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On Algebraic and Definable Closures for Theories of Abelian Groups

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Abstract. Classifying abelian groups and their elementary theories, a series of characteristics arises that describe certain features of the objects under consideration. Among these characteristics, an important role is played by Szmielew invariants, which define the possibilities of divisibility of elements, orders of elements, dimension of subgroups, and allow describing given abelian groups up to elementary equivalence. Thus, in terms of Szmielew invariants, the syntactic properties of Abelian groups are represented, i.e. properties that depend only on their elementary theories. The work, based on Szmielew invariants, provides a description of the behavior of algebraic and definable closure operators based on two characteristics: degrees of algebraization and the difference between algebraic and definable closures. Thus, possibilities for algebraic and definable closures, adapted to theories of Abelian groups, are studied and described. A theorem on trichotomy for degrees of algebraization is proved: either this degree is minimal, if in the standard models, except for the only two-element group, there are no positively finitely many cyclic and quasi-cyclic parts, or the degree is positive and natural, if in a standard model there are no positively finitely many cyclic and quasi-cyclic parts, except a unique copy of a two-element group and some finite direct sum of finite cyclic parts, and the degree is infinite if the standard model contains unboundedly many non-isomorphic finite cyclic parts or positively finitely many of copies of quasi-finite parts. In addition, a dichotomy of the values of the difference between algebraic closures and definable closures for abelian groups defined by Szmielew invariants for cyclic parts is established. In particular, it is shown that torsion-free abelian groups are quasi-Urbanik.

Keywords: algebraic closure, definable closure, degree of algebraization, abelian group

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Научная статья

Об алгебраических и определимых замыканиях для теорий абелевых групп

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Аннотация. Отмечено, что при классификации абелевых групп и их элементарных теорий возникает ряд характеристик, описывающих те или иные особенности рассматриваемых объектов. Среди этих характеристик особую роль играют шмелевские инварианты, задающие возможности делимости элементов, порядков элементов, размерности подгрупп и позволяющие описывать данные абелевы группы с точностью до элементарной эквивалентности. Указано, что в терминах шмелевских инвариантов представляются синтаксические свойства абелевых групп, т. е. свойства, зависящие лишь от их элементарных теорий. На базе шмелевских инвариантов приведено описание поведения операторов алгебраического и определимого замыканий на основе двух характеристик: степеней алгебраизации и разницы между алгебраическими и определимыми замыканиями. Тем самым изучены и описаны возможности для алгебраических и определимых замыканий, адаптированные к теориям абелевых групп. Доказана теорема о трихотомии для степеней алгебраизации: либо эта степень минимальная, если в стандартных моделях, кроме единственной двухэлементной группы, нет положительно конечного числа циклических и квазициклических частей, либо степень положительная и натуральная, если в стандартной модели нет положительного конечного числа циклических и квазициклических частей, кроме единственной копии двухэлеметной группы и некоторой конечной прямой суммы конечных циклических частей, и степень бесконечна, если стандартная модель содержит неограниченное число неизоморфных конечных циклических частей или положительное конечное число копий квазиконечных частей. Кроме того, установлена дихотомия значений разности между алгебраическими замыканиями и определимыми замыканиями для абелевых групп, определяемых шмелёвскими инвариантами для циклических частей. В частности, показано, что абелевы группы без кручения квазиурбаниковы.

Ключевые слова: алгебраическое замыкание, определимое замыкание, степень алгебраизации, абелева группа

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1. Introduction

The classes of abelian groups and their elementary theories are broadly known, rich enough and productive both in Classification Theory and applications [3;4;6]. They have various tools and characteristics dividing objects into clear parts. Classification of elementary theories of abelian groups is based on Szmielew invariants [1; 2; 16] allowing to define abelian groups up to the elementary equivalence by standard representations constructed as direct sums of cyclic, quasi-cyclic and torsion-free abelian groups.

In the present paper, we continue to study families of abelian groups [8–10] and describe possibilities for algebraic and definable closures [5;12;14; 15; 17] adapted for theories of abelian groups. These possibilities are based on cardinalities of finite orbits with respect to automorphism groups [6; 7] including their description for finite abelian groups [11], and on Szmielew invariants.

We prove that a trichotomy for degrees of algebraization holds: either this degree is minimal, if there are no positively finitely many cyclic and quasi-cyclic parts in standard models $\mathcal{M}$ besides unique copy of $\mathbb{Z}_2$, or the degree is positive, natural and non-minimal, if there are no positively finitely many cyclic and quasi-cyclic parts in $\mathcal{M}$ besides unique copy of $\mathbb{Z}_2$ and some finite direct sum of finite cyclic parts, and the degree is infinite, if $\mathcal{M}$ contains unboundedly many non-isomorphic finite cyclic parts or positively finitely many copies of quasi-finite parts (Theorem 2).

Besides we clarify a dichotomy for the values of difference between algebraic closures and definable closures for abelian groups defined by Szmielew invariants for cyclic parts (Theorem 3). In particular, it is shown that torsion-free abelian groups are quasi-Urbanik (Corollary 3).

Throughout we consider complete first-order theories.

2. Algebraic and definable closures. Degrees of algebraization

Definition 1. [5; 14; 17] 1. A tuple $\overline{b}$ is *defined* by a formula $\varphi(\overline{x}, \overline{a})$ of T with parameters $\overline{a}$, if $\varphi(\overline{x}, \overline{a})$ has the unique solution $\overline{b}$.

A tuple $\overline{b}$ is *defined* by a type p if $\overline{b}$ is the unique tuple which realizes p. It is *definable* over a set A if $\mathrm{tp}(\overline{b}/A)$ defines it.

2. For a set A of a theory T the union of sets of solutions of formulae $\varphi(x, \overline{a})$, $\overline{a} \in A$, such that $\models \exists^{=n} x\, \varphi(x, \overline{a})$ for some $n \in \omega$ (respectively $\models \exists^{=1} x\, \varphi(x, \overline{a})$) is said to be an *algebraic* (*definable* or *definitional*) *closure* of A. The algebraic closure of A is denoted by $\mathrm{acl}(A)$ and its definable (definitional) closure, by $\mathrm{dcl}(A)$.

In such a case we say that the formulae $\varphi(x, \overline{a})$ *witness* that algebraic / definable (definitional) closure, and these formulae are called *algebraic* / *defining*.

Any element $b \in \mathrm{acl}(A)$ (respectively, $b \in \mathrm{dcl}(A)$) is called *algebraic* (*definable* or *definitional*) over A. If the set A is fixed or empty, we just say that b is *algebraic* (*definable*, or definitional).

3. If $\mathrm{dcl}(A) = \mathrm{acl}(A)$, $\mathrm{cl}_1(A)$ denotes their common value.

4. If $A = \mathrm{acl}(A)$ (respectively, $A = \mathrm{dcl}(A)$) then A is called *algebraically* (*definably*) closed.

5. A type p is *algebraic* (*defining*) if it is realized by finitely many tuples (unique one) only, i.e., it contains an algebraic (defining) formula φ. This formula φ can be chosen with the minimal number of solutions, and in such a case φ isolates p. The number of these solutions is called the *degree* $\deg(p)$ of p.

6. The complete algebraic types $p(x) \in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is algebraic over A. The *degree* of a over A, $\deg(a/A)$ is the degree of $\mathrm{tp}(a/A)$.

Remark 1. [14] The pairs $\langle M, \mathrm{acl}\rangle$ and $\langle M, \mathrm{dcl}\rangle$ satisfy the following properties:

(i) the reflexivity: it is witnessed by the formula $x \approx y$;

(ii) the transitivity: if the formulae $\varphi_1(x_1, \overline{a}), \ldots, \varphi_n(x_n, \overline{a})$ witnessed that $b_1, \ldots, b_n \in \mathrm{acl}(A)$ (respectively, $b_1, \ldots, b_n \in \mathrm{dcl}(A)$) and the formula $\psi(x, b_1, \ldots, b_n)$ witnesses that $c \in \mathrm{acl}(\{b_1, \ldots, b_n\})$ (respectively, $c \in \mathrm{dcl}(\{b_1, \ldots, b_n\})$) then the formula

$$\exists x_1, \ldots, x_n \left(\psi(x, x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} \varphi_i(x_i, \overline{a}) \right)$$

witnesses that $c \in \mathrm{acl}(\mathrm{acl}(A))$ (respectively, $c \in \mathrm{dcl}(\mathrm{dcl}(A))$);

(iii) the finite character: if a formula $\varphi(x, \overline{a})$ witnesses that $a \in \mathrm{acl}(A)$ (respectively, $a \in \mathrm{dcl}(A)$) then $a \in \mathrm{acl}(A_0)$ for the finite $A_0 \subseteq A$ consisting of coordinates in $\overline{a}$.

Definition 2. [15] 1. For $n \in \omega \setminus \{0\}$ and a set A an element b is called *n-algebraic* over A, if $a \in \mathrm{acl}(A)$ and it is witnessed by a formula $\varphi(x, \overline{a})$, for $\overline{a} \in A$, with at most n solutions.

2. The set of all n-algebraic elements over A is denoted by $\mathrm{acl}_n(A)$.

3. If $A = \mathrm{acl}_n(A)$ then A is called *n-algebraically* closed.

4. A type p is *n-algebraic* if it is realized by at most n tuples only, i.e., $\deg(p) \leq n$.

5. The complete n-algebraic types $p(x) \in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is n-algebraic over A, i.e., with $\deg(a/A) \leq n$. Here $\deg(a/A) = k \leq n$ defines the *n-degree* $\deg_n(a/A)$ of $\mathrm{tp}(a/A)$.

6. If $\mathrm{acl}(A) = \mathrm{acl}_n(A)$ then minimal such n is called the *degree of algebraization* over the set A and it is denoted by $\deg_{\mathrm{acl}}(A)$. If that n does not exist then we put $\deg_{\mathrm{acl}}(A) = \infty$. The supremum of values $\deg_{\mathrm{acl}}(A)$

with respect to all sets A of a given theory T is denoted by $\deg_{\rm acl}(T)$ and called the *degree of algebraization* of the theory T.

7. Following [19] theories T with $\deg_{\rm acl}(T)=1$, i.e., with defined ${\rm cl}_1(A)$ for any set A of T, are called *quasi-Urbanik*, and the models $\mathcal{M}$ of T are *quasi-Urbanik*, too.

Notice that by the definition the closure operator ${\rm acl}_1$ coincides with the closure operator dcl whereas the operators ${\rm acl}_n$, for $n\geq 2$, may be or not be transitive depending on a given theory [15].

Definition 3. [15] If for a theory T, ${\rm dcl}(A)={\rm acl}(A)$ for any set A with $|A|\geq n$ then minimal such n is called the acl-dcl-*difference* and denoted by ${\rm acl\text{-}dcl}_{\rm dif}(T)$. If such natural n does not exist, i.e., for any $n\in\omega$ there exists a finite set A with $|A|\geq n$ and ${\rm acl}(A)\supset{\rm dcl}(A)$ then we put ${\rm acl\text{-}dcl}_{\rm dif}(T)=\infty$.

Notice that, by the definition, a theory T is quasi-Urbanik iff

$${\rm acl\text{-}dcl}_{\rm dif}(T)=0.$$

3. Degrees of algebraization for finite abelian groups and their theories

Definition 4. [11] Let $\mathcal{M}$ be a structure, $A\subseteq M$. Recall that an A-*automorphism* of $\mathcal{M}$ is an automorphism $f\in{\rm Aut}(\mathcal{M})$ fixing A pointwise. The set of all A-automorphisms for $\mathcal{M}$ is denoted by ${\rm Aut}(\mathcal{M}/A)$.

For an element $a\in M$, the *orbit* $O(a/A)$ with respect to the automorphism group ${\rm Aut}(\mathcal{M})$ is the set of all elements $b\in M$ connected with a by A-automorphisms $f\in{\rm Aut}(\mathcal{M}/A)$: $f(a)=b$ and $f(a')=a'$ for any $a'\in A$.

We write $O(a)$ instead of $O(a/\emptyset)$.

We denote by $o(\mathcal{M})$ the maximal cardinality of orbits $O(a)$, i.e., the value of $\deg_{\rm acl}(\emptyset)$.

Let $T={\rm Th}(\mathcal{M})$ for a finite structure $\mathcal{M}$. Since all models of T are pairwise isomorphic, the value $o(\mathcal{M})$ does not depend on the choice of model $\mathcal{M}\models T$ and it is denoted by $o(T)$.

Proposition 1. [11] *If T is a theory of a finite structure then*

$$\deg_{\rm acl}(T)=\deg_{\rm acl}(\emptyset)=o(T).$$

Recall that any finite abelian group $\mathcal{S}$ is represented as a direct sum $\oplus_{p,n}\mathbb{Z}_{p^n}^{(\alpha_{p,n})}$ [6, Theorem 8.1.2]. Recall also [18] that *Euler function* $\varphi(n)$ is defined as follows: $\varphi(n)=|\{m\in\mathbb{Z}_n\mid(m,n)=1\}|$.

Theorem 1. [11] *For any finite abelian group* $\mathcal{A} = \oplus_{p,n}\mathbb{Z}_{p^n}^{(\alpha_{p,n})}$,

$$\deg_{\mathrm{acl}}(\mathrm{Th}(\mathcal{S})) = \prod_{p,n}(p^{n\alpha_{p,n}} - (p^n - \varphi(p^n))^{\alpha_{p,n}}).$$

Corollary 1. [11] *A finite abelian group* $\mathcal{A}$ *is quasi-Urbanik iff* $\mathcal{A}$ *is either a singleton or isomorphic to* $\mathbb{Z}_2$.

4. Abelian groups, their theories and Szmielew invariants

Let $\mathcal{A}$ be an abelian group in the language $\Sigma = \langle +^{(2)}, -^{(1)}, 0^{(0)}\rangle$. Then $k\mathcal{A}$ denotes its subgroup $\{ka \mid a \in A\}$ and $\mathcal{A}[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. Let P be the set of all prime numbers. If $p \in P$ and $pA = \{0\}$ then $\dim\mathcal{A}$ denotes the dimension of the group $\mathcal{A}$, considered as a vector space over a field with p elements. The following numbers, for arbitrary $p \in P$ and $n \in \omega \setminus \{0\}$ are called the *Szmielew invariants* for the group A [2; 16]:

$$\alpha_{p,n}(\mathcal{A}) = \min\{\dim((p^n\mathcal{A})[p]/(p^{n+1}\mathcal{A})[p]), \omega\},$$

$$\beta_p(\mathcal{A}) = \min\{\inf\{\dim((p^n\mathcal{A})[p] \mid n \in \omega\}, \omega\},$$

$$\gamma_p(\mathcal{A}) = \min\{\inf\{\dim((\mathcal{A}/\mathcal{A}[p^n])/p(\mathcal{A}/\mathcal{A}[p^n])) \mid n \in \omega\}, \omega\},$$

$$\varepsilon(\mathcal{A}) \in \{0, 1\},$$

$$\text{and } \varepsilon(\mathcal{A}) = 0 \Leftrightarrow (n\mathcal{A} = \{0\} \text{ for some } n \in \omega, n \neq 0).$$

It is known [2, Theorem 8.4.10] that two abelian groups are elementarily equivalent if and only if they have the same Szmielew invariants. Besides, the following proposition holds.

Proposition 2. [2, Proposition 8.4.12]. *Let for any* p *and* n *the cardinals* $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, *and* $\varepsilon \in \{0, 1\}$ *be given. Then there is an abelian group* $\mathcal{A}$ *such that the Szmielew invariants* $\alpha_{p,n}(\mathcal{A})$, $\beta_p(\mathcal{A})$, $\gamma_p(\mathcal{A})$, *and* $\varepsilon(\mathcal{A})$ *are equal to* $\alpha_{p,n}$, β_p, γ_p, *and* ε, *respectively, if and only if the following conditions hold:*

(1) *if for prime* p *the set* $\{n \mid \alpha_{p,n} \neq 0\}$ *is infinite then* $\beta_p = \gamma_p = \omega$;

(2) *if* $\varepsilon = 0$ *then for any prime* p, $\beta_p = \gamma_p = 0$ *and the set* $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ *is finite.*

We denote by $\mathbf{Q}$ the additive group of rational numbers, $\mathbf{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbf{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbf{Q}$, $\mathbf{Z}_{p^n}$, R_p,

$\mathbf{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since abelian groups with the same Szmielew invariants have same theories, any abelian group $\mathcal{A}$ is elementarily equivalent to a group

$$\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbf{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)}, \quad (4.1)$$

where $\mathcal{B}^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group $\mathcal{B}$. Thus, any theory of an abelian group has a model being a direct sum of basic groups. The groups of form (4.1) are called *standard*.

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas [2, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y), \quad (4.2)$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \quad (4.3)$$

where $m_i \in \mathbf{Z}$, $k \in \omega$, p is a prime number [1], [2, Lemma 8.4.7]. Formulas (4.2) and (4.3) allow to witness that Szmielew invariants define theories of abelian groups modulo Proposition 2.

In view of Proposition 2 and equations (4.2) and (4.3) we have the following:

Remark 2. Theories of abelian groups are forced by sentences implied by formulas of form (4.2) and (4.3) and describing dimensions with respect to $\alpha_{p,n}$, β_p, γ_p, ε as well as bounds for orders p^k of elements and possibilities for divisions of elements by p^k. Moreover, distinct values of Szmielew invariants are separated by some sentences modulo Proposition 2.

5. Degrees of algebraization for theories of abelian groups

Proposition 3. *If* $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ *then*

$$\deg_{\rm acl}({\rm Th}(\mathcal{A})) \geq \max\{\deg_{\rm acl}({\rm Th}(\mathcal{A}_1)), \deg_{\rm acl}({\rm Th}(\mathcal{A}_2))\}. \quad (5.1)$$

Proof. Since $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, each algebraic element a over $A_0 \subseteq A$ witnessing the value $\deg_{\rm acl}({\rm Th}(\mathcal{A}))$ has maximal, by cardinality, finite orbits O_1 and O_2 over $A_0 \cap A_1$ and $A_0 \cap A_2$, respectively, with $a = a_1a_2$, $a_1 \in O_1$, $a_2 \in O_2$. Therefore, a witnesses, via a_1 and a_2, both the values $\deg_{\rm acl}({\rm Th}(\mathcal{A}_1))$ and $\deg_{\rm acl}({\rm Th}(\mathcal{A}_2))$. Again since $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, the orbits O_1 and O_2 are independent producing at least $|O_1|$ and at least $|O_2|$ possibilities for simultaneous actions for a_1 and a_2, i.e. for a, implying the inequality (5.1). $\square$

By the definition if $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is finite then both $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite. Thus, Propositions 1 and 3 immediately imply:

Corollary 2. *If $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is finite then*

$$\deg_{\rm acl}(\mathrm{Th}(\mathcal{A})) \geq \max\{o(\mathrm{Th}(\mathcal{A}_1)), o(\mathrm{Th}(\mathcal{A}_2))\}.$$

Both equalities and inequalities are realized in Corollary 2 using Theorem 1.

As degrees of algebraization for finite abelian groups are described in Section 3, below we assume that $\mathcal{A}$ is an infinite abelian group with a theory T. In view of Section 4, without loss of generality, we additionally assume that $\mathcal{A}$ has the form (4.1).

Lemma 1. *If T is the theory of an abelian group $\mathcal{A}$ with all $\alpha_{p,n} = 0$, and $\beta_p = 0$ or $\beta_p = \omega$ then T is quasi-Urbanik.*

Proof. For the structure $\mathcal{A}$, in view of Szmielew invariant classification, we have $\mathcal{A} \equiv \oplus_{\beta_p > 0} \mathbb{Z}_{p^\infty}^{(\omega)} \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)}$. By Remark 2 links between elements of $\mathcal{A}$ are defined by connections via linear combinations and their divisibilities. Now we observe that by the absence of natural $\beta_p > 0$ the only possibilities for algebraic elements over a subset $A_0 \subseteq A$ are to take linear combinations of elements in A_0. Since values of terms are unique we observe that $\mathrm{acl}(A_0) = \mathrm{dcl}(A_0)$ implying that $\mathcal{A}$ is quasi-Urbanik. □

Since the condition $\alpha_{p,n} = \beta_p = 0$ for all p, n characterizes that the abelian group is torsion-free, Lemma 1 immediately implies:

Corollary 3. *Any theory T of a torsion-free abelian group is quasi-Urbanik, i.e.,* $\text{acl-dcl}_{\rm dif}(T) = 0$.

In view of Lemma 1 the torsion-free group $\mathbb{Z}$, with Szmielew invariants $\alpha_{p,n} = 0$, $\beta_p = 0$, $\gamma_p = 1$ for each prime p, $\varepsilon = 1$ (see [13] for these values) is quasi-Urbanik.

Lemma 2. *If T is the theory of an abelian group $\mathcal{A}$ with some positive $\beta_p \in \omega$ then* $\deg_{\rm acl}(T) = \infty$.

Proof. Let $0 < \beta_p < \omega$. Then by Proposition 2 the set $\{n \mid \alpha_{p,n} = 0\}$ is finite and we can separate p-divisible and non p-divisible elements of finite orders related to $\mathbb{Z}_{p^n}$ and to $\mathbb{Z}_{p^\infty}$ in a standard model $\mathcal{M}$ of T. Since $\beta_p \in \omega$ there are finitely many and unboundedly many elements in copies of $\mathbb{Z}_{p^n}$ in $\mathcal{M}$ of orders p, p^2, p^3, etc., implying $\deg_{\rm acl}(T) = \infty$ by formulae witnessing $\mathrm{acl}(\emptyset)$. □

Lemma 3. *If T is the theory of an abelian group $\mathcal{A}$ with $\beta_p \in \{0, \omega\}$ for any p then exactly one of the following conditions holds:*

1) $\deg_{\rm acl}(T) = 1$, *if $\alpha_{p,n} \in \{0, \omega\}$ for any $(p, n) \neq (2, 1)$ and $\alpha_{2,1} \in \{0, 1, \omega\}$;*

2) $\deg_{\rm acl}(T) \in \omega \setminus \{0, 1\}$, *if some positive $\alpha_{p,n}$ is finite besides the possibility $\alpha_{2,1} = 1$, and there are finitely many natural positive $\alpha_{p,n}$; here the value* $\deg_{\rm acl}(T)$ *satisfies the equality in Theorem* 1*;*

3) $\deg_{\rm acl}(T) = \infty$, *if there are infinitely many natural positive $\alpha_{p,n}$.*

Proof. If $\alpha_{p,n} \in \{0, \omega\}$ and $\beta_p \in \{0, \omega\}$ then there are no non-zero algebraic elements in finite cyclic groups and quasi-cyclic groups which are contained in a standard group $\mathcal{A}' \equiv \mathcal{A}$, since these groups are connected by automorphisms with their copies. It implies the value $\deg_{\rm acl}(T) = 0$. Here the possibility with $\alpha_{2,1} = 1$ can be added since $\mathbb{Z}_2 \leqslant \mathcal{A}'$ is rigid and does not have isomorphic copies in $\mathcal{A}'$.

If some positive $\alpha_{p,n}$ is finite besides the possibility $\alpha_{2,1} = 1$, and there are finitely many natural positive $\alpha_{p,n}$, then elements of finite cyclic subgroups form a finite definable set with non-trivial orbits. Thus, Theorem 1 is applicable for the value $\deg_{\rm acl}(T)$ which is finite since other parts of $\mathcal{A}'$ are quasi-Urbanik in view of Lemma 1.

If there are infinitely many natural positive $\alpha_{p,n}$ then the cyclic part of $\mathcal{A}'$ has definable finite orbits of unbounded cardinalities implying $\deg_{\rm acl}(T) = \infty$. □

Summarizing Lemmas 1, 2, 3 we obtain the following:

Theorem 2. *For any theory T of an abelian group $\mathcal{A}$ exactly one of the following conditions holds:*

1) $\deg_{\rm acl}(T) = 1$, *if $\beta_p \in \{0, \omega\}$ for any p, $\alpha_{p,n} \in \{0, \omega\}$ for any $(p, n) \neq (2, 1)$, and $\alpha_{2,1} \in \{0, 1, \omega\}$;*

2) $\deg_{\rm acl}(T) \in \omega \setminus \{0, 1\}$, *if $\beta_p \in \{0, \omega\}$ for any p, some positive $\alpha_{p,n}$ is finite besides the possibility $\alpha_{2,1} = 1$, and there are finitely many natural positive $\alpha_{p,n}$;*

3) $\deg_{\rm acl}(T) = \infty$, *if the values p or n for natural $\alpha_{p,n} > 0$ are unbounded, or $\beta_p \in \omega \setminus \{0\}$ for some p.*

In Theorem 2 containing a trichotomy for $\deg_{\rm acl}(T)$, finite values $\deg_{\rm acl}(T)$ are described in terms of Szmielew invariants on the base of Theorem 1.

The following theorem asserts a dichotomy for the values $\text{acl-dcl}_{\rm dif}(T)$ of acl-dcl-difference for theories T of abelian groups.

Theorem 3. *For any abelian group $\mathcal{A}$ and its theory T either* $\text{acl-dcl}_{\rm dif}(T)$ *is finite, if $\mathcal{A}$ is finite, or all $\alpha_{p,n} \in \{0, \omega\}$, except may be $\alpha_{2,1} = 1$, and $\beta_p \in \{0, \omega\}$, or* $\text{acl-dcl}_{\rm dif}(T) = \infty$ *otherwise, i.e., if $\mathcal{A}$ is infinite and there are positive $\alpha_{p,n} \in \omega$ or $\beta_p \in \omega$ besides $\alpha_{2,1} = 1$.*

Proof. In view of Corollary 1 and Lemmas 1, 3 we have

$$\text{acl-dcl}_{\text{dif}}(T) = 0$$

if all $\alpha_{p,n} \in \{0, \omega\}$ for all $(p, n) \neq (2, 1)$, $\alpha_{2,1} \in \{0, 1, \omega\}$ and all $\beta_p \in \{0, \omega\}$. Besides $\text{acl-dcl}_{\text{dif}}(T) \in \omega$ if $\mathcal{A}$ is finite and this value is defined following the value of degree of algebraization in Theorem 1. Now by Lemma 2 having some $\beta_p \in \omega \setminus \{0\}$ we obtain $\text{acl-dcl}_{\text{dif}}(T) = \infty$. Since closures in standard abelian groups are formed by closures in basic subgroups, the only possibility for infinite $\mathcal{A}$ to have a finite value $\text{acl-dcl}_{\text{dif}}(T)$ is to have the cyclic part with $\alpha_{p,n} \in \{0, \omega\}$ for $(p, n) \neq (2, 1)$ and with $\alpha_{2,1} \in \{0, 1, \omega\}$, and to have a quasi-finite part with all $\beta_p \in \{0, \omega\}$ which imply here that T is quasi-Urbanik. □

6. Conclusion

We studied possibilities for algebraic and definable closures for abelian groups and their theories. A theorem on trichotomy for degrees of algebraization is proved: either the degree is minimal, if in the standard models, except for the only two-element group, there are no positively finitely many cyclic and quasi-cyclic parts, or the degree is positive and natural, if in a standard model there are no positively finitely many cyclic and quasi-cyclic parts, except a unique copy of a two-element group and some finite direct sum of finite cyclic parts, and the degree is infinite if the standard model contains unboundedly many non-isomorphic finite cyclic parts or positively finitely many of copies of quasi-finite parts. Besides, a dichotomy of the values of the difference between algebraic closures and definable closures for abelian groups defined by Szmielew invariants for cyclic parts is established. In particular, it is shown that torsion-free abelian groups are quasi-Urbanik and the separation of the class of quasi-Urbanik abelian groups is established in terms of Szmielew invariants.

It would be interesting to study a hierarchy of algebraic closures for abelian groups relative to various sets of formulas.

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