi(t)\,dt, \tag{3.2}$$

where $\mu,\eta,\lambda\in\mathbb{R}$ with $\lambda<0,\mu>0$and $\eta\le 0$ such that $\mu+\eta>0,\lambda>\eta-\frac{1}{4}$. ${}_2R_1(\mathfrak{a},\mathfrak{b};\mathfrak{c};\tau;\mathfrak{z})$ is the generalized Gauss hypergeometric function [17]. For $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathbb{C},\Re(\mathfrak{a}),\Re(\mathfrak{b}),\Re(\mathfrak{c})>0;\tau>0$ and $|\mathfrak{z}|<1$, we have

$${}_2R_1(\mathfrak{a},\mathfrak{b};\mathfrak{c};\tau;\mathfrak{z})=\frac{\Gamma(\mathfrak{c})}{\Gamma(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{(\mathfrak{a})_\ell\Gamma(\mathfrak{b}+\tau\ell)}{\Gamma(\mathfrak{c}+\tau\ell)}\frac{\mathfrak{z}^\ell}{\ell!},\quad (\mathfrak{z}\in\mathbb{C}). \tag{3.3}$$

Remark 1. Special cases

1) Taking $\eta = -\mu$ in (3.1) and (3.2), we get left sided Riemann-Liouville and Weyl fractional integral operators [4; 11; 13], respectively.
2) Taking $\eta = 0$ in (3.1) and (3.2), we get left and right sided Erdély-Kober fractional integral operators [4; 11; 13], respectively.
3) When $\tau = 1$, (3.1) and (3.2) reduce to left and right sided fractional integral operators [16], respectively.

3.2. Some results on extended Saigo integral operators

Theorem 1. *If $\mu, \eta, \lambda \in \mathbb{R}$ with $\lambda < 0, \mu > 0, \eta \leq 0$ such that $\mu + \eta > 0$ and ξ be a function with $\xi \in L^2[0, c]$ then $\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi \in L^2[0, c]$ and*

$$\left\|\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi\right\|_2 \leq \left(\int_0^c |\xi(t)|^2 dt\right)^{\frac{1}{2}} S, \tag{3.4}$$

where

$$S = \left|\frac{c^{\frac{1}{2}-\eta}(-2\eta-1)^{-\frac{1}{2}}}{\Gamma(-\lambda)\Gamma(\mu+\eta)}\right| \left({}_4\Psi_3\left[\begin{matrix}(-\lambda,1),(-\lambda,1),(\mu+\eta,\tau),(\mu+\eta,\tau)\\(\mu,\tau),(\mu,\tau),(1,1)\end{matrix};1\right]\right)^{\frac{1}{2}}.$$

Proof. Consider,

$$\left(\left\|\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi\right\|_2\right)^2 = \int_0^c \left|\frac{y^{-\mu-\eta}}{\Gamma(\mu)}\int_0^y \frac{{}_2R_1\left(-\lambda, \mu+\eta; \mu; \tau; \left(1-\frac{t}{y}\right)\right)}{(y-t)^{1-\mu}}\xi(t)\,dt\right|^2 dy \leq$$

$$\leq \left|\frac{1}{\Gamma(\mu+\eta)}\right|^2 \sum_{k=0}^{\infty}\left|\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}\right|^2$$

$$\int_0^c |\xi(t)|^2 \int_c^t \left(y^{-\mu-\eta-k}\right)^2\left((y-t)^{\mu+k-1}\right)^2 dydt =$$

$$= \left|\frac{1}{\Gamma(\mu+\eta)}\right|^2 \sum_{k=0}^{\infty}\left|\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{c^{(\mu+\eta+k)}\Gamma(\mu+\tau k)\,k!}\right|^2 \beta(1, 2(\mu+k)-1)$$

$$\int_0^c \frac{|\xi(t)|^2}{(t-c)^{-2(\mu+k)}}\,{}_2F_1\left(2(\mu+\eta+k), 1; 2(\mu+k); \frac{c-t}{c}\right)dt$$

$$\leq \left|\frac{1}{\Gamma(\mu+\eta)}\right|^2 \sum_{k=0}^{\infty}\left(\left|\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{c^{(\mu+\eta+k)}\Gamma(\mu+\tau k)\,k!}\right|^2 \frac{1}{(2(\mu+k)-1)}\right.$$

$$\left.\sum_{m=0}^{\infty}\frac{(2(\mu+\eta+k))_m(1)_m}{c^{2m}(2(\mu+k))_m m!}\int_0^c |\xi(t)|^2|(c-t)|^{2(\mu+k+m)}dt\right).$$

Which implies

$$\left\|\mathcal{I}_{\tau}^{\mu,\eta,\lambda}\xi\right\|_{2}$$

$$\leq\left(\left|\frac{1}{\Gamma(\mu+\eta)}\right|^{2}\sum_{k=0}^{\infty}\left(\left|\frac{(-\lambda)_{k}\Gamma(\mu+\eta+\tau k)}{c^{(\mu+\eta+k)}\Gamma(\mu+\tau k)\,k!}\right|^{2}\frac{1}{(2(\mu+k)-1)}\right.\right.$$

$$\left.\left.\sum_{m=0}^{\infty}\frac{(2(\mu+\eta+k))_{m}(1)_{m}}{c^{2m}(2(\mu+k))_{m}m!}\int_{0}^{c}|\xi(t)|^{2}|(c-t)|^{2(\mu+k+m)}dt\right)\right)^{\frac{1}{2}}$$

$$\leq\left(\left|\frac{1}{\Gamma(\mu+\eta)}\right|^{2}\sum_{k=0}^{\infty}\left(\left|\frac{(-\lambda)_{k}\Gamma(\mu+\eta+\tau k)}{c^{(\mu+\eta+k)}\Gamma(\mu+\tau k)\,k!}\right|^{2}\frac{1}{(2(\mu+k)-1)}\right.\right.$$

$$\left.\left.\sum_{m=0}^{\infty}\frac{(2(\mu+\eta+k))_{m}(1)_{m}}{c^{2m}(2(\mu+k))_{m}m!}\int_{0}^{c}|(c-t)|^{2(\mu+k+m)}dt\right)\right)^{\frac{1}{2}}\left(\int_{0}^{c}|\xi(t)|^{2}dt\right)^{\frac{1}{2}}$$

$$\leq\left(\left|\frac{1}{\Gamma(\mu+\eta)}\right|^{2}\sum_{k=0}^{\infty}\left(\left|\frac{(-\lambda)_{k}\Gamma(\mu+\eta+\tau k)}{2(\mu+k)\Gamma(\mu+\tau k)\,k!}\right|^{2}\right.\right.$$

$$\left.\left.\frac{c^{1-2\eta}}{(2(\mu+k)-1)}{}_{2}F_{1}\left(2(\mu+\eta+k),1;2(\mu+k);1\right)\right)\right)^{\frac{1}{2}}\left(\int_{0}^{c}|\xi(t)|^{2}dt\right)^{\frac{1}{2}}.$$

Using the Gauss summation theorem [14]; ${}_{2}F_{1}(\mathfrak{s},\mathfrak{b};\mathfrak{r};1)=\frac{\Gamma(\mathfrak{r})\Gamma(\mathfrak{r}-\mathfrak{s}-\mathfrak{b})}{\Gamma(\mathfrak{r}-\mathfrak{s})\Gamma(\mathfrak{r}-\mathfrak{b})}$, and further simplifications, the above expression reduces to

$$\left(\int_{0}^{c}|\xi(t)|^{2}dt\right)^{\frac{1}{2}}\left|\frac{1}{\Gamma(\mu+\eta)}\right|\left(\frac{c^{1-2\eta}}{(\Gamma(-\lambda))^{2}(-2\eta-1)}\right.$$

$$\left.\sum_{k=0}^{\infty}\frac{\Gamma(-\lambda+k)\Gamma(-\lambda+k)\Gamma(\mu+\eta+\tau k)\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\Gamma(\mu+\tau k)\Gamma(1+k)}\frac{1^{k}}{k!}\right)^{\frac{1}{2}}.$$

Which completes the proof. ☐

Theorem 2. *If $\mu,\eta,\lambda\in\mathbb{R}$ with $\lambda<0,\mu>0,\eta\leq 0$ such that $\mu+\eta>0,\lambda>\eta-\frac{1}{4},|\omega|<4$ and $\xi\in L^{2}[0,\infty)$, then*

$$\left\|\mathcal{J}_{\tau}^{\mu,\eta,\lambda}\xi\right\|_{2}\leq\left(\int_{0}^{\infty}|\xi(t)|^{2}dt\right)^{\frac{1}{2}}\mathcal{G},\tag{3.5}$$

where $\mathcal{G}=\left|\frac{1}{\Gamma(\mu+\eta)}\right|\left(\frac{\omega^{2(\mu-\eta)-1}}{\Gamma(-\lambda)\Gamma(-\lambda)}\right.$

$$\left.{}_{6}\Psi_{4}\left[\begin{matrix}(-\lambda,1),(-\lambda,1),(\mu+\eta,\tau),(\mu+\eta,\tau),(2\mu,2),(2(\mu-\eta)+1,2)\\(\mu,\tau),(\mu,\tau),(1,1),(4\mu-2\eta+1,4)\end{matrix};\omega^{2}\right]\right)^{\frac{1}{2}}.$$

Proof. Let us begin with

$$
\begin{aligned}
&\left(\left\|\mathcal{J}_{\tau}^{\mu,\eta,\lambda}\ \xi\right\|_2\right)^2 \\
&= \int_{\omega}^{\infty} \left| \frac{1}{\Gamma(\mu)} \int_{y}^{\infty} (t-y)^{\mu-1} t^{-\mu-\eta} {}_2R_1\left(-\lambda, \mu+\eta; \mu; \tau; \left(1-\frac{y}{t}\right)\right) \xi(t)\, dt \right|^2 dy \\
&\leq \sum_{k=0}^{\infty} \left| \frac{(-\lambda)_k \Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\eta)\Gamma(\mu+\tau k)\, k!} \right|^2 \int_{\omega}^{\infty} \left| \frac{\xi(t)}{\left|t^{\mu+\eta+k}\right|^2} \right|^2 \int_{\omega}^{t} \left| (t-y)^{\mu+k-1} \right|^2 dy dt \\
&\leq \left| \frac{1}{\Gamma(\mu+\eta)} \right|^2 \sum_{k=0}^{\infty} \left| \frac{(-\lambda)_k \Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\, k!} \right|^2 \frac{1}{(2(\mu+k)-1)} \\
&\qquad \int_{\omega}^{\infty} |\xi(t)|^2 \left| t^{-1-2\eta} (t-\omega)^{2(\mu+k)-1} \right| dt \\
&\leq \left(\left| \frac{1}{\Gamma(\mu+\eta)} \right|^2 \sum_{k=0}^{\infty} \left| \frac{(-\lambda)_k \Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\, k!} \right|^2 \right. \\
&\qquad \left. \frac{1}{(2(\mu+k)-1)} \int_{\omega}^{\infty} |\xi(t)|^2 \left| t^{-\frac{1}{2}-\eta} (t-\omega)^{(\mu+k)-\frac{1}{2}} \right|^2 dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_{0}^{\infty} |\xi(t)|^2 dt \right)^{\frac{1}{2}} \left(\left| \frac{1}{\Gamma(\mu+\eta)} \right|^2 \sum_{k=0}^{\infty} \left| \frac{(-\lambda)_k \Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\, k!} \right|^2 \right. \\
&\qquad \left. \frac{1}{(2(\mu+k)-1)} \int_{\omega}^{\infty} \left| t^{-\frac{1}{2}-\eta} (t-\omega)^{(\mu+k)-\frac{1}{2}} \right|^2 dt \right)^{\frac{1}{2}}
\end{aligned}
$$

Using the integral representation of beta function and further simplifications leads us to (3.5). □

Theorem 3. *For $\mu, \eta, \lambda \in \mathbb{R}$ with $\lambda < 0, \mu > 0, \eta \leq 0$ such that $\mu + \eta > 0, \lambda > \eta - \frac{1}{4}, \Re(\sigma) > -1$ and $y > \alpha > 0$, we have*

$$
\begin{aligned}
\mathcal{I}_{\tau}^{\mu,\eta,\lambda}(y-\alpha)^{\sigma} = & \frac{y^{-\mu-\eta}}{\Gamma(\mu+\eta)} \sum_{k=0}^{\infty} \frac{(-\lambda)_k \Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\, k!} \\
& \frac{(y-\alpha)^{\sigma+\mu+k}}{y^k} \beta\left(\frac{y}{y-\alpha}; \mu+k, \sigma+1\right). \quad (3.6)
\end{aligned}
$$

Proof. By taking $\xi(y) = (y-\alpha)^{\sigma}$ in (3.1), we get

$$\begin{aligned}
&\mathcal{I}_{\tau}^{\mu,\eta,\lambda}(y-\alpha)^{\sigma}\\
&= \frac{y^{-\mu-\eta}}{\Gamma(\mu)}\int_{0}^{y}(y-t)^{\mu-1}\,{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{t}{y}\right)\right)(t-\alpha)^{\sigma}dt\\
&= \frac{y^{-\mu-\eta}}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}\left(\frac{1}{y}\right)^{k}\int_{0}^{y}(y-t)^{\mu+k-1}(t-\alpha)^{\sigma}dt\\
&= \frac{y^{-\mu-\eta}}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}\left(\frac{1}{y}\right)^{k}\int_{0}^{y}(t)^{\mu+k-1}(y-t-\alpha)^{\sigma}dt
\end{aligned}$$

Taking $t = (y-\alpha)u$ and further simplification leads us to

$$\frac{y^{-\mu-\eta}}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}\frac{(y-\alpha)^{\sigma+\mu+k}}{y^{k}}\beta\left(\frac{y}{y-\alpha};\mu+k,\sigma+1\right),$$

where $\beta(z;a,b)$ is incomplete beta function and is given by [12]
$\beta(z;a,b) = \int_{0}^{z} u^{a-1}(1-u)^{b-1}du.$ □

Theorem 4. *For $\mu,\eta,\lambda \in \mathbb{R}$ with $\lambda < 0, \mu > 0, \eta \leq 0$ such that $\mu+\eta > 0, \lambda > \eta - \frac{1}{4}$; $\Re(\sigma) < 0$ such that $\eta > \Re(\sigma)$ and $y > \alpha$, we have*

$$\left(\mathcal{J}_{\tau}^{\mu,\eta,\lambda}(y-\alpha)^{\sigma}\right) = \frac{1}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}I, \tag{3.7}$$

where, $I = \frac{(y-\alpha)^{\sigma}}{y^{\eta}}\beta(\mu+k,\eta-\sigma)\,{}_2F_1\left(-\sigma,\mu+k;\mu+\eta+k-\sigma;\left(\frac{-\alpha}{y-\alpha}\right)\right).$

Proof. Let us consider,

$$\begin{aligned}
&\mathcal{J}_{\tau}^{\mu,\eta,\lambda}(y-\alpha)^{\sigma}\\
&= \frac{1}{\Gamma(\mu)}\int_{y}^{\infty}(t-y)^{\mu-1}t^{-\mu-\eta}{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{y}{t}\right)\right)(t-\alpha)^{\sigma}dt\\
&= \frac{1}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}\int_{y}^{\infty}(t-y)^{\mu+k-1}t^{-\mu-\eta-k}(t-\alpha)^{\sigma}dt\\
&= \frac{1}{\Gamma(\mu+\eta)}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\Gamma(\mu+\eta+\tau k)}{\Gamma(\mu+\tau k)\,k!}I.
\end{aligned}$$

The integral, $I=\int\limits_y^\infty (t-y)^{\mu+k-1}t^{-\mu-\eta-k}(t-\alpha)^\sigma dt$, in above term can be rewritten as $I=\int\limits_y^\infty \left(\frac{t-y}{t}\right)^{\mu+k-1}t^{-\eta-1}(t-\alpha)^\sigma dt$. Now, taking $t=\frac{y}{1-u}$ and by further simplification, we have the proof. □

Theorem 5. *For $\mu,\eta,\lambda\in\mathbb{R}$ with $\lambda<0,\mu>0,\eta\leq 0$ such that $\mu+\eta>0,\lambda>\eta-\frac{1}{4},\Re(\sigma)>-1;\tau>0, and, x>0$, we have*

$$\int\limits_0^\infty \phi(x)\left[\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi(x)\right]dx=\int\limits_0^\infty \xi(x)\left[\mathcal{J}_\tau^{\mu,\eta,\lambda}\phi(x)\right]dx. \tag{3.8}$$

Proof. Consider the left hand side of (3.8),

$$\begin{aligned}&\int\limits_0^\infty \phi(x)\left[\mathcal{I}_\tau^{\mu,\eta,\lambda}\xi(x)\right]dx\\&=\int\limits_0^\infty \frac{\phi(x)}{x^{\mu+\eta}\Gamma(\mu)}\int\limits_0^x (x-t)^{\mu-1}\,{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{t}{x}\right)\right)\xi(t)\,dtdx\\&=\frac{1}{\Gamma(\mu)}\int\limits_0^\infty \frac{\phi(x)}{x^{\mu+\eta}}\int\limits_0^x (x-t)^{\mu-1}\,{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{t}{x}\right)\right)\xi(t)\,dtdx\\&=\frac{1}{\Gamma(\mu)}\int\limits_0^\infty \xi(t)\int\limits_t^\infty \frac{(x-t)^{\mu-1}}{x^{\mu+\eta}}\,{}_2R_1\left(-\lambda,\mu+\eta;\mu;\tau;\left(1-\frac{t}{x}\right)\right)\phi(x)\,dxdt.\end{aligned}$$

Which is equal to right side of (3.8). □

3.3. Extended Saigo q-integral operators

In this section, we define q-analogues of (3.1) and (3.2) as below;

$$\begin{aligned}\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda}\xi(y)=&\frac{y^{-\eta-1}}{\Gamma_q(\mu)}\int\limits_0^y\left(\frac{tq}{y};q\right)_{\mu-1}\sum_{m=0}^\infty\frac{(q^{-\lambda};q)_m(q^{\mu+\eta};q)_{\tau m}}{(q^\mu;q)_{\tau m}(q;q)_m}\\&\frac{(-1)^m q^{-\binom{m}{2}}}{q^{(\eta-\lambda)m}}\left(\frac{t}{y}-1\right)_m\xi(t)\,d_qt,\end{aligned}\tag{3.9}$$

$$\text{and, } \mathcal{J}_{\tau,q}^{\mu,\eta,\lambda}\xi\left(y\right)=\frac{q^{\frac{-\mu(\mu+1)}{2-\eta}}}{\Gamma_q\left(\mu\right)}\int_y^{\infty}\left(\frac{y}{t};q\right)_{\mu-1}t^{-\eta-1}\sum_{m=0}^{\infty}\frac{\left(q^{-\lambda};q\right)_m(q^{\mu+\eta};q)_{\tau m}}{(q^{\mu};q)_{\tau m}(q;q)_m}$$
$$\frac{(-1)^m q^{-\binom{m}{2}}}{q^{(\eta-\lambda)m}}\left(\frac{t}{y}-1\right)_m\xi\left(tq^{1-\mu}\right)d_qt, \quad (3.10)$$

where $\mu,\eta,\lambda\in\mathbb{R}$ with $\lambda<0,\mu>0$and $\eta\leq 0$ such that $\mu+\eta>0,\lambda>\eta-\frac{1}{4};\tau>0$ and $0<|q|<1$.

Remark 2. Special cases

1) Letting $q\to 1$ in (3.9) and (3.10), we get (3.1) and (3.2), respectively.

2) Clearly, q-Riemann-Liouville and q-Weyl operators [2] are special cases of (3.9) and (3.10), respectively, when $\eta=-\mu$.

3) Taking $\tau=1$ in (3.9) and (3.10), we get left and right sided Saigo fractional integral operators [16], respectively.

3.4. Some results on extended Saigo q-integral operators

Lemma 1. *The summation formulas of* (3.9) *and* (3.10) *are given as*

$$\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda}\xi\left(y\right)=y^{-\eta}(1-q)^{\mu}\times\sum_{m=0}^{\infty}\left(\frac{\left(q^{-\lambda};q\right)_m(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}\right.$$
$$\left.q^{(\lambda-\eta+1)m}\sum_{k=0}^{\infty}q^k\frac{\left(q^{\mu+m};q\right)_k}{\left(q;q\right)_k}\xi\left(yq^{k+m}\right)\right), \quad (3.11)$$

and, $$\mathcal{J}_{\tau,q}^{\mu,\eta,\lambda}\xi\left(y\right)=\frac{(1-q)^{\mu}}{q^{\frac{\mu(\mu+1)}{2}}y^{\eta}}\sum_{m=0}^{\infty}\left(\frac{\left(q^{-\lambda};q\right)_m(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}\right.$$
$$\left.q^{\lambda m}\sum_{k=0}^{\infty}q^{\eta k}\frac{\left(q^{\mu+m};q\right)_k}{\left(q;q\right)_k}\xi\left(yq^{-\mu-k-m}\right)\right). \quad (3.12)$$

Proof. By using (2.4) and (2.7) in (3.9), we get

$$\mathcal{I}^{\mu,\eta,\lambda}_{\tau,q}\xi\left(y\right)=\frac{y^{-\eta-1}(1-q)^{\mu-1}}{(q;q)_{\mu-1}}\times\sum_{m=0}^{\infty}\frac{(q^{\mu+\eta};q)_{\tau m}\left(q^{-\lambda};q\right)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}\frac{q^{(\lambda-\eta)m}(-1)^m}{q^{-\frac{m(m-1)}{2}}}$$

$$y\left(1-q\right)\sum_{j=0}^{\infty}q^j\left(yq^jq/y;q\right)_{\mu-1}\left(\frac{yq^j}{y}-1\right)_m\xi\left(yq^j\right)$$

$$=\frac{y^{-\eta}(1-q)^{\mu}}{(q;q)_{\mu-1}}\times\sum_{m=0}^{\infty}\left(\frac{(q^{\mu+\eta};q)_{\tau m}\left(q^{-\lambda};q\right)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}\right.$$

$$\left.\frac{q^{(\lambda-\eta)m}(-1)^m}{q^{-\frac{m(m-1)}{2}}}\sum_{j=0}^{\infty}q^j\left(q^{j+1};q\right)_{\mu-1}\left(q^j-1\right)_m\xi\left(yq^j\right)\right).$$

Applying (2.2) in above expression, we get

$$\mathcal{I}^{\mu,\eta,\lambda}_{\tau,q}\xi\left(y\right)=y^{-\eta}(1-q)^{\mu}\times\sum_{m=0}^{\infty}\left(\frac{(q^{\mu+\eta};q)_{\tau m}\left(q^{-\lambda};q\right)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}\right.$$

$$\left.\frac{q^{(\lambda-\eta)m}(-1)^m}{q^{-\frac{m(m-1)}{2}}}\sum_{j=0}^{\infty}q^j\frac{\left(q^{j+1};q\right)_{\mu-1}}{(q;q)_{\mu-1}}\left[\left(q^{-j};q\right)_m\left(-q^j\right)^mq^{-\frac{m(m-1)}{2}}\right]\xi\left(yq^j\right)\right).$$

Using the identity (see [8]) $(a;q)_m\left(-a^{-1}\right)^mq^{-\frac{m(m-1)}{2}}=\left(a^{-1}q^{1-m};q\right)_m$ and further simplifications leads us to

$$\mathcal{I}^{\mu,\eta,\lambda}_{\tau,q}\xi\left(y\right)=y^{-\eta}(1-q)^{\mu}\times\sum_{m=0}^{\infty}\left(\frac{(q^{\mu+\eta};q)_{\tau m}\left(q^{-\lambda};q\right)_m}{(q^{\mu};q)_{\tau m}(q;q)_m}q^{(\lambda-\eta)m}\right.$$

$$\left.\sum_{k=0}^{\infty}q^{m+k}\frac{\left(q^{m+k+1};q\right)_{\mu-1}}{(q;q)_{\mu-1}}\left(q^{k+1};q\right)_m\xi\left(yq^{m+k}\right)\right).$$

Finally, by using $(a;q)_{\alpha}=\frac{(a;q)_{\infty}}{(aq^{\alpha};q)_{\infty}}$, and, $(a;q)_{m+k}=(a;q)_m(aq^m;q)_k$, we get the right hand side of (3.11). With the help of same identity operations in (3.10), we can prove (3.12). □

Theorem 6. *For* $\mu,\eta,\lambda\in\mathbb{R}$ *with* $\lambda<0,\mu>0$*and* $\eta\leq 0$ *such that* $\mu+\eta>0,\lambda>\eta-\frac{1}{4};\tau>0$ *and* $0<|q|<1$;

$$\mathcal{I}^{\mu,\eta,\lambda}_{\tau,q}y^{\sigma}=\frac{(1-q)^{\mu}}{y^{\eta-\sigma}}\sum_{m=0}^{\infty}\frac{\left(q^{-\lambda};q\right)_m(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_m}{(q^{\mu};q)_{\tau m}(q;q)_m(q^{1+\sigma};q)_{\mu+m}}q^{(\lambda-\eta+\sigma+1)m},\quad(3.13)$$

$$\textit{and, }\mathcal{J}^{\mu,\eta,\lambda}_{\tau,q}y^{\sigma}=\frac{y^{\sigma-\eta}(1-q)^{\mu}}{q^{\frac{\mu(\mu+1)}{2}+\mu\sigma}}\sum_{m=0}^{\infty}\frac{\left(q^{-\lambda};q\right)_m(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_m}{(q^{\mu};q)_{\tau m}(q;q)_m(q^{\eta-\sigma};q)_{\mu+m}}q^{(\lambda-\sigma)m}.\quad(3.14)$$

Proof. To prove (3.13), let us begin by taking $\xi(y) = y^{\sigma}$ in (3.11), we get

$$\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda} y^{\sigma} = y^{\sigma-\eta}(1-q)^{\mu} \times \left(\sum_{m=0}^{\infty} \frac{(q^{-\lambda};q)_m (q^{\mu+\eta};q)_{\tau m} (q^{\mu};q)_m}{(q^{\mu};q)_{\tau m} (q;q)_m} q^{(\lambda-\eta+\sigma+1)m} \sum_{k=0}^{\infty} q^{k(1+\sigma)} \frac{(q^{\mu+m};q)_k}{(q;q)_k} \right).$$

Using q-binomial identity (refer [6;8]), we get the right hand side of (3.13). In the same manner, we can prove (3.14). □

Theorem 7. *If $f(x)$ and $g(x)$ are functions expressible in power series with radii of convergence R_1 and R_2 respectively, then for $\mu, \eta, \lambda \in \mathbb{R}$ with $\lambda < 0, \mu > 0$and $\eta \le 0$ such that $\mu + \eta > 0; \tau > 0; 0 < |q| < 1$, we have the following result*

$$\int_0^{\infty} f(y)\, \mathcal{J}_{\tau,q}^{\mu,\eta,\lambda} g(y)\, d_q y = q^{-\frac{\mu(\mu+1)}{2}} \int_0^{\infty} g\left(yq^{-\mu}\right) \mathcal{I}_{\tau,q}^{\mu,\eta,\lambda} f(y)\, d_q y, \tag{3.15}$$

provided the q-integrals exist.

Proof. Using (3.12) in the left hand side of (3.15), we have

$$\int_0^{\infty} f(y)\, \mathcal{J}_{\tau,q}^{\mu,\eta,\lambda} g(y)\, d_q y = \frac{(1-q)^{\mu}}{q^{\frac{\mu(\mu+1)}{2}}} \sum_{m=0}^{\infty} \left(\frac{(q^{-\lambda};q)_m (q^{\mu+\eta};q)_{\tau m} (q^{\mu};q)_m}{(q^{\mu};q)_{\tau m} (q;q)_m} q^{\lambda m} \sum_{k=0}^{\infty} q^{\eta k} \frac{(q^{\mu+m};q)_k}{(q;q)_k} \int_0^{\infty} f(y)\, y^{-\eta} g\left(yq^{-\mu-k-m}\right) d_q y \right) d_q x.$$

On interchanging the order of integration and summations and applying the formula (2.5), the above expression is equal to

$$\frac{(1-q)^{\mu}}{q^{\frac{\mu(\mu+1)}{2}}} \times (1-q) \sum_{r=-\infty}^{\infty} \left(\sum_{m=0}^{\infty} \left(\frac{(q^{-\lambda};q)_m (q^{\mu+\eta};q)_{\tau m} (q^{\mu};q)_m}{(q^{\mu};q)_{\tau m} (q;q)_m} q^{(\lambda-\eta+1)m} \sum_{k=0}^{\infty} q^{k} \frac{(q^{\mu+m};q)_k}{(q;q)_k} q^{r} q^{-\eta r} f\left(q^{r+m+k}\right) g\left(q^{r-\mu}\right) \right) \right).$$

Again, by using (2.5) and thanks to (3.11), we get the right hand side of (3.15). □

Theorem 8. *For $\mu, \eta, \lambda \in \mathbb{R}$ with $\lambda < 0, \mu > 0$ and $\eta \le 0$ such that $\mu + \eta > 0; \tau > 0; 0 < |q| < 1; \left|q^{\lambda}(1-q)\right| < 1$ and $\xi(y)$ is defined on*

$L_q^1[0,a]$*; we have*

$$\int_0^a \left|\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda}\xi(y)\right| d_qy \leq \mathcal{B}\int_0^a |\xi(y)|\, d_qy, \tag{3.16}$$

where $\mathcal{B}=\frac{a^{-\eta}(1-q)^{1-\eta}(q;q)_\infty}{(q^\eta;q)_\infty\Gamma_q(-\lambda)\Gamma_q(\mu+\eta)}{}_3\psi_2\left[\begin{matrix}(-\lambda,1),(\mu+\eta,\tau),(\mu,1)\\(\mu,\tau)(\mu+\eta,1)\end{matrix}\middle| q,\left(\frac{q^\lambda}{(1-q)^{-1}}\right)\right]$.

Proof. Consider,

$$\begin{aligned}&\int_0^a |\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda}\,\xi(y)|\, d_qy\\ &= (1-q)^\mu \times \sum_{m=0}^\infty\left(\left|\frac{(q^{-\lambda};q)_m(q^{\mu+\eta};q)_{\tau m}(q^\mu;q)_m}{(q^\mu;q)_{\tau m}(q;q)_m}q^{(\lambda-\eta+1)m}\right|\right.\\ &\quad\left.\sum_{k=0}^\infty\left|q^k\frac{(q^{\mu+m};q)_k}{(q;q)_k}\right|\times a(1-q)\sum_{r=0}^\infty q^r\left|q^{-r\eta}a^{-\eta}\xi\left(aq^{r+k+m}\right)\right|\right).\end{aligned}$$

On series rearrangement techniques and using (2.4), we get the inequality

$$\begin{aligned}&\int_0^a |\mathcal{I}_{\tau,q}^{\mu,\eta,\lambda}\,\xi(y)|\, d_qy\\ &\leq a^{1-\eta}(1-q)^{\mu+1}\sum_{r=0}^\infty\left(q^{r(1-\eta)}|\xi(aq^r)|\sum_{m=0}^\infty\left(\frac{(q^{-\lambda};q)_m(q^{\mu+\eta};q)_{\tau m}(q^\mu;q)_m}{(q^\mu;q)_{\tau m}(q;q)_m}\right.\right.\\ &\quad\left.\left. q^{\lambda m}\sum_{k=0}^r\left|q^{k\eta}\frac{(q^{\mu+m};q)_k}{(q;q)_k}\right|\right)\right)\\ &\leq a^{1-\eta}(1-q)^{\mu+1}\sum_{r=0}^\infty\left(q^r|\xi(aq^r)|\sum_{m=0}^\infty\left(\frac{(q^{-\lambda};q)_m(q^{\mu+\eta};q)_{\tau m}(q^\mu;q)_m}{(q^\mu;q)_{\tau m}(q;q)_m}\right.\right.\\ &\quad\left.\left. q^{\lambda m}\sum_{k=0}^\infty\left(q^{k\eta}\frac{(q^{\mu+m};q)_k}{(q;q)_k}\right)\right)\right),\end{aligned}$$

which reduces to the right hand side of (3.16). □

Theorem 9. *For* $\mu,\eta,\lambda\in\mathbb{R}$ *with* $\lambda<0,\mu>0$*and* $\eta\leq 0$ *such that* $\mu+\eta>0;\tau>0;0<|q|<1$ *and* $\xi\in S_{q,\kappa}$*, we have*

$$\begin{aligned}\left|\mathcal{J}_{\tau,q}^{\mu,\eta,\lambda}\xi(y)\right| \leq & \frac{C(q;q)_\infty(1-q)^{1-\eta-\Re(\upsilon)}y^{-\eta}}{q^{\frac{\mu(\mu+1)}{2}}\left(q^{\eta+\Re(\upsilon)};q\right)_\infty\Gamma_q(-\lambda)\Gamma_q(\mu+\eta)}\\ & {}_3\psi_2\left[\begin{matrix}(-\lambda,1),(\mu+\eta,\tau),(\mu,1)\\(\mu,\tau),(\mu+\eta+\Re(\upsilon),1)\end{matrix}\middle| q,\frac{q^{(\lambda+\Re(\upsilon))}}{(1-q)}\right].\end{aligned} \tag{3.17}$$

Proof. Let us begin us left hand side of (3.17).

$$\left|\mathcal{J}_{\tau,q}^{\mu,\eta,\lambda}\xi\left(y\right)\right| = q^{-\frac{\mu(\mu+1)}{2}}(1-q)^{\mu}\left|y^{-\eta}\times\sum_{m=0}^{\infty}\left(\frac{\left(q^{-\lambda};q\right)_{m}(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_{m}}{(q^{\mu};q)_{\tau m}(q;q)_{m}}\right.\right.$$

$$\left.\left.q^{\lambda m}\sum_{k=0}^{\infty}q^{\eta k}\frac{(q^{\mu+m};q)_{k}}{(q;q)_{k}}\xi\left(yq^{-\mu-k-m}\right)\right)\right|$$

$$\leq\frac{(1-q)^{\mu}y^{-\eta}}{q^{\frac{\mu(\mu+1)}{2}}}\times\sum_{m=0}^{\infty}\left(\left|\frac{\left(q^{-\lambda};q\right)_{m}(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_{m}}{(q^{\mu};q)_{\tau m}(q;q)_{m}}q^{\lambda m}\right|\right.$$

$$\left.\sum_{k=0}^{\infty}q^{\eta k}\left|\frac{(q^{\mu+m};q)_{k}}{(q;q)_{k}}\right|\left|\xi\left(yq^{-\mu-(k+m)}\right)\right|\right).$$

Due to the beautiful proposition 1, we can rewrite the above expression as

$$\left|\mathcal{J}_{\tau,q}^{\mu,\eta,\lambda}\xi\left(y\right)\right|\leq\frac{(1-q)^{\mu}y^{-\eta}}{q^{\frac{\mu(\mu+1)}{2}}}\times\sum_{m=0}^{\infty}\left(\left|\frac{\left(q^{-\lambda};q\right)_{m}(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_{m}}{(q^{\mu};q)_{\tau m}(q;q)_{m}}q^{\lambda m}\right|\right.$$

$$\left.\sum_{k=0}^{\infty}q^{\eta k}\left|\frac{(q^{\mu+m};q)_{k}}{(q;q)_{k}}\right|\left|Cq^{(k+m)v}\right|\right.$$

$$\leq C\frac{(1-q)^{\mu}y^{-\eta}}{q^{\frac{\mu(\mu+1)}{2}}}\times\sum_{m=0}^{\infty}\left(\left|\frac{\left(q^{-\lambda};q\right)_{m}(q^{\mu+\eta};q)_{\tau m}(q^{\mu};q)_{m}}{(q^{\mu};q)_{\tau m}(q;q)_{m}}\right|\right.$$

$$\left.\left|q^{(\lambda+\Re(v))m}\right|\frac{\left(q^{\mu+m+\eta+\Re(v)};q\right)_{\infty}}{\left(q^{\eta+\Re(v)};q\right)_{\infty}}\right).$$

Which reduces to (3.17). □

4. Conclusion

The present research work aims to study properties of extension of left and right Saigo fractional integral operators [16]; which includes boundedness of these operators and other results inspired from Garg and Chanchlani [7], also authors elegantly and successfully have established in the Theorem-3.5 about the relationships between both the left and right type extended Saigo integral operators. Q-analogues of these operators have been invented in terms of new definitions and analysed, which along with other observations, adds flavour by obtaining summation formulae for these q-analogues. Finally relations between these q-analogues of extended Saigo operators have been obtained along with results relating to their boundedness.

Significance of the present study lies in the fact that, in the era where people are working on new techniques with sophisticated mathematics, authors here tried to highlight the theoretical beauty of relatively old definitions to the mathematical community, appealing mathematicians to explore such historical ideas in applications. As an aesthetical point of view also, the present work serves the purpose of torch-bearer again about intersection of traditional fractional calculus and q-calculus.

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Поступила в редакцию / Received 02.06.2024
Поступила после рецензирования / Revised 20.08.2024
Принята к публикации / Accepted 21.08.2024