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Variations of Rigidity

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Abstract. One of the main derived objects of a given structure is its automorphism group, which shows how freely elements of the structure can be related to each other by automorphisms. Two extremes are observed here: the automorphism group can be transitive and allow any two elements to be connected to each other, or can be one-element, when no two different elements are connected by automorphisms, i.e., the structure is rigid. The rigidity given by a one-element group of automorphisms is called semantic. It is of interest to study and describe structures that do not differ much from semantically rigid structures, i.e., become semantically rigid after selecting some finite set of elements in the form of constants. Another, syntactic form of rigidity is based on the possibility of getting all elements of the structure into a definable closure of the empty set. It is also of interest here to describe "almost" syntactically rigid structures, i.e., structures covered by the definable closure of some finite set. The paper explores the possibilities of semantic and syntactic rigidity. The concepts of the degrees of semantic and syntactic rigidity are defined, both with respect to existence and with respect to the universality of finite sets of elements of a given cardinality. The notion of a rigidity index is defined, which shows an upper bound for the cardinalities of algebraic types, and its possible values are described. Rigidity variations and their degrees are studied both in the general case, for special languages, including the one-place predicate signature, and for some natural operations with structures, including disjunctive unions and compositions of structures. The possible values of the degrees for a number of natural examples are shown, as well as the dynamics of the degrees when taking the considered operations.

Keywords: definable closure, semantic rigidity, syntactic rigidity, degree of rigidity

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Аннотация. Отмечено, что одним из основных производных объектов данной структуры является ее группа автоморфизмов, показывающая насколько свободно элементы структуры могут быть между собой связаны автоморфизмами. Здесь наблюдаются две крайности: группа автоморфизмов может быть транзитивной и позволяющей связывать между собой любые два элемента, или одноэлементной, когда никакие два различных элемента не связаны между собой автоморфизмами, т.е. структура является жесткой. Жесткость, задаваемая одноэлементной группой автоморфизмов, называется семантической. Представляет интерес изучение и описание структур, которые несильно отличаются от семантически жестких структур, т.е. становятся семантически жесткими после выделения некоторого конечного множества элементов в виде констант. Другой, синтаксический вид жесткости основан на возможности попадания всех элементов структуры в определимое замыкание пустого множества. Здесь также представляет интерес описания «почти» синтаксически жестких структур, т.е. структур, покрываемых определимым замыканием некоторого конечного множества. В работе изучены возможности семантической и синтаксической жесткости. Рассмотрены понятия степени семантической и синтаксической жесткости как относительно существования, так и относительно всеобщности конечных множеств элементов заданной мощности. Определено понятие индекса жесткости, показывающее верхнюю оценку для мощностей алгебраических типов, и описаны его возможные значения. Исследованы вариации жесткости и их степеней как в общем случае для специальных сигнатур, включая сигнатуру одноместных предикатов, так и для некоторых естественных операций со структурами, включая дизъюнктные объединения и композиции структур. Показаны возможные значения степеней для ряда естественных примеров, а также динамика степеней при взятии рассматриваемых операций.

Ключевые слова: определимое замыкание, семантическая жесткость, синтаксическая жесткость, степень жесткости

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1. Introduction

We continue to study variations of algebraic closures [10] considering and describing semantic and syntactic possibilities for definable closures.

In Section 2, we introduce variations and degrees for semantic and syntactic rigidity of structures, describe properties, possibilities, and dynamics for these characteristics, in general and for theories of unary predicates. In Section 3, indexes of rigidity are introduced and their possibilities are described. In Sections 4 and 5, possibilities for degrees of rigidity and for indexes of rigidity are described for disjoint unions of structures and for compositions of structures are studied.

We use the standard model-theoretic terminology [3–6; 11], notions and notations in [10].

2. Variations of rigidity and their characteristics

Definition. For a set A in a structure $\mathcal{M}$, $\mathcal{M}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is identical. The structure $\mathcal{M}$ is called *syntactically A-rigid* if $M = \mathrm{dcl}(A)$.

A structure $\mathcal{M}$ is called *$\forall$-semantically / $\forall$-syntactically n-rigid* (respectively, *$\exists$-semantically / $\exists$-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{M}$ is semantically / syntactically A-rigid for any (some) $A \subseteq M$ with $|A| = n$.

Clearly, as above, syntactical A-rigidity and n-rigidity imply semantical ones, and vice versa for finite structures, but not vice versa for some infinite ones. Besides, if $\mathcal{M}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, then $\mathcal{M}$ is Q-semantically / Q-syntactically m-rigid for any $m \geq n$.

The least n such that $\mathcal{M}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg^{Q\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{Q\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exists we put $\deg^{Q\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = \infty$ and $\deg^{Q\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) = \infty$, respectively.

Notice that all these characteristics have the upper bound $|M| - 1$ if the structure $\mathcal{M}$ is finite. Moreover, if $M \setminus \mathrm{dcl}(\emptyset)$ is finite then the cardinality $|M \setminus \mathrm{dcl}(\emptyset)| - 1$ is the upper bound for both $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$.

We have the following obvious characterizations for finite values of degrees:

Proposition 1. 1. $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = 0$ *iff* $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = 0$*, and iff the structure $\mathcal{M}$ is semantically rigid.*

2. $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M})=0$ *iff* $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M})=0$*, and iff the structure* $\mathcal{M}$ *is syntactically rigid.*

3. $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M})=n\in\omega$ *iff for any set* $A\subseteq M$ *with* $|A|\geq n$ *there is minimal* $B\subseteq A$*, under inclusion, such that* $|B|=n$ *and any automorphism* $f\in{\rm Aut}(\mathcal{M})$ *fixing* B *pointwise fixes all elements in* $\mathcal{M}$*, too, and there are no sets of cardinalities* $n'<n$ *with that property. Here* $B\subseteq A$ *can be taken arbitrary with* $|B|=n$*.*

4. $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M})=n\in\omega$ *iff for some set* $A\subseteq M$ *with* $|A|\geq n$ *there is minimal* $B\subseteq A$*, under inclusion, such that* $|B|=n$ *and any automorphism* $f\in{\rm Aut}(\mathcal{M})$ *fixing* B *pointwise fixes all elements in* $\mathcal{M}$*, too, and there are no sets of cardinalities* $n'<n$ *with that property.*

5. $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M})=n\in\omega$ *iff for any set* $A\subseteq M$ *with* $|A|\geq n$ *there is minimal* $B\subseteq A$*, under inclusion, such that* $|B|=n$ *and* $M={\rm dcl}(B)$*, and there are no sets of cardinalities* $n'<n$ *with that property. Here* $B\subseteq A$ *can be taken arbitrary with* $|B|=n$*.*

6. $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M})=n\in\omega$ *iff for some set* $A\subseteq M$ *with* $|A|\geq n$ *there is minimal* $B\subseteq A$*, under inclusion, such that* $|B|=n$ *and* $M={\rm dcl}(B)$*, and there are no sets of cardinalities* $n'<n$ *with that property.*

By the definition, we have the following *monotonicity property*: if $\mathcal{M}$ is semantically / syntactically A-rigid and $A\subseteq A'\subseteq M$ then $\mathcal{M}$ is semantically / syntactically A'-rigid.

Using the definition and the monotonicity property, for any structure $\mathcal{M}$ the following inequalities hold:

$$\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M})\leq\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}),\tag{2.1}$$

the equality in (2.1) means that either there are no finite sets A with identical A-automorphisms only, or minimal finite sets A with identical A-automorphisms only have unbounded cardinalities, or all finite $A\subseteq M$ of some fixed cardinality n satisfy $M={\rm dcl}(A)$ and some A with $|A|=n$ does not have proper subsets A' such that there are identical A'-automorphisms only;

$$\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M})\leq\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}),\tag{2.2}$$

the equality in (2.2) means that either there are no finite sets A with identical A-automorphisms only, or there is finite $A\subseteq M$ such that $M={\rm dcl}(A)$, and there are no sets A' with less cardinalities such that there are identical A'-automorphisms only;

$$\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M})\leq\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}),\tag{2.3}$$

the equality in (2.3) means that either there are no finite sets A with identical A-automorphisms only, or there is finite $A\subseteq M$ with identical A-automorphism only and each finite $A'\subseteq M$ with $|A'|\geq|A|$ has a minimal restriction A'', under inclusion, with $|A''| = |A|$ and with identical A''-automorphism only;

$$\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}) \leq \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}). \quad (2.4)$$

the equality in (2.4) means that either there are no finite sets A with $\mathrm{dcl}(A) = M$, or there is finite $A \subseteq M$ with $\mathrm{dcl}(A) = M$ and each finite $A' \subseteq M$ with $|A'| \geq |A|$ has a minimal restriction A'', under inclusion, with $|A''| = |A|$ and with $\mathrm{dcl}(A'') = M$.

Example 1. The structure $\mathcal{M} = \langle \omega, \leq \rangle$ is both semantically and syntactically rigid, therefore $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}) = \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}) = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}) = 0$. We observe the same effect for arbitrary structures in which each element is marked by a constant.

Example 2. If $\mathcal{M}$ has the empty language then

$$\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}) = \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}) = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}) = |M| - 1$$

if $\mathcal{M}$ is finite, and and these values equal ∞ if $\mathcal{M}$ is infinite.

Example 3. If $\mathcal{V}$ is a vector space over a field F then we have the following criterion for the semantic/syntactic rigidity: $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{V}) = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{V}) = 0$ iff $\dim(\mathcal{V}) \leq 1$ and $|F| = 2$ for $\dim(\mathcal{V}) = 1$. If $\mathcal{V}$ is not rigid then $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{V}) = \dim(\mathcal{V})$ for finite $\dim(\mathcal{V})$, and $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{V}) = \infty$, otherwise. Besides, $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{V}) = \infty$ if $\dim(\mathcal{V})$ is infinite, or $\dim(\mathcal{V}) \geq 1$ and F is infinite. Finally for $\dim(\mathcal{V}) = n \in \omega \setminus \{0\}$ and $|F| = m \in \omega \setminus \{0\}$ with $(n, m) \neq (1, 2)$, we have $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{V}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{V}) = (n-1)m + 1$, since we obtain the rigidity taking all vectors in a $(n-1)$-dimensional subspace $\mathcal{V}'$, with $(n-1)m$ elements, and a vector in $\mathcal{V} \setminus \mathcal{V}'$.

Example 4. Let $\mathcal{M}$ be a structure of disjoint infinite unary predicates P_i, $i \in I$, expanded by constants for all elements in $\bigcup\limits_{i \in I} P_i$. Since $\mathcal{M}$ is both semantically and syntactically rigid we have $\deg_{\rm rig}^{Q\text{-sem}}(\mathcal{M}) = \deg_{\rm rig}^{Q\text{-synt}}(\mathcal{M}) = 0$ for $Q \in \{\forall, \exists\}$. At the same time extending n predicates P_i by new elements a_i we obtain $\mathcal{N} \succ \mathcal{M}$ with $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{N}) = \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{N}) = 0$, $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{N}) = n$, $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{N}) = \infty$. Moreover, if infinitely many P_i are extended by new elements a_i then the correspondent elementary extension $\mathcal{N}'$ of $\mathcal{M}$ has the following characteristics: $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{N}') = 0$, $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{N}') = n$ and $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{N}') = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{N}') = \infty$. Besides, if some extended P_i are again extended by m new elements in total then an appropriate elementary extension $\mathcal{N}_{m,n}$ has the following characteristics: $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{N}_{m,n}) = m$, $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{N}_{m,n}) = m + n$, $\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{N}_{m,n}) = \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{N}_{m,n}) = \infty$ including the possibility

$$\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{N}_{\mu,n}) = \\ = \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{N}_{\mu,n}) = \deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{N}_{\mu,n}) = \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{N}_{\mu,n}) = \infty$$

if $\mu \geq \omega$ new elements are added.

Thus by Example 4 the difference between

$$\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}) \text{ and } \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$$

can be arbitrary large. In view of Proposition 1 and inequality 2.2 we obtain the following theorem on distributions for these characteristics:

Theorem 1. 1. *The pairs* $\left(\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})\right)$ *belong to the set* $\mathrm{DEG}^{\exists\text{-sem},\exists\text{-synt}}_{\rm rig} = \{(\mu,\nu) \mid \mu,\nu \in \omega \cup \{\infty\}, \mu \leq \nu\}$.

2. *For each pair* $(\mu,\nu) \in \mathrm{DEG}^{\exists\text{-sem},\exists\text{-synt}}_{\rm rig}$ *there exists a structure* $\mathcal{M}_{\mu,\nu}$ *such that*

$$\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}_{\mu,\nu}) = \mu,\ \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}_{\mu,\nu}) = \nu.$$

Example 4 shows that values in $\mathrm{DEG}^{\exists\text{-sem},\exists\text{-synt}}_{\rm rig}$ in Theorem 1 are covered by structures in countable languages Σ_1 of unary predicates. Now we describe possibilities for the pairs $\left(\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})\right)$ in these languages Σ_1.

Proposition 2. *For any structure* $\mathcal{M}$ *in a language* Σ_1 *of unary predicates the pair*

$$\left(\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})\right)$$

has one of the following possibilities:

1) $(0,0)$*, if* $\mathcal{M}$ *is both semantically and syntactically rigid;*

2) (n,n)*, if* $\mathcal{M}$ *is finite with* $n+1$ *elements and it is not semantically rigid that is not syntactically rigid;*

3) $(0,\infty)$*, if* $\mathcal{M}$ *is infinite, semantically rigid but not syntactically rigid;*

4) (∞,∞)*, if* $\mathcal{M}$ *is infinite and both not semantically rigid and not syntactically rigid.*

Proof. If $\mathcal{M}$ is syntactically rigid then we have

$$\left(\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}), \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})\right) = (0,0)$$

by the inequality (2.1). Now we assume that $\mathcal{M}$ is not syntactically rigid and consider the following cases.

Case 1: $\mathcal{M}$ is semantically rigid, i.e., $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}) = 0$. In such a case $\mathcal{M}$ is infinite since finite structures have isolated 1-types only and there are complete 1-types over empty set with at least two realizations that contradicts the semantic rigidity for the language Σ_1. Again using the unary language Σ_1 and the arguments of [2, Section 8.1] that all 1-types, over empty set, are forced by formulae of quantifier free diagrams and formulae describing estimations for cardinalities of their solutions, with independent actions of automorphisms in distinct sets of realizations of 1-types. Thus each 1-type has at most one realization in $\mathcal{M}$. Since $\mathcal{M}$ is not syntactically rigid, $\mathcal{M}$ realizes at least one nonisolated 1-type $p(x)$ by some unique element a. Now for any $n \in \omega$ we can take n realizations of other 1-types forming a set A such that $a \notin {\rm dcl}(A)$. It implies $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}) = \infty$.

Case 2: $\mathcal{M}$ is not semantically rigid and $|M| = n + 1 \in \omega$. In such a case $\mathcal{M}$ has a complete 1-type $p(x)$ with at least two realizations a and b. Since there is an $(M \setminus \{a, b\})$-automorphism f with $f(a) = b$, we obtain $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}) = n$ implying $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}) = n$ by the inequality (2.1) and the syntactic rigidity of $\mathcal{M}$ over each n-element set.

Case 3: $\mathcal{M}$ is not semantically rigid and it is infinite. In such a case $\mathcal{M}$ has a complete 1-type $p(x)$ with at least two realizations a and b and such that realizations of other 1-types allow to form arbitrarily large finite set A such that some A-automorphism transforms a in b. It means that $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}) = \infty$ implying $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M}) = \infty$ by the inequality (2.1).

Combining arguments for Theorems 1 and 2 we obtain the following possibilities for tetrads

$$\deg_4(\mathcal{M}) \rightleftharpoons \left(\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}), \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}), \deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}), \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M})\right)$$

in a language of unary predicates:

Corollary 1. *For any structure $\mathcal{M}$ in a language Σ_1 of unary predicates the tetrad $\deg_4(\mathcal{M})$ has one of the following possibilities:*

1) $(0, 0, 0, 0)$*, if $\mathcal{M}$ is both semantically and syntactically rigid;*

2) (m, m, n, n)*, if $\mathcal{M}$ is finite with $n + 1$ elements and it is not semantically rigid that is not syntactically rigid with some minimal m-elements set $A \subset M$, $1 \leq m \leq n$, producing ${\rm dcl}(A) = M$;*

3) $(0, \nu, 0, \infty)$*, if $\mathcal{M}$ is infinite, semantically rigid but not syntactically rigid, with $1 \leq \nu \leq \infty$;*

4) $(\mu, \nu, \infty, \infty)$*, if $\mathcal{M}$ is infinite and both not semantically rigid and not syntactically rigid, with $1 \leq \mu \leq \nu \leq \infty$.*

Example 5. Let $\mathcal{M}$ be a finitely generated algebra by a set X. Then by the definition we have $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}) \leq |X|$ which implies $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}) \leq |X|$ by the inequality (2.2). Here, if additionally the generating set X admits substitutions by any $Y \subseteq M$ with $|Y| = |X|$ and these substitutions preserve the generating property then we have $\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M}) \leq |X|$ which implies $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}) \leq |X|$ by the inequality (2.1). For instance, if $\mathcal{M}$ is a directed graph forming a finite cycle of positive length then $\deg_4(\mathcal{M}) = (1,1,1,1)$.

Since algebras, with constants and unary operations, can define arbitrary configurations of unary predicates, possibilities for characteristics $\deg_4(\mathcal{M})$ in Corollary 1 can be realized in the class of algebras, too.

Example 6. Let $\mathrm{pm} = \mathrm{pm}(G_1, G_2, \mathcal{P})$ be a connected polygonometry of a group pair (G_1, G_2) on an exact pseudoplane $\mathcal{P}$, and $\mathcal{M} = \mathcal{M}(\mathrm{pm})$ be a ternary structure for pm [7]. Since all points a in $\mathcal{M}$ are connected by automorphisms we have $\mathrm{acl}(\{a\}) = \{a\}$. At the same time any two distinct points $a, b \in M(\mathrm{pm})$ (laying in a common line) define all points in $\mathcal{M}$ by line and angle parameters of broken lines. It implies $M(\mathrm{pm}) = \mathrm{dcl}(\{a, b\})$. If line and angle parameters of shortest broken lines connecting arbitrary distinct points a and b are defined uniquely then $M(\mathrm{pm}) = \mathrm{dcl}(\{a, b\})$ for these points, too. Hence, in such a case, $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}) = \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}) = \deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}) = \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M}) \leq 2$. Moreover, these degree values equal 1 iff pm consists of unique line and with at least two points, i.e., $|G_1| > 1$ and $|G_2| = 1$. Finally, for a polygonometry pm, the degrees equal 0 iff pm consists of unique point.

If parameters of broken lines do not define these broken lines by endpoints then finite cardinalities of points in these lines can be unbounded. Indeed, taking opposite vertices a and b in an n-cube [7; 8] or in its polygonometry pm we obtain n adjacent vertices $c_1, \dots, c_n$ for a and these vertices are connected by $\{a, b\}$-automorphisms. Moreover, in such a case, $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}) = \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}) = n + 1$ witnessed, for instance, by the set $A = \{a, b, c_1, \dots, c_{n-1}\}$.

The value $\deg_4(\mathcal{M}_2) = (2,2,2,2)$ for $\mathcal{M}_2 = \mathcal{M}(\mathrm{pm})$ can be increased till $\deg_4(\mathcal{M}_n) = (n,n,n,n)$, $n \geq 3$, generalizing group trigonometries in the following way. We construct a $(n+1)$-dimensional space consisting of points and n-dimensional hyperplanes. We introduce an incidence n-ary relation I_n for n distinct points to lay on a common hyperplane. Now fixing a hyperplane H and $n-1$ pairwise distinct points $a_1, \dots, a_{n-1} \in H$ we define an exact transitive action of a group G_1 on $H \setminus \{a_1, \dots, a_{n-1}\}$, i.e., on H with respect to $a_1, \dots, a_{n-1}$, such that this action is transformed for any pairwise distinct points $a'_1, \dots, a'_{n-1} \in H$. Since each H can be defined by its $n-1$ distinct points with actions, we can fix $a_1, \dots, a_{n-1}$ and move $a_n \in H \setminus \{a_1, \dots, a_{n-1}\}$ into points a'_n in other hyperplanes H' containing $a_1, \dots, a_{n-1}$. Collecting these movements we define an action of a group G_2 on that bundle of hyperplanes containing $a_1, \dots, a_{n-1}$. Then we spread actions of G_1 and G_2 for any hyperplanes and bundles of hyperplanes, respectively, such that all pairwise distinct $a_1, \ldots, a_{n-1}$ and $a'_1, \ldots, a'_{n-1}$ are connected by automorphisms with respect to these actions.

For instance, taking the set P of planes in $\mathbb{R}^3$, a plane $\pi \in P$ and distinct points $a_1, a_2 \in P$ the action of G_1 can be defined as $\mathbb{R} \times A$ with the side group $\mathbb{R}$ and angle group A defining both the directed distance $d \in \mathbb{R}$ from a_1 to a point $a_3 \in \pi$ and the angle value α from the side $a_1\hat{\ }a_2$ to the side $a_1\hat{\ }a_3$. And G_2 is the rotation group for the planes in P around the lines $l(a_1, a_2)$.

Now we extend the language $\{I_n\}$ by $(n+1)$-ary predicates Q_{g_1}, $g_1 \in G_1$, such that first $(n-1)$-coordinates $\overline{a}$ in $\langle \overline{a}, b, c\rangle \in Q_{g_1}$ are exhausted by $a_1, \ldots, a_{n-1}$ and $c = bg_1$ with respect to $a_1, \ldots, a_{n-1}$. Simultaneously we define predicates R_{g_2}, $g_2 \in G_2$, of arities $n+1$ such that each R_{g_2} realizes a rotation of a hyperplane with respect to $a_1, \ldots, a_{n-1}$ by the element g_2. We obtain a structure $\mathcal{M}_n$ whose values $\deg^{Q\text{-sem}}_{\rm rig}(\mathcal{M}_n)$ and $\deg^{Q\text{-synt}}_{\rm rig}(\mathcal{M}_n)$, for $Q \in \{\forall, \exists\}$ equal n.

The construction above admits a generalization for polygonometries $\mathrm{pm}(G_1, G_2, \mathcal{P})$ of group pairs transforming (G_1, G_2) a pseudoplane $\mathcal{P}$ to a pseudospace $\mathcal{S}$ with hyperplanes H such that $H = \mathrm{dcl}(\{a_1, \ldots, a_n\})$ for any pairwise distinct points $a_1, \ldots, a_n \in H$ and with $\mathrm{dcl}(\{b_1, \ldots, b_{n-1}\}) = \{b_1, \ldots, b_{n-1}\}$ for any $b_1, \ldots, b_{n-1} \in \mathcal{S}$.

Comparing characteristics $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})/\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$ and $\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})$ / $\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})$ we observe that the first ones produce cardinalities of "best", i.e., minimal sets generating the structure $\mathcal{M}$ and the second ones give cardinalities of "worst" generating sets. It is natural to describe possibilities of "intermediate" generating sets. For this aim we define the degrees of rigidity with respect to a subset A of M as follows:

Definition. For a set A in $\mathcal{M}$ and an expansion $\mathcal{M}_A$ of $\mathcal{M}$ by constants in A, the least n such that $\mathcal{M}_A$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the (Q, A)*-semantical* / (Q, A)*-syntactical degree of rigidity*, it is denoted by $\deg^{Q\text{-sem}}_{{\rm rig},A}(\mathcal{M})$ and $\deg^{Q\text{-synt}}_{{\rm rig},A}(\mathcal{M})$, respectively. If such n does not exists we put $\deg^{Q\text{-sem}}_{{\rm rig},A}(\mathcal{M}) = \infty$ and $\deg^{Q\text{-synt}}_{{\rm rig},A}(\mathcal{M}) = \infty$, respectively.

Any expansion $\mathcal{M}_A$ of $\mathcal{M}$ with $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M}_A) = 0$, for $s \in \{\mathrm{sem}, \mathrm{synt}\}$, is called a *s-rigiditization* or simply a *rigiditization* of $\mathcal{M}$.

We have the following properties for (Q, A)-semantical and (Q, A)-syntactical degrees of rigidity:

Proposition 3. *Let* $\mathcal{M}$ *be a structure,* $A \subseteq M$*,* $Q \in \{\forall, \exists\}$*,* $s \in \{\mathrm{sem}, \mathrm{synt}\}$*. Then the following assertions hold:*

1. (Preservation of degrees of rigidity) *If* $A \subseteq \mathrm{dcl}(\emptyset)$ *then* $\deg^{Q\text{-}s}_{\rm rig}(\mathcal{M}) = \deg^{Q\text{-}s}_{{\rm rig},A}(\mathcal{M})$.

2. (Rigiditization) *If A contains a witnessing set for the finite value* $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})$ *then* $\deg^{\exists\text{-}s}_{{\rm rig},A}(\mathcal{M})=0$.

3. (Monotony) *If $A\subseteq B\subseteq M$ then* $\deg^{Q\text{-}s}_{{\rm rig},A}(\mathcal{M})\geq\deg^{Q\text{-}s}_{{\rm rig},B}(\mathcal{M})$.

4. (Additivity) *If A witnesses the finite value* $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})$ *then for any* $A'\subseteq A$,

$$\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})=\deg^{\exists\text{-}s}_{{\rm rig},A'}(\mathcal{M})+\deg^{\exists\text{-}s}_{{\rm rig},A\setminus A'}(\mathcal{M}).$$

5. (Cofinite character) *If A is cofinite in $\mathcal{M}$ then* $\deg^{\exists\text{-sem}}_{{\rm rig},A}(\mathcal{M})$ *and* $\deg^{\exists\text{-synt}}_{{\rm rig},A}(\mathcal{M})$ *are natural.*

6. (Finite rigiditization) *Any cofinite set A in $\mathcal{M}$ has a minimal finite extension A' such that $\mathcal{M}_{A'}$ is semantically / syntactically rigid.*

Proof. 1. If $A\subseteq{\rm dcl}(\emptyset)$ then ${\rm Aut}(\mathcal{M})={\rm Aut}(\mathcal{M}_A)$ and therefore the equalities $\deg^{Q\text{-}s}_{\rm rig}(\mathcal{M})=\deg^{Q\text{-}s}_{{\rm rig},A}(\mathcal{M})$ hold for $s=\rm sem$. For the case $s=\rm synt$ the required equalities are satisfied in view of ${\rm dcl}(B)={\rm dcl}(A\cup B)$ for any $B\subseteq M$.

2. If A contains a witnessing set for the finite value $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})$ then there exists identical A-automorphism of $\mathcal{M}$ only implying $\deg^{\exists\text{-sem}}_{{\rm rig},A}(\mathcal{M})=0$. Similarly if A contains a witnessing set for the finite value $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$ then ${\rm dcl}(A)=M$ producing $\deg^{\exists\text{-synt}}_{{\rm rig},A}(\mathcal{M})=0$.

3. If $A\subseteq B\subseteq M$ then ${\rm Aut}(\mathcal{M}_B)\leq{\rm Aut}(\mathcal{M}_A)$ therefore the inequalities $\deg^{Q\text{-}s}_{{\rm rig},A}(\mathcal{M})\geq\deg^{Q\text{-}s}_{{\rm rig},B}(\mathcal{M})$ hold for $s=\rm sem$. For the case $s=\rm synt$ the required equalities are satisfied in view of ${\rm dcl}(A\cup C)\subseteq{\rm dcl}(B\cup C)$ for any $C\subseteq M$.

4. If A witnesses the finite value $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})$ then we divide A into two disjoint parts A_1 and A_2 and by the definition of $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})$, both A_1 and A_2 are extended till minimal A witnessing the semantic / syntactic rigidity. Thus A_1 witnesses the value $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}_{A_2})$ and A_2 witnesses the value $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}_{A_1})$ producing the required equation $\deg^{\exists\text{-}s}_{\rm rig}(\mathcal{M})=\deg^{\exists\text{-}s}_{{\rm rig},A'}(\mathcal{M})+\deg^{\exists\text{-}s}_{{\rm rig},A\setminus A'}(\mathcal{M})$.

5. If A is cofinite in $\mathcal{M}$ then there are only finitely many elements, all in $M\setminus A$, witnessing the values $\deg^{\exists\text{-sem}}_{{\rm rig},A}(\mathcal{M})$ and $\deg^{\exists\text{-synt}}_{{\rm rig},A}(\mathcal{M})$. Thus these values are natural.

6. It is immediately implied by Items 2 and 5.

In view of Proposition 3 fixing a subset in $\mathcal{M}$ large enough we obtain its rigiditization. At the same time the following assertion clarifies that small subsets can produce the rigiditization for structures in bounded cardinalities only.

Proposition 4. 1. *If* $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$ *is finite then* $|M|\leq\max\{\Sigma(\mathcal{M}),\omega\}$.

1. *If $\mathcal{M}$ is homogeneous and* $\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})$ *is finite then*

$$|M|\leq 2^{\max\{\Sigma(\mathcal{M}),\omega\}}.$$

Proof. 1. If $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M})$ is finite then there is a finite set $A \subseteq M$ witnessing that value, with $M = {\rm dcl}(A)$. This equality is witnessed by at most by $\max\{\Sigma(\mathcal{M}), \omega\}$ formulae such that each element in $\mathcal{M}$ is defined by a formula in the language $\Sigma(\mathcal{M}_A)$. Since there are $\max\{\Sigma(\mathcal{M}), \omega\}$ $\Sigma(\mathcal{M}_A)$-formulae we obtain at most $\max\{\Sigma(\mathcal{M}), \omega\}$ elements in $\mathcal{M}$.

2. If a finite set $A \subseteq M$ witnesses the finite value $\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M})$ and $\mathcal{M}$ is homogeneous possibilities for A-automorphisms fixing elements of $\mathcal{M}$ are exhausted by single realizations of types in $S^1(A)$. Since there are at most $2^{\max\{\Sigma(\mathcal{M}),\omega\}}$ these types that value is the required upper bound for the cardinality of semantically rigid structure $\mathcal{M}_A$.

Proposition 4 immediately implies the following:

Corollary 2. 1. *If* $\deg_{{\rm rig},A}^{\exists\text{-synt}}(\mathcal{M})$ *is finite then* $|M| \leq \max\{\Sigma(\mathcal{M}), |A|, \omega\}$.

1. *If* $\mathcal{M}$ *is homogeneous and* $\deg_{{\rm rig},A}^{\exists\text{-sem}}(\mathcal{M})$ *is finite then*

$$|M| \leq 2^{\max\{\Sigma(\mathcal{M}),|A|,\omega\}}.$$

3. Indexes of rigidity

Definition. For a set A in a structure $\mathcal{M}$ the *index of rigidity* of $\mathcal{M}$ over A, denoted by ${\rm ind}_{\rm rig}(\mathcal{M}/A)$ is the supremum of cardinalities for the set of solutions of algebraic types ${\rm tp}(a/A)$ for $a \in M$. We put ${\rm ind}_{\rm rig}(\mathcal{M}) = {\rm ind}_{\rm rig}(\mathcal{M}/\emptyset)$. Here we assume that ${\rm ind}_{\rm rig}(\mathcal{M}) = 0$ if $\mathcal{M}$ does not have algebraic types ${\rm tp}(a)$ for $a \in M$.

Remark 1. By the definition we have ${\rm ind}_{\rm rig}(\mathcal{M}/A) \in \omega + 1$.

Example 7. 1. If $\mathcal{M}$ is a structure of unary predicates P_i, $i \in I$, then ${\rm ind}_{\rm rig}(\mathcal{M}) = 0$ iff there are no finite nonempty intersections $P_{i_1}^{\delta_1} \cap \ldots \cap P_{i_k}^{\delta_k}$, $\delta_1, \ldots, \delta_k \in \{0, 1\}$. We have ${\rm ind}_{\rm rig}(\mathcal{M}) = 1$ iff ${\rm dcl}(\emptyset) \neq \emptyset$ and there are no maximal finite intersections $P_{i_1}^{\delta_1} \cap \ldots \cap P_{i_k}^{\delta_k}$ with at least two elements. Besides, ${\rm ind}_{\rm rig}(\mathcal{M}) \in \omega$ iff these finite intersections have bounded cardinalities, and all natural possibilities n are realized by predicates with exactly n elements and infinite complements. Otherwise, i.e., for ${\rm ind}_{\rm rig}(\mathcal{M}) = \omega$, these finite intersections have unbounded cardinalities.

2. If $\mathcal{M}$ is a structure of an equivalence relation E, then ${\rm ind}_{\rm rig}(\mathcal{M}) = 0$ iff there are no finite E-classes. We have ${\rm ind}_{\rm rig}(\mathcal{M}) = 1$ iff ${\rm dcl}(\emptyset) \neq \emptyset$ and there are no finite E-classes with at least two elements. Besides, ${\rm ind}_{\rm rig}(\mathcal{M}) \in \omega$ iff these E-classes have bounded cardinalities, and all natural possibilities n are realized by infinitely many E-classes with exactly n elements. Otherwise, i.e., for ${\rm ind}_{\rm rig}(\mathcal{M}) = \omega$, these E-classes have unbounded cardinalities.

3. If $\mathcal{M} = \mathcal{M}(\mathrm{pm})$ for a polygonometry pm then $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = 0$ iff pm has infinitely many points. Otherwise, if pm has $n \in \omega$ points then $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = n$.

More generally, we have the following possibilities for a model $\mathcal{M}$ of transitive theory T, i.e., of a theory with $|S^1(\emptyset)| = 1$:

i) $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = 0$, if $\mathcal{M}$ is infinite;

ii) $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = |\mathcal{M}|$, if $\mathcal{M}$ is finite.

In view of Remark 1 the following assertion describes possibilities of indexes of rigidity:

Proposition 5. *For any $\lambda \in \omega + 1$ there is a structure $\mathcal{M}_\lambda$ such that $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}_\lambda) = \lambda$.*

Proof follows by Example 7.

4. Variations of rigidity for disjoint unions of structures

Definition [12]. The *disjoint union* $\bigsqcup\limits_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n \in \omega$, is the structure of language $\bigcup\limits_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} \mid n \in \omega\}$ with the universe $\bigsqcup\limits_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in $\mathcal{M}_n$, $n \in \omega$. The *disjoint union of theories T_n* for pairwise disjoint languages Σ_n accordingly, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons \mathrm{Th}\left(\bigsqcup_{n\in\omega} \mathcal{M}_n\right),$$

where $\mathcal{M}_n \models T_n$, $n \in \omega$.

Theorem 2. *For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$, and $s \in \{\mathrm{sem}, \mathrm{synt}\}$ the following conditions hold:*

1. $\deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_1) + \deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_2)$*, in particular,*

$$\deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2)$$

is finite iff $\deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_1)$ *and* $\deg_{\mathrm{rig}}^{\exists\text{-}s}(\mathcal{M}_2)$ *are finite.*

2. $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = 0$ *iff* $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1) = 0$ *and* $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_2) = 0$.

3. *If* $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) > 0$ *then it is finite iff* $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1) > 0$ *is finite and $\mathcal{M}_2$ is finite, or* $\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_2) > 0$ *is finite and $\mathcal{M}_1$ is finite. Here,*

$$\deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \max\{|M_1| + \deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_2), |M_2| + \deg_{\mathrm{rig}}^{\forall\text{-}s}(\mathcal{M}_1)\}.$$

Proof. 1. Let $A_i \subset M_i$ be sets witnessing values $\deg_{\rm rig}^{\exists\text{-}s}(\mathcal{M}_i)$, $i = 1, 2$. By the definition of $\mathcal{M}_1 \sqcup \mathcal{M}_2$, A_1 and A_2 are disjoint and $A_1 \cup A_2$ witnesses the value $\deg_{\rm rig}^{\exists\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2)$. Thus $\deg_{\rm rig}^{\exists\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \deg_{\rm rig}^{\exists\text{-}s}(\mathcal{M}_1) + \deg_{\rm rig}^{\exists\text{-}s}(\mathcal{M}_2)$.

2. If $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = 0$ then the empty set witnesses that $\mathcal{M}_1 \sqcup \mathcal{M}_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are s-rigid, i.e., rigid with respect to s, implying $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1) = 0$ and $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2) = 0$. Conversely, if $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1) = 0$ and $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2) = 0$ then the empty set witnesses that $\mathcal{M}_1$ and $\mathcal{M}_2$ are s-rigid. Now by the definition of $\mathcal{M}_1 \sqcup \mathcal{M}_2$ we observe that $\mathcal{M}_1 \sqcup \mathcal{M}_2$ is s-rigid, too, implying $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = 0$.

3. Let $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2) > 0$ be finite, then by Item 2, $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1) > 0$ or $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2) > 0$. Assuming that $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_i) > 0$ we can not witness that value by subsets of M_{3-i}, $i = 1, 2$. Thus M_{3-i} should be finite. Conversely, let $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1) > 0$ be finite and $\mathcal{M}_2$ be finite, or $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2) > 0$ be finite and $\mathcal{M}_1$ be finite. Then we can take $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1)$ elements of M_1 and all elements of M_2 obtaining the s-rigidity of $\mathcal{M}_1 \sqcup \mathcal{M}_2$. Similarly we can take $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2)$ elements of M_2 and all elements of M_1 obtaining the s-rigidity of $\mathcal{M}_1 \sqcup \mathcal{M}_2$, too. Thus, the finite value $\max\{|M_1| + \deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_2), |M_2| + \deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1)\}$ equals $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}_1 \sqcup \mathcal{M}_2)$.

Theorem 2 and Corollary 1 immediately imply:

Corollary 3. *For any structures $\mathcal{M}_1$ and $\mathcal{M}_2$ in a language Σ_1 of unary predicates the tetrad $\deg_4(\mathcal{M}_1 \sqcup \mathcal{M}_2)$ has one of the following possibilities:*

1) $(0, 0, 0, 0)$, *if $\mathcal{M}_1$ and $\mathcal{M}_2$ are both semantically and syntactically rigid;*

2) (m, m, n, n), *if $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite with $|M_1 \dot\cup M_2| = n+1$ elements and some $\mathcal{M}_i$ is not semantically rigid that is not syntactically rigid with some minimal m_1-elements set $A_1 \subset M_1$ producing $\mathrm{dcl}(A_1) = M_1$ and some minimal m_2-elements set $A_2 \subset M_2$ producing $\mathrm{dcl}(A_2) = M_2$, where $m = m_1 + m_2 \leq n - 1$;*

3) $(0, \nu, 0, \infty)$, *if $\mathcal{M}_1 \sqcup \mathcal{M}_2$ is infinite, $\mathcal{M}_1$ and $\mathcal{M}_2$ are semantically rigid but some of them is not syntactically rigid, with $1 \leq \nu \leq \infty$, $\nu = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}_1) + \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}_2)$;*

4) $(\mu, \nu, \infty, \infty)$, *if $\mathcal{M}_1 \sqcup \mathcal{M}_2$ is infinite, $\mathcal{M}_1$ or $\mathcal{M}_2$ is not semantically rigid, $\mathcal{M}_1$ or $\mathcal{M}_2$ is not syntactically rigid, with $1 \leq \mu \leq \nu \leq \infty$, $\mu = \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}_1) + \deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}_2)$, $\nu = \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}_1) + \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}_2)$.*

Theorem 3. *For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$, and a set $A \subseteq M_1 \cup M_2$,*

$$\mathrm{ind}_{\rm rig}((\mathcal{M}_1 \sqcup \mathcal{M}_2)/A) = \max\{\mathrm{ind}_{\rm rig}(\mathcal{M}_1/(M_1 \cap A)), \mathrm{ind}_{\rm rig}(\mathcal{M}_2)/(M_2 \cap A)\}.$$

Proof. By the definition of disjoint union types in $S^1(A)$ are locally realized either in $\mathcal{M}_1$ or in $\mathcal{M}_2$. Moreover, they are forced by their restrictions to M_1 or M_2. So algebraic types $p(x) \in S^1(A)$ are defined in $\mathcal{M}_1$ or in $\mathcal{M}_2$ by their restrictions to $M_1 \cap A$ and to $M_2 \cap A$. Now we collect possibilities for cardinalities of sets of realizations of algebraic types in $S^1(M_1 \cap A)$ and in $S^1(M_2 \cap A)$. We either choose a maximal natural cardinality obtaining natural $n = \mathrm{ind}_{\mathrm{rig}}((\mathcal{M}_1 \sqcup \mathcal{M}_2)/A)$ with $n = \max\{\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}_1/(M_1 \cap A)), \mathrm{ind}_{\mathrm{rig}}(\mathcal{M}_2)/(M_2 \cap A)\}$ or there are no maximal natural cardinality with both $\mathrm{ind}_{\mathrm{rig}}((\mathcal{M}_1 \sqcup \mathcal{M}_2)/A) = \omega$ and

$$\max\{\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}_1/(M_1 \cap A)), \mathrm{ind}_{\mathrm{rig}}(\mathcal{M}_2)/(M_2 \cap A)\} = \omega.$$

5. Variations of rigidity for compositions of structures

Recall the notions of composition for structures and theories.

Definition [1]. Let $\mathcal{M}$ and $\mathcal{N}$ be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of $\mathcal{M}$ and $\mathcal{N}$ satisfying the following conditions:

1) $\Sigma_{\mathcal{M}[\mathcal{N}]} = \Sigma_{\mathcal{M}} \cup \Sigma_{\mathcal{N}}$;

2) $M[N] = M \times N$, where $M[N]$, M, N are universes of $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$ respectively;

3) if $R \in \Sigma_{\mathcal{M}} \setminus \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$;

4) if $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$;

5) if $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$, or $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$.

The theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is called the *composition* $T_1[T_2]$ of the theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

By the definition, the composition $\mathcal{M}[\mathcal{N}]$ is obtained replacing each element of $\mathcal{M}$ by a copy of $\mathcal{N}$.

Definition [1]. The composition $\mathcal{M}[\mathcal{N}]$ is called *E-definable* if $\mathcal{M}[\mathcal{N}]$ has an $\emptyset$-definable equivalence relation E whose E-classes are universes of the copies of $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$.

Remark 2. It is shown in [1] that E-definable compositions $\mathcal{M}[\mathcal{N}]$ uniquely define theories $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ by theories $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$ and types of elements in copies of $\mathcal{N}$ are defined by types in these copies and types for connections between these copies.

Proposition 6. *For E-definable compositions $\mathcal{M}[\mathcal{N}]$ the automorphism group* $\mathrm{Aut}(\mathcal{M}[\mathcal{N}])$ *is isomorphic to the wreath product of* $\mathrm{Aut}(\mathcal{M})$ *and* $\mathrm{Aut}(\mathcal{N})$*:*

$$\mathrm{Aut}(\mathcal{M}[\mathcal{N}]) \simeq \mathrm{Aut}(\mathcal{M}) \wr \mathrm{Aut}(\mathcal{N}).$$

Proof. Since all copies of $\mathcal{N}$ are isomorphic in $\mathcal{M}[\mathcal{N}]$ and form definable E-classes each automorphism $f \in \mathrm{Aut}(\mathcal{M}[\mathcal{N}])$ is defined both by the action on the set of E-classes, which corresponds to an automorphism $g \in \mathrm{Aut}(\mathcal{M})$, and by the the actions on the E-classes, which corresponds to an automorphism h for copies of $\mathcal{N}$. Therefore f is situated in the one-to-one correspondence with the pair (g, h) producing a correspondent element of $\mathrm{Aut}(\mathcal{M}) \wr \mathrm{Aut}(\mathcal{N})$.

In view of Remark 2 and Proposition 6 we have the following:

Theorem 4. *For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:*

$$\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}[\mathcal{N}]) = \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}),$$

if $\mathcal{N}$ is semantically rigid, and

$$\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}[\mathcal{N}]) = |M| \cdot \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{N}),$$

if $\mathcal{N}$ is not semantically rigid. In particular, $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}[\mathcal{N}])$ *is finite iff* $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M})$ *and $\mathcal{N}$ are finite, if $\mathcal{N}$ is semantically rigid, and* $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{N})$ *and $\mathcal{M}$ are finite, if $\mathcal{N}$ is not semantically rigid.*

Proof. If $\mathcal{N}$ is semantically rigid then it suffices to find possibilities for automorphisms of $\mathcal{M}$ since in such a case the semantical rigidity of an inessential expansion of $\mathcal{M}$ implies the semantical rigidity of correspondent inessential expansion of $\mathcal{M}[\mathcal{N}]$. Thus, here $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}[\mathcal{N}]) = \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M})$. If $\mathcal{N}$ is not semantically rigid then copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ are automorphically independent, i.e., fixing automorphisms for $\mathcal{M}[\mathcal{N}]$ one have to fix all automorphisms for these copies. Since the smallest set fixing automorphisms for $\mathcal{N}$ contains $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{N})$, we have at least and minimally at most $|M| \cdot \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{N})$ elements to fix automorphisms for $\mathcal{M}[\mathcal{N}]$ implying $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{M}[\mathcal{N}]) = |M| \cdot \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{N})$.

Theorem 5. *For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:*

$$\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}[\mathcal{N}]) = \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}),$$

if $N = \mathrm{dcl}(\emptyset)$, and

$$\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}[\mathcal{N}]) = |M| \cdot \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{N}),$$

if $N \neq \mathrm{dcl}(\emptyset)$. In particular, $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M}[\mathcal{N}])$ *is finite iff* $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{M})$ *and $\mathcal{N}$ are finite, for $N = \mathrm{dcl}(\emptyset)$, and* $\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{N})$ *and $\mathcal{M}$ are finite, for $N \neq \mathrm{dcl}(\emptyset)$.*

Proof repeats the proof of Theorem 4 replacing automorphism groups by definable closures.

Proposition 1, (1), (2) and Theorems 4, 5 immediately imply:

Corollary 4. *For any E-definable composition $\mathcal{M}[\mathcal{N}]$ and $s \in \{\mathrm{sem}, \mathrm{synt}\}$ the following conditions are equivalent:*

(1) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = 0$*;*

(2) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) = 0$ *and* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$.

Theorem 6. *For any $s \in \{\mathrm{sem}, \mathrm{synt}\}$ and E-definable composition $\mathcal{M}[\mathcal{N}]$ with*

$$\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) > 0$$

the following conditions are equivalent:

(1) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}])$ *is finite;*

(2) *one of the following conditions hold:*

i) $\mathcal{M}$ *and* $\mathcal{N}$ *are finite, i.e.* $\mathcal{M}[\mathcal{N}]$ *is finite;*

ii) $\mathcal{M}$ *is infinite with* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) = 1$ *and* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$*;*

iii) $\mathcal{M}$ *is infinite and* $\mathcal{N}$ *is finite with* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) \in \omega \setminus \{0, 1\}$ *and* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$*;*

iv) $\mathcal{M}$ *is a singleton and* $\mathcal{N}$ *is infinite with* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) \in \omega \setminus \{0\}$.

Here there are the following possibilities:

a) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = (\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) - 1) \cdot |N| + 1$, *if the case* i) *or* iii) *is satisfied with* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$*;*

b) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = (|M| - 1) \cdot |N| + \deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N})$, *if the case* i) *is satisfied with* $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) > 0$*;*

c) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = 1$, *if the case* ii) *is satisfied;*

d) $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N})$, *if the case* iv) *is satisfied.*

Proof. At first we notice that $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) > 0$ or $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) > 0$ in view of Corollary 4.

Now by the definition $\mathcal{M}[\mathcal{N}]$ is finite iff $\mathcal{M}$ and $\mathcal{N}$ are finite. In such a case we have the following possibilities:

• $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = (\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) - 1) \cdot |N| + 1$, if $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$, since the rigidity of $\mathcal{M}[\mathcal{N}]$ can be achieved here taking all elements in $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) - 1$ copies of $\mathcal{N}$ with one additional element witnessing the degree $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M})$ defining rigidly all E-classes for copies of $\mathcal{N}$ which are rigid by $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$; it corresponds the case i) with a);

• $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) = (|M| - 1) \cdot |N| + \deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N})$, if $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) > 0$, since the rigidity of $\mathcal{M}[\mathcal{N}]$ can be achieved here taking all elements in $(|M| - 1)$ copies of $\mathcal{N}$ with $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N})$ additional elements in the last copy of $\mathcal{N}$; it corresponds the case i) with b).

(1) $\Rightarrow$ (2). Let $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}[\mathcal{N}]) > 0$ is finite. We can assume that $\mathcal{M}$ is infinite or $\mathcal{N}$ is infinite. We have the following possibilities:

• $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) = 1$ and $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{N}) = 0$, that is any element of $\mathcal{M}[\mathcal{N}]$ rigidly defines its E-class and all E-classes, too, by $\deg^{\forall\text{-}s}_{\mathrm{rig}}(\mathcal{M}) = 1$, such that all copies of $\mathcal{N}$ in these E-classes are rigid by $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{N}) = 0$; it corresponds the case ii) with c);

• $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}) \in \omega \setminus \{0,1\}$ and $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{N}) = 0$; here we require that $\mathcal{N}$ is finite, since otherwise we can take arbitrary many elements in some E-classes which do not imply the rigidity in view of $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}) \geq 2$; here we have the case iii) with a).

• $\mathcal{M}$ is a singleton and $\mathcal{N}$ is infinite with $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{N}) \in \omega \setminus \{0\}$, here $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}) = 0$, $\mathcal{M}[\mathcal{N}] \simeq \mathcal{N}$ and therefore $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}[\mathcal{N}]) = \deg_{\rm rig}^{\forall\text{-}s}(\mathcal{N})$.

If $\mathcal{N}$ is infinite with $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{N}) \in \omega \setminus \{0\}$ and $|\mathcal{M}| \geq 2$ then we can not obtain the rigidity for all E-classes taking arbitrary many elements in some E-classes that contradicts the condition $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}[\mathcal{N}]) \in \omega$.

(2) $\Rightarrow$ (1). Since each finite structure has finite degrees of rigidity it suffices to show that $\deg_{\rm rig}^{\forall\text{-}s}(\mathcal{M}[\mathcal{N}])$ is finite if $\mathcal{M}$ is infinite or $\mathcal{N}$ is infinite with the conditions ii), iii), iv). We observe that ii) implies c), iii) implies a), and iv) implies d) confirming a finite value of that degree.

6. Conclusion

We studied possibilities for the degrees and indexes of rigidity, both for semantical and syntactical cases. Links of these characteristics and their possible values are described. We studied these values and dynamics for structures in some languages, for some natural operations including disjoint unions and compositions of structures. A series of examples illustrates possibilities of these characteristics. It would be interesting to continue this research describing possible values of degrees and indexes for natural classes of structures and their theories.

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