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Layer-finiteness of Some Groups

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Abstract. Infinite groups with finiteness conditions for an infinite system of subgroups are studied. Groups with a condition: the normalizer of any non-trivial finite subgroup is a layer-finite group or the normalizer of any non-trivial finite subgroup has a layer-finite periodic part are studied for beginning in the locally finite class of group, then in the class of periodic groups of Shunkov and finally in the class of Shunkov groups which are contain a strongly embedded subgroup with an almost layer-finite periodic part. The group G is called the Shunkov group if for any prime p and for every finite subgroups H from G any two conjugate elements of order p from the factor-group $N_G(H)/H$ generate a finite subgroup. Results for almost layer-finite groups and groups with almost layer-finite periodic part are transferred to layer-finite groups and groups with layer-finite periodic part. New characterizations of layer-finite groups and groups with layer-finite periodic part are obtained.

Keywords: periodic part, layer-finite group, almost layer-finite group, Shunkov group

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Научная статья

Слойная конечность в некоторых группах

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Аннотация. Изучаются бесконечные группы с условиями конечности для бесконечной системы подгрупп. Группы с условием: нормализатор любой нетривиальной конечной подгруппы является слойно конечной группой или нормализатор любой нетривиальной конечной подгруппы имеет слойно конечную периодическую часть, изучаются сначала в классе локально конечных групп, затем в классе периодических групп Шункова и, наконец, в классе групп Шункова, содержащих сильно вложенную подгруппу с почти слойно конечной периодической частью. Группа G называется группой Шункова, если для любого простого числа p и для любой конечной подгруппы H из G любые два сопряженных элемента порядка p из фактор-группы $N_G(H)/H$ порождают конечную подгруппу. Результаты для почти слойно конечных групп и групп с почти слойно конечной периодической частью переносятся на слойно конечные группы и группы со слойно конечной периодической частью. Получены новые характеризации слойно конечных групп и групп со слойно конечной периодической частью.

Ключевые слова: периодическая часть, слойно конечная группа, почти слойно конечная группа, группа Шункова

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1. Introduction

Infinite groups with finiteness conditions include periodic, locally finite groups. Later, finiteness conditions appeared for a system of subgroups, which include the conjugately biprimitively finite groups [16] we are studying, or, in modern terminology, Shunkov groups. This paper continues to study the question of how the properties of subgroups influence the structure of the group. O. Yu. Schmidt was the first to consider such issues in our country (see, for example, [15]).

We recall some necessary definitions:

Definition 1. *A layer-finite group is a group whose set of elements of any given order is finite.*

The concept of a layer-finite group first appeared without a name in the work of S. N. Chernikov [8], then in his subsequent works layer-finite groups received a complete description (see, for example, [9]).

Definition 2. *An almost layer-finite group is an extension of a layer-finite group by a finite group.*

Definition 3. *By the periodic part of a group we mean the set of its elements of finite order if the latter is a group [3].*

Definition 4. *The group G is called the Shunkov group if for any prime p and for every finite subgroups H from G any two conjugate elements of order p from the factor-group $N_G(H)/H$ generate a finite subgroup.*

The author noted that a number of results for almost layer-finite groups and groups with almost layer-finite periodic part are easily transferred to layer-finite groups and groups with layer-finite periodic part.

Recently, a number of works have also been devoted to the study of Shunkov groups [1; 2; 10–14].

2. Main results

V. P. Shunkov proved the next Theorem

Theorem 1. *(V. P. Shunkov, Theorem 1 from [4]). A locally finite group G if and only if is almost layer-finite if G satisfies the condition: the normalizer of any non-trivial finite subgroup of G is an almost layer-finite group.*

On the base of this result we will prove

Theorem 2. *A locally finite group G if and only if is layer-finite if G satisfies the condition: the normalizer of any non-trivial finite subgroup of G is a layer-finite group.*

Proof. Let G be a locally finite group and the normalizer of any non-trivial finite subgroup of G is a layer-finite group. By Theorem 1, the group G is almost layer-finite. Since in an almost layer-finite group there is a finite normal subgroup, then the normalizer of this subgroup is layer-finite by conditions of the theorem and so G is layer-finite group. The theorem is proved. □

Previously, the author characterized groups with a condition: the normalizer of any non-trivial finite subgroup of G is an almost layer-finite group or has an almost layer-finite periodic part. Shunkov's groups with such condition was studied under additional restrictions: either the periodic group contains a strongly embedded subgroup [5], or the periodic group does not contain subgroups of the form $PSL_2(q)$ [6], or the group contains a strongly embedded subgroup with a Chernikov periodic part, the periodic group contains a strongly embedded almost layer-finite subgroup, or the group contains a strongly embedded subgroup with an almost layer-finite periodic part [7].

Theorem 3. *(V. I. Senashov, [5]). Let periodic group of Shunkov G contains a strongly embedded subgroup. If in G the normalizer of any non-trivial finite subgroups is almost layer-finite, then the group G is almost layer-finite.*

This Theorem we can formulate for layer finiteness of group and subgroups instead of almost layer finiteness.

Theorem 4. *Let periodic group of Shunkov G contains a strongly embedded subgroup. If in G the normalizer of any non-trivial finite subgroups is layer-finite, then the group G is layer-finite.*

Proof. Let G be a periodic group of Shunkov and in G the normalizer of any non-trivial finite subgroups is layer-finite. By Theorem 3 the group G is almost layer-finite. Since in an almost layer-finite group there is a finite normal subgroup, then the normalizer of this subgroup is layer-finite by conditions of the theorem and so G is layer-finite group. The theorem is proved. □

The following two theorems characterize almost layer-finite groups in the class of periodic Shunkov groups under additional restrictions: in Theorem 5 in the absence of third-order elements in the group, and in Theorem 6 under a weaker restriction in the absence in the group of subgroups of the form $PSL_2(q)$.

Theorem 5. *(V. I. Senashov, [6]). Let G be a periodic Shunkov group without third order elements. If in G the normalizer of any non-trivial finite subgroup is almost layer-finite, then the group G is also almost layer-finite.*

Theorem 6. *(V. I. Senashov, [6]). Let G be a periodic Shunkov group without subgroups of the form $PSL_2(q)$. If in G the normalizer of any non-trivial finite subgroup is almost layer-finite, then the group G is also almost layer-finite.*

These two theorems can similarly be reformulated for layer-finite groups instead of almost layer-finite groups.

Theorem 7. *Let G be a periodic Shunkov group without third order elements. If in G the normalizer of any non-trivial finite subgroup is layer-finite, then the group G is also layer-finite.*

Theorem 8. *Let G be a periodic Shunkov group without subgroups of the form $PSL_2(q)$. If in G the normalizer of any non-trivial finite subgroup is layer-finite, then the group G is also layer-finite.*

The proofs of Theorems 7 and 8 are similar to the proof of Theorem 4.

Let us present another theorem for the Shunkov group with strongly embedded subgroup with an almost layer-finite periodic part in which normalizer of any non-trivial finite subgroup has an almost layer-finite periodic part.

Three articles of the author are devoted to the proof of the following theorem.

Theorem 9. *(V. I. Senashov, [7]). Let a Shunkov group contains a strongly embedded subgroup with an almost layer-finite periodic part. If in the group the normalizer of any non-trivial finite subgroup has an almost layer-finite periodic part, then the group itself has an almost layer-finite periodic part.*

This Theorem we can formulate for layer finiteness of periodic parts of group and periodic parts of subgroups instead of almost layer finiteness of them.

Theorem 10. *Let a Shunkov group contains a strongly embedded subgroup with an almost layer-finite periodic part. If in the group the normalizer of any non-trivial finite subgroup has a layer-finite periodic part, then the group has a layer-finite periodic part.*

Proof. Let G be a group of Shunkov with strongly embedded subgroup with an almost layer-finite periodic part in which normalizer of any non-trivial finite subgroup has a layer-finite periodic part. By Theorem 9 the group G has an almost layer-finite periodic part P. Since in an almost layer-finite group P there is a finite characteristic subgroup K, then the normalizer of the subgroup K coincide with the group G and has a layer-finite periodic part by conditions of the theorem. So G has a layer-finite periodic part. The theorem is proved. □

Despite the ease of proof, the results obtained are new. Previously, similar results were obtained with restrictions on soluble subgroups. Without using Theorem 9, the proof of Theorem 10 would be quite long (the proof of the similar theorem of the author takes up 12 journal pages).

3. Conclusion

New characterizations of layer-finite groups and groups with layer-finite periodic part are obtained.

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