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A Note on Pseudofinite Acyclic Graphs

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Abstract. Acyclic graphs approximated by finite acyclic graphs are considered. It is proved that any countably categorical acyclic graph is smoothly approximable. An example of pseudofinite acyclic graph theory is given, which has an even, odd, and infinite number of rays.

Keywords: approximation of theory, tree, acyclic graph, pseudofinite theory, smoothly approximated structure, pseudofinite graph

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Научная статья

Заметка о псевдоконечных ациклических графах

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Аннотация. Рассматриваются ациклические графы, аппроксимируемые конечными ациклическими графами. Доказано, что любой счетно категоричный ациклический граф гладко аппроксимируем. Приведен пример псевдоконечной ациклической теории графов, имеющей четное, нечетное и бесконечное число лучей.

Ключевые слова: аппроксимация теории, дерево, ациклический граф, псевдоконечная теория, гладко аппроксимируемая структура, псевдоконечный граф

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1. Introduction

This note continues the research begun in [8], about approximations of the theories [14] of acyclic graphs. Let us remind you that a *pseudofinite graph* is an infinite graph that satisfies every first order sentence in a language $L = \{=, R\}$ that is true for some finite graphs. Recently, pseudofinite graphs have been actively studied by various authors in various fields. In [15], graphs interpreted in pseudofinite fields were studied. The papers [16; 17] introduce various ultraflat Hrushovski-Fraïssé classes of acyclic graphs whose limits are strictly superstable and pseudofinite. In [10], generic pseudofinite graph theories are noted. Tao's work [15] concerns graphs definable in (pseudo)finite fields.

In the works [3;4;13] (super)stable graphs and their important properties were studied. The article [6] is devoted to the description of types of pregeometries with an algebraic closure operator for acyclic graph theories.

In [9], the following theorem was proved

Theorem 1. *Any theory T of a regular graph with an infinite model is pseudofinite.*

This theorem immediately implies

Theorem 2. *[1] A theory T_r is the theory of an infinite tree, each vertex of which has degree r. The theory T_∞ (also known as the everywhere infinite forest theory) is the theory of an infinite tree in which each vertex has infinite degree. Both theories and are pseudofinite.*

Note also that the theory T_∞ is the Fraïssé limit of finite trees.

Let us recall the basic definitions of approximations of the theory and the construction of the so-called "coproduct". And all other terminalologies from graph theory and model theory can be found in [2; 7; 8].

Definition 1. [14] *Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be* $\mathcal{T}$-approximated, *or* approximated by the family $\mathcal{T}$, *or a* pseudo-$\mathcal{T}$-theory, *if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ such that $\varphi \in T'$.*

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T.

Definition 2. *[18] A disjoint union $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a structure of signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n that coincide with their interpretations in structures $\mathcal{M}_n$, $n \in \omega$.*

A disjoint union of theories T_n, pairwise disjoint predicate signatures Σ_n, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n, n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on the choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of the models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

2. Smoothly approximability of acyclic graphs

The study of countably infinite and countably categorical smoothly approximable structures is relevant in many areas of mathematics, including topology, analysis, and algebra.

Definition 3. *[5] Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures. $\mathcal{N}$ is a finite homogeneous substructure of $\mathcal{M}$, notationally $\mathcal{N} \leqslant_{hom} \mathcal{M}$, if $\mathcal{N} \leqslant \mathcal{M}$ and for every $k \in \mathbb{N}$ and every pair $\bar{a}, \bar{b} \in N^k$, $\bar{a}$ and $\bar{b}$ lie in the same $Aut(\mathcal{M})$-orbit if and only if $\bar{a}$ and $\bar{b}$ lie in the same $Aut_{\{\mathcal{N}\}}(\mathcal{M})$-orbit, where $Aut_{\{\mathcal{N}\}}(\mathcal{M}) := \{\sigma \in Aut(\mathcal{M}) : \sigma(\mathcal{N}) = \mathcal{N}\}$.*

Definition 4. *[5] An L-structure $\mathcal{M}$ is smoothly approximable if $\mathcal{M}$ is ω-categorical and there exists a sequence $(\mathcal{M}_i)_{i<\omega}$ of finite homogeneous substructures of $\mathcal{M}$ such that $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ for all $i < \omega$ and $\bigcup_{i\in\omega} \mathcal{M}_i = \mathcal{M}$. We say that $\mathcal{M}$ is smoothly approximated by the $\mathcal{M}_i$.*

By the classical theorems of B.Zilber, G. Cherlin, L. Harrington and A. H. Lachlan, any sentence that true in a smoothly approximable structure $\mathcal{M}$ is true of some finite homogeneous substructure of $\mathcal{M}$.

Smoothly approximability is a stronger property than pseudofiniteness, although it implies pseudofiniteness. Not all countably categorical structures are smoothly approximable. For example, the countable model of random (bipartite) graph theory is homogeneous, countably categorical, pseudofinite, but not smoothly approximable.

The work [8] defines the class $\mathcal{G}_{fin}(\lambda)$ of acyclic graphs with bounded diameters in the aggregate. It is proved that any graph from this class is pseudofinite. A more generalized result, including countably categorical graphs, was obtained by A.T. Nurtazin in [11]. Countably categorical graphs were studied by E.V. Ovchinnikova and Yu.E. Shishmarev in [12].

Following [12], the set $O(a,k) = \{x \in G | \rho(a,x) \leq k\}$ is called *k-neighborhood* of vertex a of graph G. A connected graph G is called *bounded*, if there is a natural n that $O(a,n) = G$.

Theorem 3. *[12] Let Γ be a countable acyclic graph. Then Γ is countably categorical if and only if Γ is bounded and finitely many 1-types are realized in it.*

Let us distinguish a subclass $\mathcal{G}_{cc}(\lambda)$ of the class $\mathcal{G}_{fin}(\lambda)$ as the class of all countably categorical acyclic graphs.

Theorem 4. *Any theory $Th(G)$, $G \in \mathcal{G}_{cc}(\lambda)$ is smoothly approximable.*

Proof. Let G be a countably categorical acyclic graph that does not have a vertex of infinite degree. Since by [12] in G the set of degrees of vertices is finite and G is bounded, either all connected components are isomorphic, or G consists of a countable number of copies of non-isomorphic connected components. Each connected component can be considered as a finite homogeneous substructure. Thus, G can be represented as a union of finite homogeneous substructures, $G = \bigcup_{i\in\omega} G_i = \bigsqcup_{i\in\omega} G_i$, where G_i are finite homogeneous connected components.

Now G have the connected components with vertices of infinite degree. Let $G \setminus M$ is the connected components with vertices of infinite degree and M is the union of finite connected components. Then there are $N_0, N_1, \ldots,$: $|N_i| < \omega$, $N_0 \subset N_1 \subset \ldots$, and $G \setminus M = \bigsqcup_{i\in\omega} N_i$. □

3. An example of pseudofinite theory of an acyclic graph with an arbitrary number of rays

The work [8] also defines the class $\mathcal{G}_{inf}(\lambda)$ of acyclic graphs with unbounded diameter. It is proven that if $G \in \mathcal{G}_{inf}(\lambda)$ has an even number of rays then $Th(G)$ is pseudofinite.

In this note, we give an example of a complete, pseudofinite graph theory T in a language $L = \{=, R\}$ with equality and a binary predicate R.

We immediately move on to listing the axioms of our theory. We present them in groups, sometimes providing corresponding semantic consequences. The groups of axioms are numbered in the order of their appearance in our presentation. We avoid formal writing of axioms in the language of first-order logic, limiting ourselves to their semantic description.

(1) The theory says that R is an irreflexive and symmetric binary relation. Thus, we can consider that the models of this theory are simple, undirected graphs. Therefore, we will freely use the terms of graph theory (such as "vertex degree", "distance between vertices", "chain", "cycle", etc.) for the semantic description of the given axioms.

(2) For every natural number $i \geq 2$ there is an axiom φ_i, which says that in the models of the theory T there are no cycles of length i. This means that every connected component of the graph that is a model of T will be a tree. And for any two vertices from one connected component, the distance is defined as the length of the only chain connecting these vertices.

(3) There is one and only one vertex of degree ≥ 3, the remaining vertices have degree 2 or 1. Moreover, for each $i \geq 3$ there is an axiom δ_i, which says that if the degree of a vertex $\geq i$, then the degree of this vertex $\geq i+1$. As a consequence, in the graph that is a model of T, there is exactly one vertex of infinite degree, all other vertices have degree 2 or 1. Note that a vertex of infinite degree is formulaically definable (as a "vertex of degree ≥ 3"), so we will call it a root vertex, and in the future we will use this name to describe other axioms.

(4) For every natural number $i \geq 1$ there is an axiom ψ_i, which says that there is a unique vertex of degree 1, the distance from which to the root vertex is i.

(5) There is one axiom of the theory that says that the chain connecting two vertices of degree 1 has length ≥ 3. For every natural number $i \geq 3$ there is an axiom ϵ_i that says that every chain of length i connecting two vertices of degree 1 passes through the root vertex.

The list of axioms of the theory T is complete. First, let us highlight some properties of the models of this theory. Let's call the component containing the root vertex the *root component*. Let's imagine the root component as a tree with a root (of infinite degree), from which branches grow (upward) (i.e., maximal chains starting from the root vertex). Each branch either leads to some vertex of degree 1, in which case it represents a finite chain, or the branch represents a ray, that is an infinite $\mathbb{N}$-chain. Note that different branches do not have common vertices due to the fact that the degrees of non-root vertices ≤ 2 and there are no finite cycles. Since each vertex of the root component is connected to the root vertex by some finite chain, then the vertices on all branches exhaust all non-root vertices of the root component.

The group of axioms (4) says that for every natural number $i \geq 1$ there is exactly one branch of length i. On the other hand, the model of our theory can contain any number of infinite branches in the root component or may not contain them at all: our axioms do not contain any restrictions on the number of infinite branches.

Now, let's move on to describing the non-root components of the potential model of our theory T. In such components, all vertices have degree ≤ 2.

For convenience, we introduce the following definitions. Standard models $\langle \mathbb{N}; =, R\rangle$ and $\langle \mathbb{Z}; =, R\rangle$, where R is the neighborhood relation, i.e. $\models iRj \Leftrightarrow |i-j| = 1$, we will call it the $\mathbb{N}$-model and the $\mathbb{Z}$-model, respectively. We call a graph component a $\mathbb{N}$-component and a $\mathbb{Z}$-component if it is isomorphic to the $\mathbb{N}$-model and $\mathbb{Z}$-model, respectively.

Let us assume that the non-root component of model of the theory T contains $k \geq 2$ vertices of degree 1. Let us choose two vertices of degree 1 in the component; by the definition of connectivity there is a finite chain that connects these two vertices and, obviously, this chain does not contain a root vertex. We obtain a contradiction with the group of axioms (5).

So, the non-root component of of model of the theory T can either contain exactly one vertex of degree 1, then it will obviously be a $\mathbb{N}$-component, or not contain a vertex of degree 1 at all, then obviously it will be a $\mathbb{Z}$-component, since due to the axiom group (2) there are no finite cycles in the graph.

Thus, we have practically given a description of all models of the theory of T: it consists of a root component, where there is exactly one branch of length i for each natural number $i \geq 1$ and any number of infinite branches, and any number of $\mathbb{N}$-components and $\mathbb{Z}$-components.

Theorem 5. *Any model of theory T has an elementary extension with a countable number of $\mathbb{N}$-components, a countable number of $\mathbb{Z}$-components, and a countable number of infinite branches in the root component.*

Proof. Consider the following constant enrichment of the language, obtained by adding the constants $a_{ij}, b_{ij}, i, j \in \omega$, and $c_{kl}, k \in \mathbb{Z}, l \in \omega$, together with the following extension of the theory T obtained by adding the axioms:

$$a_{ij} R a_{kl} \Leftrightarrow j = l \wedge |i-k| = 1,$$

$$\text{"}Vertex\ a_{0j}\ is\ connected\ to\ the\ root\ vertex,\text{"} j \in \omega,$$

$$b_{ij} R b_{kl} \Leftrightarrow j = l \wedge |i-k| = 1,$$

$$\text{"}The\ vertex\ b_{0j}\ has\ degree\ 1,\text{"} j \in \omega,$$

$$\text{"}The\ vertex\ b_{ij}\ has\ degree\ 2,\text{"} 0 < i \in \omega, j \in \omega$$

$$c_{ij} R c_{kl} \Leftrightarrow j = l \wedge |i-k| = 1,$$

$$\text{"}The\ vertex\ c_{ij}\ has\ degree\ 2,\text{"} i \in \mathbb{Z}, j \in \omega$$

It is clear that the chain $\{a_{ij}\}_{i\in\omega}$ is an infinite branch, the chain $\{b_{ij}\}_{i\in\omega}$ is the $\mathbb{N}$-component, and the chain $\{c_{ij}\}_{i\in\mathbb{Z}}$ is the $\mathbb{Z}$-component for each fixed $j \in \omega$ in any model of the extended theory. Let us assume that we are given some model $\langle G;=,R\rangle$ of the theory T. Let's further enrich our language by highlighting each element of G as a constant. As a result, our language is represented as $L \cup \{a_{ij}, b_{ij}, c_{kj}, d_g\}_{i,j\in\omega,k\in\mathbb{Z},g\in G}$. It is enough for us that the extended theory is consistent with the elementary diagram $T_\Delta = Th(\langle G;=,R,\{d_g\}_{g\in G}\rangle)$ of the model $\langle G;=,R\rangle$.

The latter is a consequence of the compactness theorem. Let T_0 be a finite set of sentences consisting of several sentences of the elementary diagram T_Δ and several axioms of the extension T given above. We can assume that these axioms (extensions) mention all the constants from the set $a_{ij}, b_{ij}, c_{ij}, 0 \leq i,j \leq n \in \omega$, and only them, and T_0 includes all axioms oms describing the relationship between these constants and their properties; we will show that these constants can be interpreted in the $\langle G;=,R\rangle$ so that all expansion axioms included in T_0 will be true.

As for the finite fragment T_Δ, we can assume that they only mention elements from the final branches of the root component, since other elements do not interfere with us when interpreting the constants $a_{ij}, b_{ij}, c_{ij}, 0 \leq i,j \leq n$, in the model $\langle G;=,R\rangle$.

First we select all finite branches of length $\leq m$ of the root component, covering all constants from $\{d_g\}_{g\in G}$ mentioned in sentences from T_0, and here we interpret each constant d_g as an element $g \in G$. Let $N = max\{m+1, n+1\}$. Next, we use n branches with lengths $N+1, N+2, \ldots, N+n$ to interpret the constants $a_{ij}, 0 \leq i,j \leq n$. After this, n branches with lengths $N+n+1, N+n+2, \ldots, N+2n$ are used to interpret the constants $b_{ij}, 0 \leq i,j \leq n$. And finally, branches with lengths $N+2n+1, N+2n+2, \ldots, N+3n$ are used to interpret the constants $c_{ij}, 0 \leq i,j \leq n$.

Corollary. *Theory T is complete.*

Indeed, all models of theory T that have a countable number $\mathbb{N}$-component, a countable number of $\mathbb{Z}$-components and a countable number of infinite branches in the root component are isomorphic to each other.

It is easy to see that a connected graph containing only finite branches in the root component is a prime model of the theory, and the model mentioned in the proposition proved above (with a countable number of $\mathbb{N}$-components, a countable number of $\mathbb{Z}$-components and a countable number of infinite branches in the root component) is countably saturated model of the theory T. □

Direct counting of types over a saturated model and study of the orthogonality property between them shows that the theory T is ω-stable and has dimension 3.

4. Concluding remarks

Acyclic graphs approximated by finite acyclic graphs are considered. It is proved that any countably categorical acyclic graph is smoothly approximable. An example of pseudofinite acyclic graph theory is given, which has an even, odd, and infinite number of rays. Any finite number of axioms from each group of axioms (1-5) are true for finite models. You will notice that the simple model has no rays. Theorem Nurtazin A.T. [11] the fact that the elementary theory of a graph with finite simple chains coincides with the elementary theory of finite graphs entails the pseudofiniteness of the constructed theory.

It can be noted that acyclic graph theories with an even number of rays can be approximated by finite acyclic graph theories, but if an acyclic graph has an odd number of rays, then its theory cannot be approximated by finite graph theories.

In the future, we will be interested not only in the model-theoretic and topological properties of acyclic graphs, but also in the combinatorial properties of pseudofinite graphs.

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