Серия «Математика»
2025. Т. 51. С. 50—65

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 517.9
MSC 34A08, 39A12, 39A27
DOI https://doi.org/10.26516/1997-7670.2025.51.50

Positive Solutions of Nabla Fractional Sturm–Liouville Problems

Jagan Mohan Jonnalagadda[1✉], **Juan E. Nápoles Valdés**[2,3]

[1] Birla Institute of Technology and Science, Pilani, Hyderabad, Telangana, India
[2] National University of the Northeast, Corrientes, Argentina
[3] National Technological University, Resistencia, Argentina
✉ j.jaganmohan@hotmail.com

Abstract. This article discusses the existence of positive solutions to Sturm–Liouville boundary value problems for Riemann–Liouville nabla fractional difference equations. The results obtained here shall generalize the existing ones. We provide a few examples to illustrate the applicability of established results.

Keywords: Riemann–Liouville fractional difference, Sturm–Liouville problem, cone, existence, positive solution

For citation: Jonnalagadda J. M., Valdés J. E. N. Positive Solutions of Nabla Fractional Sturm-Liouville Problems. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 51, pp. 50–65.
https://doi.org/10.26516/1997-7670.2025.51.50

Научная статья

Положительные решения дробных задач Штурма – Лиувилля

Д. М. Джонналагадда[1✉], **Х. Э. На́полес Вальде́с**[2,3]

[1] Институт технологий и науки Бирлы, Пилани, Хайдарабад, Телангана, Индия
[2] Северо-Восточный национальный университет, Корриентес, Аргентина
[3] Национальный технологический университет, Ресистенсия, Аргентина
✉ j.jaganmohan@hotmail.com

Аннотация. Обсуждается вопрос о существовании положительных решений краевых задач Штурма – Лиувилля для дробно-разностных уравнений Римана – Лиувилля. Полученные результаты обобщают уже существующие. Приведены несколько примеров, иллюстрирующих применимость установленных результатов.

Ключевые слова: дробная разность Римана – Лиувилля, задача Штурма – Лиувилля, конус, существование, положительное решение

Ссылка для цитирования: Jonnalagadda J. M., Valdés J. E. N. Positive Solutions of Nabla Fractional Sturm-Liouville Problems // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 51. С. 50–65.
https://doi.org/10.26516/1997-7670.2025.51.50

1. Introduction

Discrete fractional calculus is a unified theory of sums and differences of arbitrary order. We find two approaches in the literature on fractional differences: the Δ point of view (called the delta fractional difference) and the ∇ perspective (called the nabla fractional difference). In this article, we confine ourselves to the second approach.

The non-local character of nabla fractional differences has attracted a lot of attention to the theory and applications of nabla fractional calculus in the last ten years. Nabla fractional calculus is an ideal tool for simulating non-local phenomena in time or space. There is a long-term memory effect in the nabla fractional difference of any function since it holds information about the function at previous times. Many natural systems, including those with non-local effects, are better described by nabla fractional difference equations than by integer-order difference equations [2; 3; 10]. A strong theory of nabla fractional calculus for discrete-variable, real-valued functions was developed as a consequence of the contributions of multiple mathematicians. We refer to [5; 6] and its sources for a thorough introduction to the development of nabla fractional calculus.

The analysis of positive solutions to nabla fractional boundary value problems has drawn more attention in the last ten years, because this topic is essential for advancing both theoretical mathematics and practical applications across various disciplines, from mathematical analysis to engineering and beyond. Among the noteworthy works cited are [4; 7; 8; 11]. The following Sturm–Liouville boundary value problem for the Riemann–Liouville nabla fractional difference equation is discussed in this article, and adequate conditions for the existence of positive solutions are established. These findings will enhance and expand upon the ones that already exist.

$$\begin{cases} -\left(\nabla^{\vartheta}_{k_0-1}y\right)(k) = \mathfrak{F}(k, y(k-1)), \quad k \in \mathbb{N}^{l}_{k_0+2}, \\ \gamma y(k_0) - \delta\left(\nabla y\right)(k_0+1) = 0, \\ \zeta y(l) + \eta\left(\nabla y\right)(l) = 0, \end{cases} \tag{1.1}$$

where k_0, $l \in \mathbb{R}$ with $l - k_0 \in \mathbb{N}_3$; $y : \mathbb{N}_{k_0}^{l} \to \mathbb{R}$; $\mathfrak{F} : \mathbb{N}_{k_0+2}^{l} \times [0, \infty) \to [0, \infty)$ is a continuous function; ∇y stands for the first-order nabla difference of y; $1 < \vartheta < 2$ and $\nabla_{k_0-1}^{\vartheta} y$ stands for the ϑ^{th}-order Riemann–Liouville nabla fractional difference of y based at $k_0 - 1$. We assume the following condition:

(C) $\gamma \geq \delta \geq 0$, $\zeta \geq 0$, $\eta \geq 0$ such that $\gamma + \delta > 0$ and $\zeta + \eta > 0$.

The current article is structured as follows. Preliminaries on nabla fractional calculus are presented in Section 2. We build the Green's function related to (1.1) in Section 3. Additionally, we include some of its fundamental characteristics, such as positivity. Enough conditions are established in Sections 4 and 5 for the presence of positive solutions to (1.1). The Guo–Krasnoselskii fixed point theorem is utilized to demonstrate the presence of positive solutions. We express this theorem in the following manner for convenience:

Theorem 1. *[12] Let $Y = (Y, \|\cdot\|)$ be a Banach space, and $\mathsf{P} \subset Y$ be a cone in Y. Let ω_1 and ω_2 be two bounded open sets in Y with $0 \in \omega_1$, $\overline{\omega}_1 \subset \omega_2$. Let $B : \mathsf{P} \cap (\overline{\omega}_2 \backslash \omega_1) \to \mathsf{P}$ be a completely continuous operator. Suppose that one of the two conditions*

1) $\|By\| \leq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_1$, and $\|By\| \geq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_2$; and

2) $\|By\| \geq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_1$, and $\|By\| \leq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_2$,

is satisfied. Then, B has a minimum of one fixed point y in $\mathsf{P} \cap (\overline{\omega}_2 \backslash \omega_1)$.

Theorem 2. *[12] Let $Y = (Y, \|\cdot\|)$ be a Banach space, and $\mathsf{P} \subset Y$ be a cone in Y. Let ω_1, ω_2 and ω_3 be three bounded open sets in Y such that $0 \in \omega_1$, $\overline{\omega}_1 \subset \omega_2$, and $\overline{\omega}_2 \subset \omega_3$. Let $B : \mathsf{P} \cap (\overline{\omega}_3 \backslash \omega_1) \to \mathsf{P}$ be a completely continuous operator. Suppose that one of the two conditions*

1) $\|By\| \geq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_1$, $\|By\| \leq \|y\|$ and $By \neq y$, $y \in \mathsf{P} \cap \partial\overline{\omega}_2$, and $\|By\| \geq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_3$; and

2) $\|By\| \leq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_1$; $\|By\| \geq \|y\|$ and $By \neq y$, $y \in \mathsf{P} \cap \partial\overline{\omega}_2$, and $\|By\| \leq \|y\|$, $y \in \mathsf{P} \cap \partial\overline{\omega}_3$,

is satisfied. Then, B has a minimum of two fixed points y_1 and y_2 in $\mathsf{P} \cap (\overline{\omega}_3 \backslash \omega_1)$; moreover, $y_1 \in \overline{\omega}_2 \backslash \omega_1$ and $y_2 \in \overline{\omega}_3 \backslash \omega_2$.

2. Preliminaries

We use the following fundamentals of discrete fractional calculus [6] throughout the article. Denote by $\mathbb{N}_p = \{p, p+1, p+2, \dots\}$ and $\mathbb{N}_p^q = \{p, p+1, p+2, \dots, q\}$ for any real numbers p and q such that $q - p \in \mathbb{N}_1$.

Definition 1. *[6] The Euler gamma function is defined by*

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can also be extended to the half-plane $\Re(z) \leq 0$, except for $z \in \{\cdots, -2, -1, 0\}$.

Definition 2. *[6] For any α, $\beta \in \mathbb{R}$, the generalized rising function is defined by*

$$\alpha^{\overline{\beta}} = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}, & \alpha, \alpha+\beta \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, \\ 1, & \alpha = \beta = 0, \\ 0, & \alpha = 0, \ \beta \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, \\ \textit{undefined}, & \textit{otherwise}. \end{cases}$$

Definition 3. *[6] The υ^{th}-order nabla fractional Taylor monomial is*

$$\mathsf{H}_\upsilon(k, k_0) = \frac{(k-k_0)^{\overline{\upsilon}}}{\Gamma(\upsilon+1)} = \frac{\Gamma(k-k_0+\upsilon)}{\Gamma(k-k_0)\Gamma(\upsilon+1)}, \quad \upsilon \in \mathbb{R} \setminus \{\dots, -3, -2, -1\},$$

provided the expression on the right-hand side is well-defined.

Definition 4. *[6] Let $y : \mathbb{N}_{k_0} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first-order nabla difference of y is defined by*

$$(\nabla y)(k) = y(k) - y(k-1), \quad k \in \mathbb{N}_{k_0+1},$$

and the N^{th}-order nabla difference of y is defined recursively by

$$\left(\nabla^N y\right)(k) = \left(\nabla \left(\nabla^{N-1} y\right)\right)(k), \quad k \in \mathbb{N}_{k_0+N}.$$

We collect some important properties of nabla fractional Taylor monomials in the following lemma.

Lemma 1. *[6] The following properties hold, provided the expressions in this lemma are well-defined.*

1) $\mathsf{H}_\upsilon(k_0, k_0) = 0$;

2) $\mathsf{H}_0(k, k_0) = 1$;

3) $\mathsf{H}_\upsilon(k, k_0) = 0$ *for* $k \in \mathbb{N}_{k_0}$ *and* $\upsilon \in \{\dots, -3, -2, -1\}$;

4) $\nabla \mathsf{H}_{\upsilon}(k, k_0) = \mathsf{H}_{\upsilon-1}(k, k_0)$*;*

5) $\displaystyle\sum_{s=k_0+1}^{k} \mathsf{H}_{\upsilon}(s, k_0) = \mathsf{H}_{\upsilon+1}(k, k_0)$*.*

Definition 5. *[6] Let* $\vartheta > 0$ *and* $y : \mathbb{N}_{k_0+1} \to \mathbb{R}$*. The* ϑ^{th}*-order nabla fractional sum of* y *based at* k_0 *is*

$$\left(\nabla_{k_0}^{-\vartheta} y\right)(k) = \sum_{s=k_0+1}^{k} \mathsf{H}_{\vartheta-1}(k, s-1) y(s), \quad k \in \mathbb{N}_{k_0},$$

where by convention $\left(\nabla_{k_0}^{-\vartheta} y\right)(k_0) = 0$*.*

Definition 6. *[6] Let* $\vartheta > 0$ *and* $y : \mathbb{N}_{k_0+1} \to \mathbb{R}$*. The* ϑ^{th}*-order Riemann–Liouville nabla fractional difference of* y *based at* k_0 *is*

$$\left(\nabla_{k_0}^{\vartheta} y\right)(k) = \left(\nabla^N \left(\nabla_{k_0}^{-(N-\vartheta)} y\right)\right)(k), \quad N - 1 < \vartheta \le N, \quad N \in \mathbb{N}_1,$$

for $k \in \mathbb{N}_{k_0+N}$*.*

In the following result, we show that the traditional definition of a nabla fractional difference can be rewritten in a form similar to the definition for a nabla fractional sum.

Theorem 3. *[1] Let* $\vartheta > 0$ *and* $y : \mathbb{N}_{k_0} \to \mathbb{R}$*. The* ϑ^{th}*-order Riemann–Liouville nabla fractional difference of* y *based at* k_0 *is*

$$\left(\nabla_{k_0}^{\vartheta} y\right)(k) = \sum_{s=k_0+1}^{k} \mathsf{H}_{-\vartheta-1}(k, s-1) y(s), \ N - 1 < \vartheta < N, \ N \in \mathbb{N}_1,$$

for $k \in \mathbb{N}_{k_0+1}$*.*

Remark 1. Let $0 < \vartheta < 1$. From Theorem 3, we observe that the value of $\left(\nabla_{k_0}^{\vartheta} y\right)(k)$ depends on the values of y on $\mathbb{N}_{k_0+1}^{k}$. This full history nature of the ϑ^{th}-order nabla fractional difference of y is one of its important features. In contrast the value of $(\nabla y)(k)$ depends on the values of y at the points $k - 1$ and k only.

We present some important composition rules of nabla fractional sums and differences.

Lemma 2. *[6] Let* $y : \mathbb{N}_{k_0+1} \to \mathbb{R}$ *and* μ*,* ϑ*,* $\upsilon > 0$*. Then,*

1) $\left(\nabla_{k_0}^{-\mu} \left(\nabla_{k_0}^{-\upsilon} y\right)\right)(k) = \left(\nabla_{k_0}^{-(\mu+\upsilon)} y\right)(k), \quad k \in \mathbb{N}_{k_0}$*.*

2) $\left(\nabla_{k_0}^{\vartheta}\left(\nabla_{k_0}^{-\mu}y\right)\right)(k)=\left(\nabla_{k_0}^{\vartheta-\mu}y\right)(k)$, *where* $N-1<\vartheta\leq N$, $N\in\mathbb{N}_1$, $k\in\mathbb{N}_{k_0+N}$.

Theorem 4. *[6] Let* $\vartheta>0$*. The homogeneous nabla fractional difference equation*

$$\left(\nabla_{k_0-1}^{\vartheta}y\right)(k)=0,\quad N-1<\vartheta\leq N,\quad N\in\mathbb{N}_1,\quad k\in\mathbb{N}_{k_0+N},$$

has a general solution

$$y(k)=\sum_{i=1}^{N}D_i\mathsf{H}_{\vartheta-i}(k,k_0-1),\quad k\in\mathbb{N}_{k_0}.$$

Here, $D_1,\ D_2,\ \cdots,\ D_N$ *are constants.*

Theorem 5. *Let* $\vartheta>0$*. The nonhomogeneous nabla fractional difference equation*

$$\left(\nabla_{k_0-1}^{\vartheta}y\right)(k)=p(k),\quad N-1<\vartheta\leq N,\quad N\in\mathbb{N}_1,\quad k\in\mathbb{N}_{k_0+N},\tag{2.1}$$

has a general solution

$$y(k)=\sum_{i=1}^{N}D_i\mathsf{H}_{\vartheta-i}(k,k_0-1)+\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)(k),\quad k\in\mathbb{N}_{k_0}.$$

Here, $D_1,\ D_2,\ \cdots,\ D_N$ *are constants and* $p:\mathbb{N}_{k_0+N}\to\mathbb{R}$.

Proof. In view of Theorem 4, it is sufficient to show that $\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)(k)$ is a particular solution of (2.1). Denote by

$$q(k)=\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)(k),\quad k\in\mathbb{N}_{k_0}.$$

It is enough to show that q satisfies the nonhomogeneous nabla fractional difference equation (2.1). That is,

$$\left(\nabla_{k_0-1}^{\vartheta}q\right)(k)=p(k),\quad k\in\mathbb{N}_{k_0+N}.\tag{2.2}$$

To see this, for $k\in\mathbb{N}_{k_0+N}$, we have

$$\begin{aligned}\left(\nabla_{k_0-1}^{\vartheta}q\right)(k)&=\sum_{s=k_0}^{k}\mathsf{H}_{-\vartheta-1}(k,s-1)q(s)\quad\text{(By Theorem 3)}\\&=\sum_{s=k_0}^{k_0+N-1}\mathsf{H}_{-\vartheta-1}(k,s-1)q(s)+\sum_{s=k_0+N}^{k}\mathsf{H}_{-\vartheta-1}(k,s-1)q(s)\\&=\sum_{s=k_0}^{k_0+N-1}\mathsf{H}_{-\vartheta-1}(k,s-1)q(s)+\left(\nabla_{k_0+N-1}^{\vartheta}q\right)(k).\end{aligned}$$

We observe that

$$\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)(s)=0, \quad s\in\mathbb{N}_{k_0}^{k_0+N-1}. \tag{2.3}$$

Now, for $k\in\mathbb{N}_{k_0+N}$, consider

$$\begin{aligned}\left(\nabla_{k_0-1}^{\vartheta}q\right)(k) &= \left(\nabla_{k_0-1}^{\vartheta}\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)\right)(k)=\\ &= \sum_{s=k_0}^{k_0+N-1}\mathsf{H}_{-\vartheta-1}(k,s-1)\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)(s)+\\ &\qquad+\left(\nabla_{k_0+N-1}^{\vartheta}\left(\nabla_{k_0+N-1}^{-\vartheta}p\right)\right)(k)=p(k),\end{aligned}$$

(By (2.3) and Lemma 2) which completes the proof. □

3. Green's Function and Its Positiveness

In this section, we construct the Green's function for the linear boundary value problem

$$\begin{cases}-\left(\nabla_{k_0-1}^{\vartheta}y\right)(k)=x(k), \quad k\in\mathbb{N}_{k_0+2}^{l},\\ \gamma y(k_0)-\delta\left(\nabla y\right)(k_0+1)=0,\\ \zeta y(l)+\eta\left(\nabla y\right)(l)=0,\end{cases} \tag{3.1}$$

associated with (1.1), and state a few of its essential properties, including positivity. Here, $x:\mathbb{N}_{k_0+2}^{l}\to\mathbb{R}$. We introduce the following notations for this purpose.

$$\begin{aligned}a_1 &= \gamma+\delta(1-\vartheta),\\ a_2 &= \gamma+\delta(2-\vartheta)=a_1+\delta,\\ \omega &= \zeta\mathsf{H}_{\vartheta-2}(l,k_0-1)+\eta\mathsf{H}_{\vartheta-3}(l,k_0-1),\\ \lambda &= a_2\phi(k_0)-a_1\omega,\\ \phi(m) &= \zeta\mathsf{H}_{\vartheta-1}(l,m-1)+\eta\mathsf{H}_{\vartheta-2}(l,m-1), \quad m\in\mathbb{N}_{k_0}^{l},\\ w(m) &= a_2\mathsf{H}_{\vartheta-1}(m,k_0-1)-a_1\mathsf{H}_{\vartheta-2}(m,k_0-1), \quad m\in\mathbb{N}_{k_0}^{l}.\end{aligned}$$

We begin with the following lemma.

Lemma 3. *The following properties hold:*

1) $a_1>0$, $a_2>0$ *and* $\phi(m)>0$ *for* $m\in\mathbb{N}_{k_0}^{l}$*;*

2) $\lambda>0$*;*

3) $w(k_0) \geq 0$, and w is a positive nondecreasing function on $\mathbb{N}_{k_0+1}^{l}$.

Theorem 6. *The unique solution of the linear boundary value problem (3.1) is*

$$y(k) = \sum_{s=k_0+2}^{l} \mathcal{G}(k,s)x(s), \quad k \in \mathbb{N}_{k_0}^{l}, \tag{3.2}$$

where

$$\mathcal{G}(k,s) = \begin{cases} \mathcal{G}_1(k,s), & k \in \mathbb{N}_{k_0}^{s-1}, \\ \mathcal{G}_2(k,s), & k \in \mathbb{N}_{s}^{l}, \end{cases} \tag{3.3}$$

with

$$\mathcal{G}_1(k,s) = \frac{w(k)}{\lambda}\phi(s),$$

and

$$\mathcal{G}_2(k,s) = \mathcal{G}_1(k,s) - \mathsf{H}_{\vartheta-1}(k,s-1).$$

Proof. From Theorem 5, a general solution of the nonhomogeneous nabla fractional difference equation in (3.1) is

$$y(k) = C_1\mathsf{H}_{\vartheta-1}(k,k_0-1)+C_2\mathsf{H}_{\vartheta-2}(k,k_0-1)-\big(\nabla_{k_0+1}^{-\vartheta}x\big)(k), \; k \in \mathbb{N}_{k_0}^{l}, \tag{3.4}$$

where C_1 and C_2 are arbitrary constants. Now, applying ∇ on both sides of equality (3.4), using Lemma 1 and Lemma 2, we obtain

$$\big(\nabla y\big)(k) = C_1\mathsf{H}_{\vartheta-2}(k,k_0-1)+C_2\mathsf{H}_{\vartheta-3}(k,k_0-1)-\big(\nabla_{k_0+1}^{1-\vartheta}x\big)(k), \; k \in \mathbb{N}_{k_0+1}^{l}. \tag{3.5}$$

From the first boundary condition $\gamma y(k_0) - \delta\big(\nabla y\big)(k_0+1) = 0$ in (3.4) - (3.5), we get

$$a_1C_1 + a_2C_2 = 0. \tag{3.6}$$

From the second boundary condition $\zeta y(l) + \eta\big(\nabla y\big)(l) = 0$ in (3.4) - (3.5), we obtain

$$\phi(k_0)C_1 + \omega C_2 = \sum_{s=k_0+2}^{l} \phi(s)x(s). \tag{3.7}$$

From (3.6) and (3.7), we have

$$C_1 = \frac{a_2}{\lambda}\sum_{s=k_0+2}^{l} \phi(s)x(s), \tag{3.8}$$

and

$$C_2 = -\frac{a_1}{\lambda}\sum_{s=k_0+2}^{l} \phi(s)x(s). \tag{3.9}$$

Substituting the equalities (3.8) and (3.9) in (3.4), we obtain (3.2). □

Lemma 4. *The Green's function defined in* (3.3) *obeys the following characteristics:*

1) $0 \leq \mathcal{G}(k,s) \leq \mathcal{G}(s-1,s), \quad (k,s) \in \mathbb{N}_{k_0}^{l} \times \mathbb{N}_{k_0+2}^{l}$;

2) There exists a positive number $\sigma \in (0,1)$ *such that*

$$\mathcal{G}(k,s) \geq \sigma \mathcal{G}(s-1,s), \quad (k,s) \in \mathbb{N}_{k_0+1}^{l-1} \times \mathbb{N}_{k_0+2}^{l},$$

where

$$\sigma = \frac{1}{w(l-1)} \min\left\{ w(k_0+1), w(l-1) - \frac{\lambda}{\zeta + (\zeta+\eta)\left(\frac{\vartheta-1}{l-k_0-2}\right)} \right\}. \tag{3.10}$$

Proof. We refer to Lemma 6 and Lemma 7 of [9] for the proof of (1). The proof of (2) is also available as Lemma 8 of [9]. We rewrite it here when $(k,s) \in \mathbb{N}_{k_0+1}^{l-1} \times \mathbb{N}_{k_0+2}^{l}$. For this purpose, consider

$$\frac{\mathcal{G}(k,s)}{\mathcal{G}(s-1,s)} = \begin{cases} \dfrac{\mathcal{G}_1(k,s)}{\mathcal{G}_1(s-1,s)}, & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ \dfrac{\mathcal{G}_2(k,s)}{\mathcal{G}_1(s-1,s)}, & k \in \mathbb{N}_{s}^{l-1}. \end{cases} \tag{3.11}$$

It follows from Lemma 7 of [9] that

$$\mathcal{G}_1(k,s) \geq \mathcal{G}_1(k_0+1,s), \quad k \in \mathbb{N}_{k_0+1}^{s-1}, \quad s \in \mathbb{N}_{k_0+2}^{l},$$

and

$$\mathcal{G}_2(k,s) \geq \mathcal{G}_2(l-1,s), \quad k \in \mathbb{N}_{s}^{l-1}, \quad s \in \mathbb{N}_{k_0+2}^{l}.$$

Then, from (3.11), we obtain

$$\begin{aligned} \frac{\mathcal{G}(k,s)}{\mathcal{G}(s-1,s)} &\geq \begin{cases} \dfrac{\mathcal{G}_1(k_0+1,s)}{\mathcal{G}_1(s-1,s)}, & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ \dfrac{\mathcal{G}_2(l-1,s)}{\mathcal{G}_1(s-1,s)}, & k \in \mathbb{N}_{s}^{l-1}. \end{cases} \\ &= \begin{cases} \dfrac{w(k_0+1)}{w(s-1)}, & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ \dfrac{w(l-1)}{w(s-1)} - \dfrac{\lambda \mathsf{H}_{\vartheta-1}(l-1,s-1)}{w(s-1)\phi(s)}, & k \in \mathbb{N}_{s}^{l-1}. \end{cases} \\ &= \frac{1}{w(s-1)} \begin{cases} w(k_0+1), & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ w(l-1) - \lambda \dfrac{\mathsf{H}_{\vartheta-1}(l-1,s-1)}{\phi(s)}, & k \in \mathbb{N}_{s}^{l-1}. \end{cases} \end{aligned} \tag{3.12}$$

Now, for $k \in \mathbb{N}_s^{l-1}$ and $s \in \mathbb{N}_{k_0+2}^{l}$, consider

$$\begin{aligned}\frac{\mathsf{H}_{\vartheta-1}(l-1,s-1)}{\phi(s)} &= \frac{\mathsf{H}_{\vartheta-1}(l-1,s-1)}{\zeta\mathsf{H}_{\vartheta-1}(l,s-1)+\eta\mathsf{H}_{\vartheta-2}(l,s-1)} = \\ = \frac{1}{\zeta\frac{\mathsf{H}_{\vartheta-1}(l,s-1)}{\mathsf{H}_{\vartheta-1}(l-1,s-1)}+\eta\frac{\mathsf{H}_{\vartheta-2}(l,s-1)}{\mathsf{H}_{\vartheta-1}(l-1,s-1)}} &= \frac{1}{\zeta\left(\frac{l-s+\vartheta-1}{l-s}\right)+\eta\left(\frac{\vartheta-1}{l-s}\right)} = \\ = \frac{1}{\zeta+(\zeta+\eta)\left(\frac{\vartheta-1}{l-s}\right)} &\leq \frac{1}{\zeta+(\zeta+\eta)\left(\frac{\vartheta-1}{l-k_0-2}\right)}. \end{aligned} \tag{3.13}$$

Using (3.13) in (3.12), we get

$$\frac{\mathcal{G}(k,s)}{\mathcal{G}(s-1,s)} \geq \frac{1}{w(s-1)}\begin{cases} w(k_0+1), & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ w(l-1) - \dfrac{\lambda}{\zeta+(\zeta+\eta)\left(\frac{\vartheta-1}{l-k_0-2}\right)}, & k \in \mathbb{N}_s^{l-1}. \end{cases} \tag{3.14}$$

Since w is a positive increasing function on $\mathbb{N}_{k_0+2}^{l}$, from (3.14), we have

$$\frac{\mathcal{G}(k,s)}{\mathcal{G}(s-1,s)} \geq \frac{1}{w(l-1)}\begin{cases} w(k_0+1), & k \in \mathbb{N}_{k_0+1}^{s-1}, \\ w(l-1) - \dfrac{\lambda}{\zeta+(\zeta+\eta)\left(\frac{\vartheta-1}{l-k_0-2}\right)}, & k \in \mathbb{N}_s^{l-1}, \end{cases}$$

which completes the proof. Clearly, $0 < \sigma < 1$. □

4. A Positive Solution

This section deduces adequate conditions on the existence of a positive solution to (1.1) using the Guo–Krasnoselskii fixed point theorem. From Theorem 6, we obtain the equivalence between the solutions of (1.1) and the solutions of the equation

$$y(k) = \sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s,y(s-1)), \quad k \in \mathbb{N}_{k_0}^{l}.$$

Let

$$\mathcal{B} = \left\{ y : \mathbb{N}_{k_0+1}^{l-1} \to \mathbb{R} \middle| \gamma y(k_0) - \delta\left(\nabla y\right)(k_0+1) = 0,\ \zeta y(l) + \eta\left(\nabla y\right)(l) = 0 \right\}.$$

Then, $\mathcal{B}$ is a $(l-k_0-1)$-dimensional Banach space equipped with the maximum norm defined by

$$\|y\| = \max_{k \in \mathbb{N}_{k_0+1}^{l-1}} |y(k)|,$$

for any $y \in \mathcal{B}$. Define the operator $S : \mathcal{B} \to \mathcal{B}$ by

$$(Sy)(k) = \sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s, y(s-1)), \quad k \in \mathbb{N}_{k_0}^{l}.$$

Obviously, y is a fixed point of $S \iff y$ is a solution of (1.1). Let

$$K = \left\{ y \in \mathcal{B} \mid y(k) \geq 0 \text{ for } k \in \mathbb{N}_{k_0+1}^{l-1}, \ \min_{k \in \mathbb{N}_{k_0+1}^{l-1}} y(k) \geq \sigma \|y\| \right\},$$

where σ is given by (3.10). Clearly, K is a cone in $\mathcal{B}$.

Lemma 5. *$S(K) \subset K$.*

Proof. Let $y \in K$. Clearly, $Sy \geq 0$ for all $k \in \mathbb{N}_{k_0+1}^{l-1}$. Consider

$$\min_{k \in \mathbb{N}_{k_0+1}^{l-1}} (Sy)(k) = \min_{k \in \mathbb{N}_{k_0+1}^{l-1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s, y(s-1)) \right] \geq$$

$$\geq \sum_{s=k_0+2}^{l} \min_{k \in \mathbb{N}_{k_0+1}^{l-1}} [\mathcal{G}(k,s)]\, \mathfrak{F}(s, y(s-1)) \geq \sigma \sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s)\mathfrak{F}(s, y(s-1)) \geq$$

$$\geq \sigma \sum_{s=k_0+2}^{l} \max_{k \in \mathbb{N}_{k_0+1}^{l-1}} [\mathcal{G}(k,s)]\, \mathfrak{F}(s, y(s-1)) \geq$$

$$\geq \sigma \max_{k \in \mathbb{N}_{k_0+1}^{l-1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s, y(s-1)) \right] = \sigma \|Sy\|.$$

Therefore, $Sy \in K$. □

Denote by

$$\alpha = \sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s).$$

Theorem 7. *Suppose we have two positive numbers $0 < m < M < \infty$ such that*

(A1) $\mathfrak{F}(k,\xi) \leq \dfrac{m}{\alpha}, \quad (k,\xi) \in \mathbb{N}_{k_0+2}^{l} \times [0,m]$;

(A2) $\mathfrak{F}(k,\xi) \geq \dfrac{M}{\alpha\sigma}, \quad (k,\xi) \in \mathbb{N}_{k_0+2}^{l} \times [\sigma M, M]$,

where σ is given by (3.10). Then, (1.1) has a minimum of one positive solution $y \in K$ such that $m \leq \|y\| \leq M$.

Proof. Define the sets

$$\omega_1 = \{y \in K \mid \|y\| < m\} \text{ and } \omega_2 = \{y \in K \mid \|y\| < M\}.$$

Clearly, ω_1 and ω_2 are bounded open subsets of $\mathcal{B}$ with $0 \in \omega_1$, $\overline{\omega}_1 \subset \omega_2$, and $S : K \cap (\overline{\omega}_2 \backslash \omega_1) \to K$ is a completely continuous operator. In order to apply Theorem 1, we separate the proof into the following two steps:
Step 1: Let $y \in K \cap \partial\overline{\omega}_1$. Then, we have $0 \le y(s) \le m$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$. It follows from (A1) and Lemma 4 that

$$\begin{aligned}\|Sy\| = \max_{k\in\mathbb{N}_{k_0+1}^{l-1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s,y(s-1))\right] \le \\ \le \frac{m}{\alpha} \max_{k\in\mathbb{N}_{k_0+1}^{l-1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\right] \le \frac{m}{\alpha} \sum_{s=k_0+2}^{l} \max_{k\in\mathbb{N}_{k_0+1}^{l-1}} [\mathcal{G}(k,s)] \le \\ \le \frac{m}{\alpha} \sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s) = m = \|y\|.\end{aligned}$$

Step 2: Let $y \in K \cap \partial\overline{\omega}_2$. Then, we have $0 \le y(s) \le M$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$, and

$$\min_{s\in\mathbb{N}_{k_0+1}^{l-1}} y(s) \ge \sigma\|y\| = \sigma M,$$

implying that $\sigma M \le y(s) \le M$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$. Consider

$$\begin{aligned}(Sy)(k) &= \sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s,y(s-1)) \\ &\ge \frac{M}{\alpha\sigma} \sum_{s=k_0+2}^{l} \mathcal{G}(k,s) \quad \text{(By (A2))} \\ &\ge \frac{M}{\alpha} \sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s) \quad \text{(By Lemma 4)} \\ &= M = \|y\|,\end{aligned}$$

implying that $\|Sy\| \ge \|y\|$ for $y \in K \cap \partial\overline{\omega}_2$. Hence, by the first condition of Theorem 1, (1.1) has a minimum of one positive solution $y \in K$ such that $m \le \|y\| \le M$. □

Theorem 8. *Suppose we have two positive numbers* $0 < m < M < \infty$ *such that*

(B1) $\mathfrak{F}(k,\xi) \ge \dfrac{m}{\alpha\sigma}, \quad (k,\xi) \in \mathbb{N}_{k_0+2}^{l} \times [0,m];$

(B2) $\mathfrak{F}(k,\xi) \leq \dfrac{M}{\alpha}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [\sigma M, M]$,

where σ is given by (3.10)*. Then,* (1.1) *has a minimum of one positive solution $y \in K$ such that $m \leq \|y\| \leq M$.*

Proof. The proof is similar to the proof of Theorem 7. So, we omit it □

5. Twin Positive Solutions

Here, we deduce adequate conditions on the existence of two positive solutions to (1.1) using the Guo–Krasnoselskii fixed point theorem.

Theorem 9. *Suppose we have three positive numbers $0 < m < p < M < \infty$ such that*

(A1) $\mathfrak{F}(k,\xi) \leq \dfrac{m}{\alpha}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [0, m]$;

(A3) $\mathfrak{F}(k,\xi) > \dfrac{p}{\alpha\sigma}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [\sigma p, p]$;

(A4) $\mathfrak{F}(k,\xi) \leq \dfrac{M}{\alpha}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [\sigma M, M]$,

where σ is given by (3.10)*. Then,* (1.1) *has a minimum of two positive solutions y_1, $y_2 \in K$ such that $m \leq \|y_1\| < p < \|y_2\| \leq M$.*

Proof. Define the sets $\omega_1 = \{y \in K \mid \|y\| < m\}$, $\omega_2 = \{y \in K \mid \|y\| < p\}$, and $\omega_3 = \{y \in K \mid \|y\| < M\}$. Clearly, ω_1, ω_2 and ω_3 are three bounded open sets in $\mathcal{B}$ such that $0 \in \omega_1$, $\overline{\omega}_1 \subset \omega_2$, $\overline{\omega}_2 \subset \omega_3$, and $S : K \cap (\overline{\omega}_3 \backslash \omega_1) \to K$ is a completely continuous operator. In order to apply Theorem 2, we separate the proof into the following three steps:
Step 1: Let $y \in K \cap \partial\overline{\omega}_1$. Then, we have $0 \leq y(s) \leq m$ for all $s \in \mathbb{N}^{l-1}_{k_0+1}$. It follows from (A1) and Lemma 4 that

$$
\begin{aligned}
\|Sy\| = \max_{k \in \mathbb{N}^{l-1}_{k_0+1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s, y(s-1)) \right] \leq \\
\leq \frac{m}{\alpha} \max_{k \in \mathbb{N}^{l-1}_{k_0+1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s) \right] \leq \frac{m}{\alpha} \sum_{s=k_0+2}^{l} \max_{k \in \mathbb{N}^{l-1}_{k_0+1}} [\mathcal{G}(k,s)] \\
\leq \frac{m}{\alpha} \sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s) = m = \|y\|.
\end{aligned}
$$

Step 2: Let $y \in K \cap \partial\overline{\omega}_2$. Then, we have $0 \leq y(s) \leq p$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$, and

$$\min_{s\in\mathbb{N}_{k_0+1}^{l-1}} y(s) \geq \sigma\|y\| = \sigma p,$$

implying that $\sigma p \leq y(s) \leq p$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$. Consider

$$\begin{aligned}(Sy)(k) &= \sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s, y(s-1)) \\ &> \frac{p}{\alpha\sigma}\sum_{s=k_0+2}^{l} \mathcal{G}(k,s) \quad (\text{By} \quad (\text{A3})) \\ &\geq \frac{p}{\alpha}\sum_{s=k_0+2}^{l} \mathcal{G}(s-1,s) \quad (\text{By} \quad \text{Lemma 4}) \\ &= p = \|y\|,\end{aligned}$$

implying that

$$\|Sy\| > \|y\|, \quad y \in K \cap \partial\overline{\omega}_2. \tag{5.1}$$

Further, by (5.1), $Sy \neq y$ for $y \in K \cap \partial\overline{\omega}_2$.
Step 3: Let $y \in K \cap \partial\overline{\omega}_3$. Then, we have $0 \leq y(s) \leq M$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$, and

$$\min_{s\in\mathbb{N}_{k_0+1}^{l-1}} y(s) \geq \sigma\|y\| = \sigma M,$$

implying that $\sigma M \leq y(s) \leq M$ for all $s \in \mathbb{N}_{k_0+1}^{l-1}$. It follows from (A4) and Lemma 4 that

$$\begin{aligned}\|Sy\| = \max_{k\in\mathbb{N}_{k_0+1}^{l-1}} \left[\sum_{s=k_0+2}^{l} \mathcal{G}(k,s)\mathfrak{F}(s,y(s-1))\right] \leq \\ \leq \frac{M}{\alpha}\max_{k\in\mathbb{N}_{k_0+1}^{l-1}}\left[\sum_{s=k_0+2}^{l}\mathcal{G}(k,s)\right] \leq \frac{M}{\alpha}\sum_{s=k_0+2}^{l}\max_{k\in\mathbb{N}_{k_0+1}^{l-1}}[\mathcal{G}(k,s)] \leq \\ \leq \frac{M}{\alpha}\sum_{s=k_0+2}^{l}\mathcal{G}(s-1,s) = M = \|y\|.\end{aligned}$$

Hence, by the first condition of Theorem 2, S has a minimum of two fixed points y_1 and y_2 in K such that $m \leq \|y_1\| < p < \|y_2\| \leq M$. □

Theorem 10. *Suppose we have three positive numbers* $0 < m < p < M < \infty$ *such that*

(B1) $\mathfrak{F}(k,\xi) \geq \dfrac{m}{\alpha\sigma}, \quad (k,\xi) \in \mathbb{N}_{k_0+2}^{l} \times [0,m];$

(B3) $\mathfrak{F}(k,\xi) < \dfrac{p}{\alpha}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [\sigma p, p];$

(B4) $\mathfrak{F}(k,\xi) \geq \dfrac{M}{\alpha\sigma}, \quad (k,\xi) \in \mathbb{N}^{l}_{k_0+2} \times [\sigma M, M],$

where σ is given by (3.10)*. Then,* (1.1) *has a minimum of two positive solutions y_1, $y_2 \in K$ such that $m \leq \|y_1\| < p < \|y_2\| \leq M$.*

Proof. The proof is similar to the proof of Theorem 9. So, we omit it □

6. Examples

We provide a few examples to illustrate the applicability of established results.

Example 1. (Dirichlet Conditions) Consider (1.1) with $k_0 = 0$, $l = 5$, $\vartheta = 1.5$, $\delta = \eta = 0$, $\gamma = \zeta = 1$ and $\mathfrak{F}(k,\xi) = \frac{k}{50} + \frac{\sin^2 \xi}{10}$ for all $(k,\xi) \in \mathbb{N}^5_2 \times [0,\infty)$. We obtain that $\sigma = 0.0357$ and $\alpha = 4.0636$. Clearly, $\mathfrak{F}$ is a continuous function. If we take $m = \frac{1}{200}$ and $M = 1$ so that (B1) and (B2) hold. Then, by Theorem 8, (1.1) has a minimum of one positive solution $y \in K$ such that $m \leq \|y\| \leq M$.

Example 2. (Right Focal Conditions) Consider (1.1) with $k_0 = 0$, $l = 5$, $\vartheta = 1.5$, $\delta = \zeta = 0$, $\gamma = \eta = 1$ and $\mathfrak{F}(k,\xi) = \frac{k}{50} + \frac{\sin^2 \xi}{100}$ for all $(k,\xi) \in \mathbb{N}^5_2 \times [0,\infty)$. We obtain that $\sigma = 0.2501$ and $\alpha = 14.6306$. Clearly, $\mathfrak{F}$ is a continuous function. If we take $m = \frac{1}{3}$ and $M = 25$ so that (B1) and (B2) hold. Then, by Theorem 8, (1.1) has a minimum of one positive solution $y \in K$ such that $m \leq \|y\| \leq M$.

7. Conclusions

In this article, we developed a theory to study the existence of positive solutions to the problem (1.1) under natural conditions, for this we applied the Guo–Krasnoselskii fixed point theorem, after constructing and studying the function of Green associated. In the near future, we want to extend our results to other nabla fractional difference operators.

References

1. Ahrendt K., Castle L., Holm M., Yochman K. Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula. *Commun. Appl. Anal.*, 2012, vol. 16, no. 3, pp. 317–347.

2. Atici F.M., Atici M., Nguyen N., Zhoroev T., Koch G. A study on discrete and discrete fractional pharmacokinetics-pharmacodynamics models for tumor growth and anti-cancer effects. *Comput. Math. Biophys.*, 2019, vol. 7, pp. 10–24.
3. Bohner M., Jonnalagadda J.M. Discrete fractional cobweb models. *Chaos Solitons Fractals*, 2022, vol. 162, pp. 1–5.
4. Eralp B., Topal F.S. Existence of positive solutions for discrete fractional boundary value problems. *Adv. Dyn. Syst. Appl.*, 2020, vol. 15, no. 2, pp. 79–97.
5. Ferreira R.A.C. *Discrete fractional calculus and fractional difference equations.* SpringerBriefs in Mathematics. Springer, Cham, 2022.
6. Goodrich C., Peterson A.C. *Discrete fractional calculus.* Springer, Cham, 2015.
7. Ikram A. Lyapunov inequalities for nabla Caputo boundary value problems. *J. Difference Equ. Appl.*, 2019, vol. 25, no. 6, pp. 757–775.
8. Jonnalagadda J.M. On two-point Riemann-Liouville type nabla fractional boundary value problems. *Adv. Dyn. Syst. Appl.*, 2018, vol. 13, no. 2, pp. 141–166.
9. Jonnalagadda J.M. A comparison result for the nabla fractional difference operator. *Foundations*, 2023, vol. 3, pp. 181–198.
10. Ostalczyk P. *Discrete fractional calculus. Applications in control and image processing.* Series in Computer Vision, 4. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.
11. St. Goar J. A Caputo boundary value problem in Nabla fractional calculus. Thesis (Ph.D.) The University of Nebraska-Lincoln, 2016.
12. Guo D.J., Lakshmikantham V. *Nonlinear problems in abstract cones.* Notes and Reports in Mathematics in Science and Engineering, 5. Academic Press, Inc., Boston, MA, 1988.

Об авторах

Джонналагадда Джаган Мохан, PhD (математика), доц., Технологический институт Бирлы и науки, Пилани, Хайдарабад, Телангана, Индия, 500078, j.jaganmohan@hotmail.com, https://orcid.org/0000-0002-1310-8323

Наполес Вальдес Хуан Э., PhD (математика), проф., Северо-Восточный национальный университет, Корриентес, 3400, Аргентина; Национальный технологический университет, Ресистенсия, 3500, Аргентина, jnapoles@exa.unne.edu.ar, https://orcid.org/0000-0003-2470-1090

About the authors

Jagan Mohan Jonnalagadda, PhD (Math.), Assoc. Prof., Birla Institute of Technology and Science, Pilani, Hyderabad, Telangana, India, 500078, j.jaganmohan@hotmail.com, https://orcid.org/0000-0002-1310-8323

Juan E. Nápoles Valdés, PhD (Math.), Prof., National University of the Northeast, Corrientes, 3400, Argentina; National Technological University, Resistencia, 3500, Argentina, jnapoles@exa.unne.edu.ar, https://orcid.org/0000-0003-2470-1090

Поступила в редакцию / Received 03.07.2024
Поступила после рецензирования / Revised 25.08.2024
Принята к публикации / Accepted 28.09.2024