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Uniform Ultimate Boundedness of Lur'e Systems with Switchings and Delays

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Abstract. The paper investigates a hybrid system consisting of Lur'e subsystems with constant delays and time-dependent switching. It is assumed that nonlinearities from the right side of the systems have degrees less than unity. An analysis of such a property of the system as the uniform ultimate boundedness of all its solutions is conducted. The linear part of the system is supposed to be asymptotically stable. As is known, this means that there is a correspondent homogeneous Lyapunov function. Using this function, a common Lyapunov–Krasovskii functional is constructed which makes it possible to find sufficient conditions for the uniform ultimate boundedness with arbitrary choices of positive delays and switching laws. Moreover, delays can occur during switching, for example, when generating feedback. The derived conditions are found to be less conservative in the case of asynchronous switching compared to synchronous ones. The validity of the theoretical results is confirmed through numerical modeling.

Keywords: uniform ultimate boundedness, delay, synchronous and asynchronous switching

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Научная статья

Равномерная предельная ограниченность решений систем Лурье с переключениями и запаздываниями

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Аннотация. Исследуются гибридные системы, состоящие из подсистем Лурье с постоянными запаздываниями и переключениями во времени. Предполагается, что нелинейности из правых частей систем имеют степени меньшие единицы. Проводится анализ такого свойства системы, как ограниченность всех ее решений. Линейная часть системы является общей для всех подсистем и предполагается асимптотически устойчивой. Как известно, это означает, что существует соответствующая однородная функция Ляпунова. С помощью этой функции строится общий для всех подсистем функционал Ляпунова – Красовского, позволяющий найти достаточные условия равномерной предельной ограниченности решений при произвольном выборе положительных запаздываний и законе переключений. Более того, при формировании обратной связи могут возникать задержки, ведущие к возникновению запаздывания в законе переключений. Установлено, что полученные условия в случае таких асинхронных переключений оказываются менее жесткими, чем при синхронных. Достоверность теоретических результатов подтверждена посредством численного моделирования.

Ключевые слова: равномерная предельная ограниченность решений, запаздывание, синхронное и асинхронное переключения

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1. Introduction and problem statement

Consider a hybrid system with positive delays r and h

$$\begin{cases} \dot{x}(t) = By(t) + A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t)} f(x(t-r)), \\ \dot{y}(t) = Dy(t) + C_1^{\sigma(t)} f(x(t)) + C_2^{\sigma(t)} f(x(t-h)), \end{cases} \tag{1.1}$$

where $x(t)$ is an n-dimensional and $y(t)$ is ν-dimensional vectors, matrix parameters have the corresponding dimensions. Vector function $f(x) = (x_1^{\mu_1}, ..., x_n^{\mu_n})^T$ has positive rational degrees with odd denominators and numerators. Without loss of generality, assume that $\mu_1 \leq \ldots \leq \mu_n$. The matrix coefficients preceding the nonlinearities change over time. This alteration is governed by a switching law $\sigma(t)$ — a piecewise-constant right continuous function which maps the time interval $[0, +\infty)$ into the set $\{1,\dots,N\}$ of subsystems' numbers. Moreover, within each finite time period no more than a finite number of switchings can occur.

At each moment a subsystem of Lur'e-type is obtained [5; 7]. Such systems are particularly used in the theory of automatic system control and neural networks [10].

To get a solution $\left(x^T(t,t_0,\varphi); y^T(t,t_0,y_0)\right)^T$ of the system (1.1) we must specify an initial moment t_0, value $y_0 = y(t_0)$, and function φ from a set of piece-wise continuous on $[-\tau;0]$, $\tau = \max\{r,h\}$ functions with values in $\mathbb{R}^n$. For the differential–difference systems the state is denoted by $x_t(\theta) = x(t+\theta)$, where $\theta \in [-\tau;0]$. The uniform norm of some state φ is defined with the aid of euclidean one as follows: $\|\varphi\|_\tau = \sup_{\theta\in[-\tau;0]} \|\varphi(\theta)\|$.

The purpose of this work is to find the conditions for the system (1.1) to be uniformly ultimately bounded.

Definition 1. [8] *The system is uniformly ultimately bounded when there is $R>0$ such that for any $q>0$ there exists $T(q)>0$ for which $\|x(t,t_0,\varphi)\|^2 + \|y(t,t_0,y_0)\|^2 < R^2$ will hold whenever $\|\varphi\|_\tau^2 + \|y_0\|^2 < q^2$ for all $t \ge t_0+T$, $t_0 \ge 0$.*

Notice that delays in the system can occur both in nonlinear functions and switching law, e.g., while generating feedback or receiving information about the moment, when the current subsystem was changed. Therefore along with the system (1.1), a system with asynchronous switching [11–13]

$$\begin{cases} \dot{x}(t) = By(t) + A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t-r)} f(x(t-r)) \\ \dot{y}(t) = Dy(t) + C_1^{\sigma(t)} f(x(t)) + C_2^{\sigma(t-h)} f(x(t-h)) \end{cases} \tag{1.2}$$

will also be investigated. Here $\sigma(t)$ must be defined on $[-\tau,+\infty)$.

2. Analysis of the system with synchronous switching

In [1] a Lur'e system with homogeneous nonlinearities, delays, and asynchronous switching is studied via Lyapunov–Krasovskii approach. The found conditions of stability and ultimate boundedness are independent of the delay and the switching law. In this paper similar results are extended to the case of multiple homogeneous nonlinearities.

Assumption 1. *Let $\mu_n < 1$. Suppose also that there is a positive definite diagonal matrix Λ such that for all $s,p,q = 1,\dots,N$ the inequalities*

$$f_k^T \Lambda [A_1^{(s)} + A_2^{(p)} - BD^{-1}(C_1^{(s)} + C_2^{(q)})] f \le -\sum_{i=1}^{n} \beta_0^{(i)} x_i^{\mu_i(k+1)} \tag{2.1}$$

are valid; here vector $f_k = (x_1^{k\mu_1},\dots,x_n^{k\mu_n})^T$, rational $k \ge \dfrac{1}{\mu_1}$ has odd numerator and denominator, and $\beta_0^{(i)} > 0$, $i = 1,\dots,n$.

Remark 1. For condition (2.1) to be satisfied it is enough to require for the matrix in its left side to be Hurwitz and Metzler [2].

Theorem 1. *If assumption 1 is fulfilled and D is Hurwitz matrix, when the system* (1.1) *with any positive fixed delays is uniformly ultimately bounded.*

Proof. Since the system $\dot{y}(t) = Dy(t)$ is asymptotically stable, a homogeneous Lyapunov function $V_0(y)$ of order $m_0 > 1$ exists [14]. With a matrix $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ from Assumption 1, the Lyapunov–Krasovsky functional can be chosen:

$$V(x_t, y) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i(t)} \xi^{k\mu_i} d\xi - f_k^T \Lambda B D^{-1} \int_{-h}^{0} C_2^{\sigma(t+\theta+h)} f(x(t+\theta)) d\theta +$$
$$+ V_0(y) - f_k^T \Lambda B D^{-1} y + \int_{-r}^{0} (\beta_1 + \gamma_1(r+\theta)) \|f(x(t+\theta))\|^{m_1} d\theta +$$
$$+ f_k^T \Lambda \int_{-r}^{0} A_2^{\sigma(t+\theta+r)} f(x(t+\theta)) d\theta + \int_{-h}^{0} (\beta_2 + \gamma_2(h+\theta)) \|f(x(t+\theta))\|^{m_1} d\theta, \tag{2.2}$$

where $m_1, \beta_1, \beta_2, \gamma_1, \gamma_2$ are any positive numbers. Some approaches to constructing this type of functionals can be found in [3; 4].

We denote (i, j) element of matrix M by $(M)_{ij}$, and its i-th row by $(M)_i$.

To check whether (2.2) is positive, one can write down its lower bound using Hölder's inequality for the second and third integrals. This yields

$$V(x_t, y) \geq \sum_{i=1}^{n} \alpha_i x_i^{k\mu_i+1}(t) + \alpha_{n+1} \|y(t)\|^{m_0} - \sum_{i=1}^{n} \alpha_{n+i+1} |x_i(t)|^{k\mu_i} \|y(t)\| -$$
$$- r^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i' |x_i(t)|^{k\mu_i} \left(\int_{-r}^{0} \|f(x(t+\theta))\|^{m_1} d\theta \right)^{\frac{1}{m_1}} -$$
$$- h^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i'' |x_i(t)|^{k\mu_i} \left(\int_{-h}^{0} \|f(x(t+\theta))\|^{m_1} d\theta \right)^{\frac{1}{m_1}} +$$
$$+ \beta_1 \int_{-r}^{0} \|f(x(t+\theta))\|^{m_1} d\theta + \beta_2 \int_{-h}^{0} \|f(x(t+\theta))\|^{m_1} d\theta,$$

where

$$\alpha_i = \frac{\lambda_i}{k\mu_i + 1}, \ \alpha_{n+1} = \min_{\|y\|=1} V_0(y) > 0, \ \alpha_{n+i+1} = \max_{\|y\|=1} \Big| \sum_{j=1}^{\nu} (\Lambda B D^{-1})_{ij} y_j \Big|,$$

$$\alpha_i' = \max_{j=1,...,N} \big\| (\Lambda A_2^{(j)})_i \big\|, \quad \alpha_i'' = - \max_{j=1,...,N} \big\| (\Lambda B D^{-1} C_2^{(j)})_i \big\|, \quad i = 1, ..., n.$$

Now introduce the vector

$$s=\left(\tfrac{1}{k\mu_1+1},\ \ldots,\ \tfrac{1}{k\mu_n+1},\ \tfrac{1}{m_0},\ \tfrac{1}{m_1},\ \tfrac{1}{m_1}\right)$$

and define a generalized homogeneous norm $\|z\|_s = \sum_{i=1}^{n+3} |z_i|^{\frac{1}{s_i}}$ [6] for vector

$$z^T{=}\left(x_1, ..., x_n, \|y\|, \left[\int_{-r}^{0}\|f(x(t{+}\theta))\|^{m_1} d\theta\right]^{\frac{1}{m_1}}, \left[\int_{-h}^{0}\|f(x(t{+}\theta))\|^{m_1} d\theta\right]^{\frac{1}{m_1}}\right).$$

Hence, the latest functional assessment can be expressed in terms of vector s:

$$V \geq \sum_{i=1}^{n+1} \alpha_i z_i^{\frac{1}{s_i}} - \sum_{i=1}^{n} \alpha_{n+i+1} |z_i|^{\frac{1}{s_i}-1} z_{n+1} + \beta_1 z_{n+2}^{\overline{\frac{1}{s_{n+2}}}} + \beta_2 z_{n+3}^{\overline{\frac{1}{s_{n+3}}}} -$$
$$- r^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i' |z_i|^{\frac{1}{s_i}-1} z_{n+2} - h^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i'' |z_i|^{\frac{1}{s_i}-1} z_{n+3}.$$

Here nonnegative terms can be estimated as follows:

$$\sum_{i=1}^{n+1} \alpha_i z_i^{\frac{1}{s_i}} + \beta_1 z_{n+2}^{\overline{\frac{1}{s_{n+2}}}} + \beta_2 z_{n+3}^{\overline{\frac{1}{s_{n+3}}}} \geq \alpha^{(min)} \|z\|_s,\ \alpha^{(min)} = \min_{i=1,\ldots,n+1} \{\alpha_i, \beta_1, \beta_2\}.$$

Applying the property of homogeneous functions to the remaining terms and using the homogeneous norm of the (s_i, s_{n+j}) class, we obtain

$$|z_i|^{\frac{1}{s_i}-1} z_{n+j} = \left(|z_i|^{\frac{1}{2s_i}}\right)^{2-2s_i} \left(z_{n+j}^{\overline{\frac{1}{2s_{n+j}}}}\right)^{2s_{n+j}} \leq \alpha_{ij} \|z_i, z_{n+j}\|_{(s_i,\ s_{n+j})}^{1-s_i+s_{n+j}},$$

$$\alpha_{ij} = \max_{\|z_i, z_{n+j}\|_{(s_i, s_{n+j})}=1} |z_i|^{\frac{1}{s_i}-1} z_{n+j}, \qquad i = 1, \ldots, n, \qquad j = 1, 2, 3.$$

The last inequality can be enhanced:

$$|z_i|^{\frac{1}{s_i}-1} z_{n+j} \leq \alpha_{ij} \|z\|_s^{1-s_i+s_{n+j}}, \qquad i = 1, \ldots, n, \qquad j = 1, 2, 3.$$

Consequently, the assessment of the functional (2.2) can be written as:

$$V(x_t, y) \geq \alpha^{(min)} \|z\|_s - \sum_{i=1}^{n} \alpha_{n+i+1} \alpha_{i1} \|z\|_s^{1-s_i+s_{n+1}} -$$
$$- r^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i' \alpha_{i2} \|z\|_s^{1-s_i+s_{n+2}} - h^{\frac{m_1-1}{m_1}} \sum_{i=1}^{n} \alpha_i'' \alpha_{i3} \|z\|_s^{1-s_i+s_{n+3}}.$$

Taking into account $s_{n+2} = s_{n+3}$, denote

$$\overline{\alpha}_{i1} = \alpha_{n+i+1}\alpha_{i,1}, \qquad \overline{\alpha}_{i2} = r^{\frac{m_1-1}{m_1}}\alpha_i'\alpha_{i,2} + h^{\frac{m_1-1}{m_1}}\alpha_i''\alpha_{i,3}$$

and get

$$V(x_t, y) \geq \|z\|_s\Big(\alpha^{(min)} - \sum_{i=1}^{n}\overline{\alpha}_{i1}\|z\|_s^{s_{n+1}-s_i} - \sum_{i=1}^{n}\overline{\alpha}_{i2}\|z\|_s^{s_{n+2}-s_i}\Big).$$

It is necessary to satisfy $s_{n+j} \leq s_i,\ i = 1, \ldots, n,\ j = 1, 2$ for the functional to be positive within some region $\|z\|_s > H$ which will be specified further. That is

$$m_j \geq k\mu_i + 1, \quad i = 1, \ldots, n,\ j = 0, 1. \tag{2.3}$$

Note that situations may arise when conditions (2.3) turn into equalities for some (i, j) from a set I. To ensure the positivity of the functional in an arbitrary domain, we need to check additional conditions on the parameters:

$$\alpha^{(min)} > \sum_{(i,j)\in I}\overline{\alpha}_{ij}.$$

In case when the inequalities (2.3) are strict, the conditions

$$\|z\|_s > \left(\overline{\alpha}_{ij}\frac{2n}{\alpha^{(min)}}\right)^{\frac{1}{s_i - s_{n+j}}} =: H_{ij}, \quad (i, j) \notin I$$

entail that the general value $H_0 = \max_{i,j} H_{ij}$ defines the domain $\|z\|_s > H_0$ of the functionals positivity. Therefore, the right side of the estimate

$$V(x_t, y) \geq \alpha\|z\|_s \tag{2.4}$$

has a positive coefficient

$$\alpha = \alpha^{(min)} - \sum_{j=1}^{2}\sum_{i=1}^{n}\overline{\alpha}_{ij}H_0^{s_{n+j}-s_i}.$$

In a similar way an upper bound for (2.2)

$$V(x_t, y) \leq \gamma\|z\|_s \tag{2.5}$$

can be obtained in the region $\|z\|_s > H_0$; here

$$\gamma = \alpha^{(max)} + \sum_{j=1}^{2}\sum_{i=1}^{n}\overline{\alpha}_{ij}H_0^{s_{n+j}-s_i}, \qquad \alpha^{(max)} = \max_{i=1,\ldots,n+1}\{\alpha_i, \beta_1, \beta_2\}.$$

Derivative of the functional (2.2) along the solution of system (1.1) can be represented as

$$\dot V = f_k^T(x(t))\Lambda[A_1^{\sigma(t)} + A_2^{\sigma(t+r)} - BD^{-1}(C_1^{\sigma(t)} + C_2^{\sigma(t+h)})]f(x(t)) + \dot V_0 -$$
$$-[By(t) + A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t)} f_r]^T \frac{\partial f_k(t)}{\partial x}\Lambda BD^{-1} y(t) + [By(t) +$$
$$+A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t)} f_r]^T \frac{\partial f_k(t)}{\partial x}\Lambda\Big[\int_{-r}^0 A_2^{\sigma(t+\theta+r)} f(x(t+\theta))d\theta -$$
$$-BD^{-1}\int_{-h}^0 C_2^{\sigma(t+\theta+h)} f(x(t+\theta))d\theta\Big] - \beta_1\|f_r\|^{m_1} +$$
$$+\Big[\beta_1 + \gamma_1 r + \beta_2 + \gamma_2 h\Big]\|f(x(t))\|^{m_1} - \beta_2\|f_h\|^{m_1} -$$
$$-\gamma_1\int_{-r}^0 \|f(x(t+\theta))\|^{m_1} d\theta - \gamma_2\int_{-h}^0 \|f(x(t+\theta))\|^{m_1} d\theta,$$

where $f_r = f(x(t-r))$, $f_h = f(x(t-h))$, Jacobian matrix

$$\frac{\partial f(x(t))}{\partial x} = \begin{pmatrix} \frac{\partial f_1(x(t))}{\partial x_1} & \dots & \frac{\partial f_1(x(t))}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n(x(t))}{\partial x_1} & \dots & \frac{\partial f_n(x(t))}{\partial x_n} \end{pmatrix}.$$

Let us separately evaluate the following terms of the derivative:

$$\big(By + A_1^{\sigma(t)} f + A_2^{\sigma(t)} f_r\big)^T \frac{\partial f_k}{\partial x}\Lambda BD^{-1} y \geq -\sum_{i=1}^n \beta_2^{(i)} x_i^{k\mu_i - 1}\|y\|^2 -$$
$$-\sum_{l=1}^n |x_l^{\mu_l}| \sum_{i=1}^n \alpha'_{il} x_i^{k\mu_i-1}\|y\| - \sum_{i=1}^n \beta_1^{(i)} x_i^{k\mu_i-1}\|y\|\|f_r\|,$$

where

$$\beta_2^{(i)} = |\alpha_2^{(i)}|\|(B)_i\|, \qquad \alpha_2^{(i)} = k\mu_i \max_{\|y\|=1}\left|\sum_{j=1}^{\nu}(\Lambda BD^{-1})_{ij} y_j\right|,$$

$$\beta_1^{(i)} = |\alpha_2^{(i)}| \max_j \|(A_2^{(j)})_i\|, \qquad \alpha'_{il} = \max_j |(A_1^{(j)})_{il}|\alpha_2^{(i)}.$$

Likewise,

$$\dot x^T \frac{\partial f_k}{\partial x}\int_{-r}^0 \Lambda A_2^{\sigma(t+\theta+r)} f(x(t+\theta))d\theta \leq \Bigg(\sum_{i=1}^n \beta_4^{(i)} x_i^{k\mu_i-1}\|y\| +$$
$$+\sum_{l=1}^n |x_l^{\mu_l}|\sum_{i=1}^n \alpha''_{il} x_i^{k\mu_i-1} + \sum_{i=1}^n \beta_3^{(i)} x_i^{k\mu_i-1}\|f_r\|\Bigg)\int_{-r}^0 \|f(x(t+\theta))\|d\theta,$$

with parameters $\alpha_3^{(i)} = k\mu_i\alpha_i'$, $\beta_3^{(i)} = \alpha_3^{(i)} \max_s \|(A_2^{(s)})_i\|$, $\beta_4^{(i)} = \alpha_3^{(i)}\|(B)_i\|$, and $\alpha_{il}'' = \max_s |(A_1^{(s)})_{il}|\alpha_3^{(i)}$. As above,

$$\dot{x}^T\frac{\partial f_k}{\partial x}\Lambda BD^{-1}\int_{-h}^{0} C_2^{\sigma(t+\theta+h)}f(x(t+\theta))d\theta \geq -\Big(\sum_{i=1}^{n}\beta_6^{(i)}x_i^{k\mu_i-1}\|y\|+ + \sum_{l=1}^{n}|x_l^{\mu_l}|\sum_{i=1}^{n}\alpha_{il}'''x_i^{k\mu_i-1}+\sum_{i=1}^{n}\beta_5^{(i)}x_i^{k\mu_i-1}\|f_r\|\Big)\int_{-h}^{0}\|f(x(t+\theta))\|d\theta,$$

where $\alpha_4^{(i)} = k\mu_i\alpha_i''$, $\beta_5^{(i)} = |\alpha_4^{(i)}|\|(B)_i\|$, $\beta_6^{(i)} = |\alpha_4^{(i)}|\max_s \|(A_2^{(s)})_i\|$, $\alpha_{il}''' = \max_s |(A_1^{(s)})_{il}|\alpha_4^{(i)}$.

Taking into account the assessment of homogeneous function: $\left\|\frac{\partial V_0}{\partial y}\right\| \leq \gamma_{n+1}'\|y\|^{m_0-1}$, where γ_{n+1}' is a positive constant, we have

$$\dot{V}_0 = \dot{V}_0\big|_{\dot{y}=Dy} + \left(\frac{\partial V_0}{\partial y}\right)^T\Big(C_1^{\sigma(t)}f + C_2^{\sigma(t)}f\big(x(t-h)\big)\Big) \leq -\beta_{n+1}\|y\|^{m_0}+ + \gamma_{n+1}'\|y\|^{m_0-1}\Big(\max_s\|C_1^{(s)}\|\|f\|+\max_s\|C_2^{(s)}\|\|f\big(x(t-h)\big)\|\Big).$$

Now we can continue estimating the derivative of the functional:

$$\dot{V} \leq -\sum_{i=1}^{n}\beta_0^{(i)}x_i^{\mu_i(k+1)} + \gamma_{n+1}'\|y\|^{m_0-1}\max_s\|C_1^{(s)}\|\Big(\sum_{l=1}^{n}x_l^{2\mu_l}\Big)^{\frac{1}{2}} - - \beta_1\|f_r\|^{m_1} + \gamma_{n+1}'\|y\|^{m_0-1}\max_s\|C_2^{(s)}\|\|f_h\| - \gamma_1\int_{-r}^{0}\|f(x(t+\theta))\|^{m_1}d\theta - - \beta_{n+1}\|y\|^{m_0} + \sum_{i=1}^{n}\beta_2^{(i)}x_i^{k\mu_i-1}\|y\|^2 + \sum_{l=1}^{n}|x_l^{\mu_l}|\sum_{i=1}^{n}\alpha_{il}'x_i^{k\mu_i-1}\|y\|+ + \sum_{i=1}^{n}\beta_1^{(i)}x_i^{k\mu_i-1}\|y\|\|f_r\| + \Big(\sum_{i=1}^{n}\beta_4^{(i)}x_i^{k\mu_i-1}\|y\| + \sum_{l=1}^{n}|x_l^{\mu_l}|\sum_{i=1}^{n}\alpha_{il}''x_i^{k\mu_i-1}+ + \sum_{i=1}^{n}\beta_3^{(i)}x_i^{k\mu_i-1}\|f_r\|\Big)\int_{-r}^{0}\|f(x(t+\theta))\|d\theta + \Big(\sum_{i=1}^{n}\beta_6^{(i)}x_i^{k\mu_i-1}\|y\|+ + \sum_{l=1}^{n}|x_l^{\mu_l}|\sum_{i=1}^{n}\alpha_{il}'''x_i^{k\mu_i-1} + \sum_{i=1}^{n}\beta_5^{(i)}x_i^{k\mu_i-1}\|f_r\|\Big)\int_{-h}^{0}\|f(x(t+\theta))\|d\theta + + \big[\beta_1 + \gamma_1 r + \beta_2 + \gamma_2 h\big]\|f\|^{m_1} - \beta_2\|f_h\|^{m_1} - \gamma_2\int_{-h}^{0}\|f(x(t+\theta))\|^{m_1}d\theta.$$

We introduce new variables

$$\tilde{z} = \Big(z,\ \|f(x(t-r))\|, \|f(x(t-h))\|\Big)^T, \quad s'' = \big(s',\ \tfrac{1}{m_1},\ \tfrac{1}{m_1}\big),$$

where

$$s' = \left(\tfrac{1}{\mu_1(k+1)},\ \ldots,\ \tfrac{1}{\mu_n(k+1)},\ \tfrac{1}{m_0},\ \tfrac{1}{m_1},\ \tfrac{1}{m_1}\right).$$

According to the properties of homogeneous functions, there are constants $\tilde{\beta}_0 > 0$, $\beta_{0,i}^{(j)} \geq 0$, $j = 1, \ldots, 5$ and $\beta_0^{(j)} > 0$, $j = 6, 7, 8$ such that

$$\dot{V} \leq -\tilde{\beta}_0 \|\tilde{z}\|_{s''} + \sum_{i=1}^{n} \beta_{0,i}^{(1)} \|\tilde{z}\|_{s''}^{\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{2}{m_0}} + \sum_{i=1}^{n} \beta_{0,i}^{(2)} \|\tilde{z}\|_{s''}^{\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{1}{m_0} + \frac{1}{k+1}} +$$
$$+ \sum_{i=1}^{n} \beta_{0,i}^{(3)} \|\tilde{z}\|_{s''}^{\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{1}{m_0} + \frac{1}{m_1}} + \sum_{i=1}^{n} \beta_{0,i}^{(4)} \|\tilde{z}\|_{s''}^{\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{1}{k+1} + \frac{1}{m_1}} + \beta_0^{(6)} \|\tilde{z}\|_{s''}^{\frac{m_1}{k+1}} +$$
$$+ \sum_{i=1}^{n} \beta_{0,i}^{(5)} \|\tilde{z}\|_{s''}^{\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{2}{m_1}} + \beta_0^{(7)} \|\tilde{z}\|_{s''}^{\frac{2m_0 + m_0 k - k - 1}{m_0(k+1)}} + \beta_0^{(8)} \|\tilde{z}\|_{s''}^{1 + \frac{1}{m_1} - \frac{1}{m_0}}.$$

We can find conditions for the parameters k, m_0, m_1 to ensure that in the last estimate of the derivative of the functional the norms' degrees do not exceed 1. Hence,

$$\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{2}{m_j} \leq 1 \Longleftrightarrow m_j \geq \frac{2\mu_i(k+1)}{\mu_i + 1}, \qquad j = 0, 1;$$
$$\frac{1}{k+1} + \frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{1}{m_j} \leq 1 \Longleftrightarrow m_j \geq \mu_i(k+1), \qquad j = 0, 1;$$
$$\frac{k\mu_i - 1}{\mu_i(k+1)} + \frac{1}{m_0} + \frac{1}{m_1} \leq 1 \Longleftrightarrow \frac{1}{m_0} + \frac{1}{m_1} \leq \frac{\mu_i + 1}{\mu_i(k+1)};$$
$$m_0 \leq m_1 \leq k + 1.$$

Combining the obtained inequalities with the conditions for V to be positive: $m_j \geq k\mu_i + 1$, $j = 0, 1$, the total result is

$$\frac{2\mu_i(k+1)}{\mu_i + 1} \leq m_0 \leq m_1 \leq k + 1.$$

Under such conditions there is a region $\|\tilde{z}\|_{s''} > H'_0$, where

$$\dot{V}(x_t, y) \leq -\beta \|\tilde{z}\|_{s''}$$

is valid for some positive constant β. Moreover, the estimation

$$\dot{V}(x_t, y) \leq -\beta \|z\|_{s'} \tag{2.6}$$

will work in the region $\|z\|_{s'} > H'_0$.

The previously obtained regions $\|z\|_s > H_0$ of functional positivity and $\|z\|_{s'} > H'_0$ of its derivative's negativity follow from the common region $\|z\| > H$, where $H = \sqrt{n+3}\max_j \left\{H_0^{s_j}; H_0'^{s'_j}\right\}$.

Then, differential inequality

$$\dot{V}(x_t, y) \leq -cV^\rho(x_t, y) \tag{2.7}$$

could be obtained in the found unified region. Constant c is positive and $\rho = \min_{i=1,\dots,n+3} \frac{s_i}{s_i'}$ less then 1. The (2.7) follows from the fact that in $\|z\| > H$

$$\|z\|_{s'} = \sum_{i=1}^{n+3} z_i^{\frac{1}{s_i'}} \geq \sum_{i=1}^{n+3} a_i \Big(z_i^{\frac{1}{s_i}} \Big)^\rho \geq c_2 \|z\|_s^\rho,$$

where $c_2 > 0$ and $a_i > 0$ for all $i = 1, \dots, n+3$.

Further we will use the approach from [1] for proving the boundedness of a solution $\big(x^T(t, t_0, \varphi); y^T(t, t_0, y_0)\big)^T$. Differential inequality (2.7) works when $\|z\| > H$. Then it works all the more in

$$\|z\|_s > \tilde{H}, \qquad \tilde{H} = \max \left\{ \frac{H}{\sqrt{n+3}}^{\frac{1}{\min s_i}}, \ \frac{H}{\sqrt{n+3}}^{\frac{1}{\max s_i}} \right\}.$$

Using the assessment (2.4) value $\Delta = \alpha \tilde{H}$ is chosen so that for $V(x_t, y) \geq \Delta$ the inequality (2.7) is still valid.

If at the initial moment the functional $V(\varphi, y_0) \leq \Delta$, then it will remain in the domain $V(x_t, y) \leq \Delta$ because of its decreasing outside. In the case when $V(\varphi, y_0) > \Delta$ the moment T can be found when $V(x_t, y) \leq \Delta$ for all $t \geq t_0 + T$. For this purpose, we integrate the inequality (2.7) for $t \in [t_0; t_1]$ while $V(x_t, y) > \Delta$:

$$V(x_t, y) \leq \Big(V^{1-\rho}(\varphi, y_0) - c(1-\rho)(t - t_0) \Big)^{\frac{1}{1-\rho}}.$$

Now it can be noticed that the inequality $V(x_t, y) > \Delta$ will be broken for $t \geq t_0 + T$, where

$$T = \frac{1}{c(1-\rho)} \Big(V^{1-\rho}(\varphi, y_0) - \Delta^{1-\rho} \Big).$$

From (2.4) the region of ultimate boundedness for system (1.1) can be derived:

$$\sum_{i=1}^{n} x_i^{k\mu_i + 1}(t) + \|y(t)\|^{m_0} \leq \frac{\Delta}{\alpha}. \tag{2.8}$$

Let $\tilde{\Delta}$ be any positive quantity and $\omega = \sup_{\|\psi\|_\tau^2 + \|\eta\|^2 \leq \tilde{\Delta}^2} V(\psi, \eta)$. Then for $\|\psi\|_\tau^2 + \|\eta\|^2 \leq \tilde{\Delta}^2$ the solution $\big(x^T(t, t_0, \varphi); y^T(t, t_0, y_0)\big)^T$ of system (1.1) belongs to (2.8) for any $t \geq \max\{0, \tilde{t}\}$, where

$$\tilde{t} = \frac{1}{c(1-\rho)} \Big(\omega^{1-\rho} - \Delta^{1-\rho} \Big).$$

This completes the proof of Theorem 1. □

3. Analysis of a system with asynchronous switching

The subject of this section is the system with asynchronous switching (1.2).

Assumption 2. *Let $\mu_n < 1$ and suppose that there is a positive definite diagonal matrix Λ such that for every $s = 1, \dots, N$ the inequalities*

$$f_k^T \Lambda [A_1^{(s)} + A_2^{(s)} - BD^{-1}(C_1^{(s)} + C_2^{(s)})] f \leq -\sum_{i=1}^{n} \beta_0^{(i)} x_i^{\mu_i(k+1)}$$

hold, where $\beta_0^{(i)} > 0$.

Theorem 2. *If assumption 2 is fulfilled and D is Hurwitz matrix, when the system* (1.2) *with any positive fixed delays is uniformly ultimately bounded.*

Proof. Based on (2.2) a modified functional can be constructed

$$V(x_t, y) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} \xi^{k\mu_i} d\xi + V_0(y) - f_k^T \Lambda BD^{-1} y + f_k^T \Lambda \int_{-r}^{0} A_2^{\sigma(t+\theta)} \times$$

$$\times f(x(t+\theta)) d\theta - f_k^T \Lambda BD^{-1} \int_{-h}^{0} C_2^{\sigma(t+\theta)} f(x(t+\theta)) d\theta + \int_{-r}^{0} (\beta_1 +$$

$$+\gamma_1(r+\theta)) \|f(x(t+\theta))\|^{m_1} d\theta + \int_{-h}^{0} (\beta_2 + \gamma_2(h+\theta)) \|f(x(t+\theta))\|^{m_1} d\theta, \tag{3.1}$$

here λ_i are components of matrix Λ from Assumption 2.

The derivative of the functional (3.1) along the solution of system (1.2) takes the form

$$\dot{V} = f_k^T(x(t)) \Lambda [A_1^{\sigma(t)} + A_2^{\sigma(t)} - BD^{-1}(C_1^{\sigma(t)} + C_2^{\sigma(t)})] f(x(t)) + \dot{V}_0 -$$

$$-[By(t) + A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t-r)} f(x(t-r))]^T \frac{\partial f_k(t)}{\partial x} \Lambda BD^{-1} y(t) +$$

$$+[By(t) + A_1^{\sigma(t)} f(x(t)) + A_2^{\sigma(t-r)} f(x(t-r))]^T \frac{\partial f_k(t)}{\partial x} \Lambda \times$$

$$\times \left[\int_{-r}^{0} A_2^{\sigma(t+\theta)} f(x(t+\theta)) d\theta - BD^{-1} \int_{-h}^{0} C_2^{\sigma(t+\theta)} f(x(t+\theta)) d\theta \right] + \Big[\beta_1 +$$

$$+\gamma_1 r + \beta_2 + \gamma_2 h \Big] \|f(x(t))\|^{m_1} - \beta_1 \|f(x(t-r))\|^{m_1} - \beta_2 \|f(x(t-h))\|^{m_1} -$$

$$-\gamma_1 \int_{-r}^{0} \|f(x(t+\theta))\|^{m_1} d\theta - \gamma_2 \int_{-h}^{0} \|f(x(t+\theta))\|^{m_1} d\theta.$$

In the same way as it was obtained for assessments (2.4), (2.5) and (2.6) one can show the existence of positive values H_0, H_0' α, β, and γ such that

$$\alpha \|z\|_s \leqslant V(x_t, y) \leqslant \gamma \|z\|_s, \qquad \text{for } \|z\|_s > H_0$$

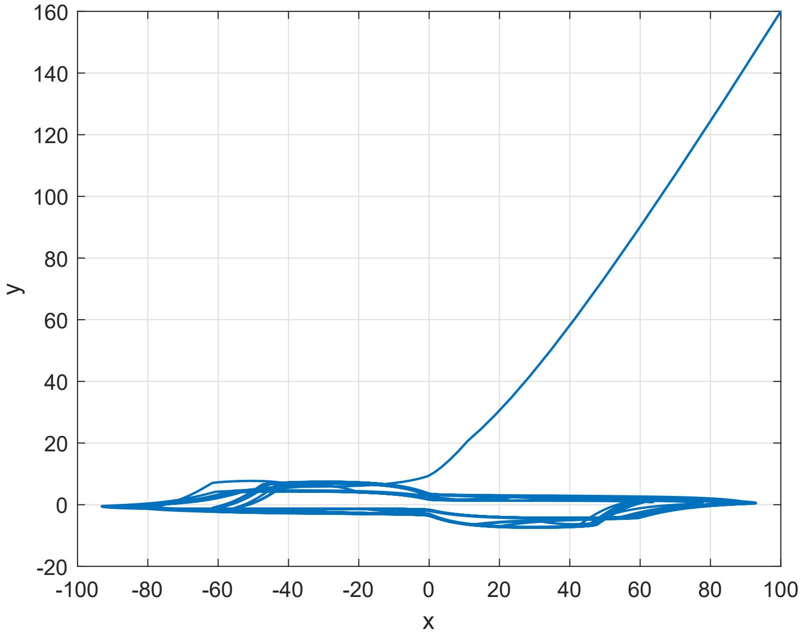

Figure 1. Solution of (1.1) in phase space.

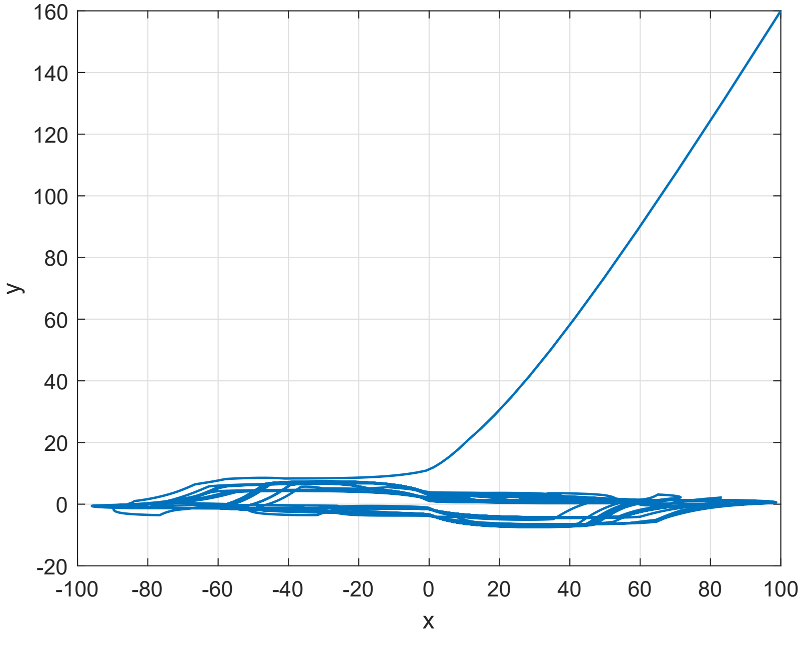

Figure 2. Solution of (1.2) in phase space.

and

$$\dot{V}(x_t, y) \leq -\beta\|z\|_{s'}, \qquad \text{for } \|z\|_{s'} > H_0'.$$

Hence, the subsequent proof follows the same logical reasoning as presented in Theorem 1. □

4. Example

To illustrate the obtained results, consider an example. Suppose the systems (1.1) and (1.2) consist of two subsystems with scalar parameters

$$B = -1,\ D = -2,\ A_1^{(1)} = -1,\ A_2^{(1)} = -5,\ C_1^{(1)} = -3,\ C_2^{(1)} = 5,$$

$$r = 5,\ h = 3,\ \mu = \frac{1}{5},\ A_1^{(2)} = 3,\ A_2^{(2)} = -5,\ C_1^{(2)} = -2,\ C_2^{(2)} = 3.$$

It can easily be checked that Assumptions 1 and 2 are valid. Initial data: $t_0 = 0$, $x_{t_0} \equiv 100$ for $[-5; 0]$, $y(t_0) = 160$. The switching law is given by

$$\sigma(t) = 1 \text{ for } t \in [-5; 1) \cup \left[\frac{i(i-1)}{2}; \frac{i(i+1)}{2}\right), \quad i = 3, 5, \ldots$$

$$\sigma(t) = 2 \text{ for } t \in \left[\frac{i(i-1)}{2}; \frac{i(i+1)}{2}\right), \quad i = 2, 4, \ldots.$$

The Runge–Kutta method of fourth-order with accuracy 0.01 is applied for solving the systems. Charts are plotted in phase spaces. Figure 1 presents the solution of the system (1.1) with synchronous switching and Figure 2 corresponds to the case of system (1.2) with asynchronous switching.

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