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Identification of a Mathematical Model of Economic Development of Two Regions of the World

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Abstract. This paper is devoted to solving the inverse problem (determining the parameters of a system of ordinary differential equations based on additional information determined at discrete points in time) and analyzing its solution for a mathematical model describing the dynamics of changes in the population and capital of two regions of the world. The inverse problem is reduced to the problem of minimizing the target functional and is solved by the method of differential evolution. A numerical method for solving direct and inverse problems is implemented. The developed method was tested on model and real data for countries such as Russia, China, India and the USA.

Keywords: mathematical model, system of ordinary differential equations, population, economic development, inverse problem, direct problem

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Научная статья

Идентификация математической модели экономического развития двух регионов мира

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Аннотация. Работа посвящена решению обратной задачи (определению параметров системы обыкновенных дифференциальных уравнений на основе дополнительной информации, определенной в дискретные моменты времени) и анализу ее решения для математической модели, описывающей динамику изменений численности населения и капитала двух регионов мира. Обратная задача сводится к задаче минимизации целевого функционала и решается методом дифференциальной эволюции. Реализован численный метод решения прямых и обратных задач. Разработанный метод был протестирован на модельных и реальных данных для таких стран, как Россия, Китай, Индия и США.

Ключевые слова: математическая модель, система обыкновенных дифференциальных уравнений, популяция, экономическое развитие, обратная задача, прямая задача

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1. Introduction

Analyzing economic growth and its causes is vital in macroeconomics. GDP dynamics result from three factors: long-term economic development trends influenced by various internal and external factors, cyclical fluctuations around the GDP trend, and abrupt changes due to economic, political, or other events, like global crises or COVID-19. Economic growth theory seeks to identify the causes driving a country's overall economic trend.

These causes fall into two categories: internal (capital and population growth, technological advancements, and domestic price changes) and external (world prices, substitute goods, and cross-border movement of production factors). Using mathematical models, we analyze two global regions economic development, deriving parameters reflecting their socio-economic progress over time, which aids in identifying growth trends.

The interaction between different countries economies, particularly the flow of production factors like capital and labor, significantly impacts economic growth rates. Numerous works in economic literature address this issue, emphasizing current trends without delving into its historical context. The econometric study of convergence between the countries of the European regions (16 countries) conducted by [21] showed that, according to previous studies by [4]:

– There is an absolute convergence across European regions: countries with a lower level of GDP per capita demonstrate higher growth rates of this indicator than the leaders, which leads to a relative equalization of their economic situation.

– The decline in inequality in Europe is mainly due to inter-country convergence, not intra-country. The gap within individual countries between developed and less developed regions persists and may widen.

In [28], a hypothesis was presented that asserts that industrialized nations primarily spearhead innovation, while developing countries predominantly engage in imitation, thus fostering a capital (in the form of technology) flow from leaders to laggards. This notion was consistent with similar models presented in [1; 2]. Consequently economic growth rates for developed economies (nearing the technological forefront) are endogenously linked to innovation costs and domestic market size.

An econometric model was used to assess the impact of foreign capital inflows on economic growth rates in low- and middle-income countries [8]. The study concluded that net capital inflows predominantly positively affect economic growth, with a more pronounced impact on low-income countries due to real exchange rate increases.

A nonparametric approach to the problem of identifying production functions was presented in the context of the Solow [10] spatial model, which admits fairly general production functions, particularly convex-concave ones. The inverse problem was reduced to minimizing the Tikhonov functional by the gradient descent method.

A new growth model [24] was proposed, based on the work of Sato [20] and the assumption of logistic growth of production factors. The corresponding production functions were obtained, along with a compatible concept of the wage share. It is shown that the new functions are quite well comparable with the corresponding economic data.

The determination of the parameters of the mathematical model of neoclassical economic growth by Solow [13] was numerically investigated.

Some inverse problems in economic measurement and financial mathematics were considered in [11; 22].

Empirical results show [12] that the specification of the Solow model, according to which stable differences in output per capita are due to international differences in the ratio of investment and output at a constant level of technology, allows us to generalize the data quite well using the indicator of institutional technology and considering the coefficient of capital return as part of the regression constant.

2. Precondition

The macroeconomic production function of the country's economy can be represented as the following dependence:

$$Y = F(A, K, L, z) \tag{2.1}$$

where Y is output (GDP) of the country, A is a parameter that characterizes the technological potential of the country's economy, K is the amount of capital used in the country's economy, L is the amount of labor used in the country's economy, z is costs for other resources used in production (in particular, raw materials).

The interrelation of production facts, spanning technological methods, production hierarchy, economic goods, and fixed capital, significantly impacts production. This influence is not purely quantitative but primarily qualitative. A variant of the closed economy, where a country has limited foreign economic relations, was occasionally seen in the USSR. Excluding mineral resource exports and economic ties with socialist countries in COMECON, the USSR primarily aimed for self-sufficiency in technology development.

Economic growth dynamics, including GDP changes in Formula (2.1), are influenced by state economic policies (fiscal, monetary, foreign economic), and shifts in the real exchange rate. Devaluation raises imported goods and services prices, affecting domestic production relying heavily on imports (e.g., as seen in Russia). This also leads to higher prices for capital (K), technology (A), and imported raw materials (z).

International data on population growth and per capita income show that countries with high population growth rates have a low level of income per capita, while countries with lower population growth rates have a higher standard of living of the population [32]. At the same time, there are examples of an inverse relationship, for example, China, where the population and economic growth rates have the same trend [19; 29]. The region's population growth may result from migration and a higher birth rate, primarily driven by economic factors. These observations prompt an exploration of the link between regional population and economic indicators. This paper utilizes a mathematical model to analyze the economic development of two world regions and obtain socio-economic parameters over time intervals.

3. Mathematical model of economy and population

To study the movement of capital and population, the work was limited to considering two regions using a system of ordinary differential equations. As a starting point, a mathematical model was used that connects the population of the two regions and their capitals. The model is based on two mathematical models: the neoclassical model of economic growth by Robert Solow [26] and the population growth model by Pierre-Francois Ferhulst [31; 33]. The model also takes into account the impact of the difference in value between capitals on the cost of capital in each of the regions. To describe the change in population size, the difference in population size between regions is taken into account, as well as the movement of labor caused by the difference in capital. The model has the following form:

$$\begin{aligned}
\frac{\mathrm{d}K_1}{\mathrm{d}t} &= d_k\left(K_2 - K_1\right) + \lambda_1 K_1^{\varphi} L_1^{1-\varphi} - \delta_1 K_1,\\
\frac{\mathrm{d}K_2}{\mathrm{d}t} &= d_k\left(K_1 - K_2\right) + \lambda_2 K_2^{\varphi} L_2^{1-\varphi} - \delta_2 K_2,\\
\frac{\mathrm{d}L_1}{\mathrm{d}t} &= d_l\left(L_2 - L_1\right) + a_1 L_1 + b_1 L_1^2 - cH(L_1, L_2, K_1, K_2),\\
\frac{\mathrm{d}L_2}{\mathrm{d}t} &= d_l\left(L_1 - L_2\right) + a_2 L_2 + b_2 L_2^2 + cH(L_1, L_2, K_1, K_2).
\end{aligned} \tag{3.1}$$

In the model (3.1), the function $H(L_1, L_2, K_1, K_2)$ has the form:

$$H(L_1, L_2, K_1, K_2) = \begin{cases} L_1(K_2 - K_1), & \text{if } \ K_2 - K_1 \geq 0,\\ L_2(K_2 - K_1), & \text{if } \ K_2 - K_1 < 0. \end{cases} \tag{3.2}$$

K_1, K_2 are the sums of capital in the first and second regions, and L_1, L_2 are the sums of the population in the first and second regions. The description of all parameters in the model is presented in the Table 1.

Table 1

Description of model parameters

Parameter	Description
d_k	The capital diffusion coefficient
d_l	The labor diffusion coefficient
c	The parameter of strength of the movement of labor force caused by capital
δ_1 (δ_2)	The coefficient of the depreciation rate of the capital of the first (second) region
φ	The output elasticities of capital
$1-\varphi$	The output elasticities of population
$\lambda_1(\lambda_2)$	Total factor productivity of the first (second) region
$a_1(a_2)$	The parameter of growth rate of first (second) region
$b_1(b_2)$	Coefficient of crowding effect or intraspecific competition in the first (second) region

It is worth noting that each region has its own technological and production facilities [16]. Moreover, each region has its own characteristics of remuneration and how the labor supply will react to changes in this remuneration [18]. For this reason, in the expressions for changing the values of K_1 and K_2, the coefficient φ was converted to φ_1 and φ_2, respectively. The final view of the model:

$$\begin{aligned}
\frac{\mathrm{d}K_1}{\mathrm{d}t} &= d_k\left(K_2-K_1\right)+\lambda_1 K_1^{\varphi_1} L_1^{1-\varphi_1}-\delta_1 K_1,\\
\frac{\mathrm{d}K_2}{\mathrm{d}t} &= d_k\left(K_1-K_2\right)+\lambda_2 K_2^{\varphi_2} L_2^{1-\varphi_2}-\delta_2 K_2,\\
\frac{\mathrm{d}L_1}{\mathrm{d}t} &= d_l\left(L_2-L_1\right)+a_1 L_1+b_1 L_1^2-cH(L_1,L_2,K_1,K_2),\\
\frac{\mathrm{d}L_2}{\mathrm{d}t} &= d_l\left(L_1-L_2\right)+a_2 L_2+b_2 L_2^2+cH(L_1,L_2,K_1,K_2).
\end{aligned} \tag{3.3}$$

According to the data of the selected real task, the following conditions are always met:

$$K_2 > K_1; L_2 > L_1$$

that is, the volume of capital in the USA, first of all, although it was so significant (the share of the USA in the volume of capital for the period under review fluctuated in the range of 17.6-30.0%, the population - in the range of 4.3-5.5%), that the country's economy was considered a large open economy, but nevertheless they were constantly smaller, than in the economy of the rest of the world.

Thus, the positive sign of the coefficient d_k means the flow of capital from the economy of the rest of the world to the US economy, and the negative sign means the outflow of capital from the US economy, i.e. characterizes international investment processes. Note that according to the Table 3, for the period under review, for 10 five-year subperiods out of 12, there was a flow of capital from the rest of the world to the United States (with the exception of only 1975-1980 and 2005-2010). The dynamics of domestic net (i.e., taking into account capital depreciation) investments in the first two equations (3.3) represent the last two terms. The term $\lambda K^{\varphi} L^{1-\varphi}$ is associated with the representation of the production function of the country's economy in the form of the Cobb-Douglas function:

$$Y = AK^{\varphi} L^{1-\varphi}$$

with a constant return to scale: with a proportional increase in the volume of factors of production (capital and labor) by a certain number of times, output grows at the same rate – mathematically, this is expressed in the fact that the sum of the exponents of the variables K and L is equal to one. Thus, the value of the coefficient λ can be interpreted as the product of the technological parameter A by the share of output, which materializes in the form of gross domestic investment, and the whole term $\lambda K^{\varphi} L^{1-\varphi}$ can be seen as the value of gross domestic investment. In turn, another term, δK, is meaningfully interpreted as depreciation charges or depreciation of the capital used. In total, these two terms represent the value of domestic (excluding cross-country capital flows) net investments. As a result of model calculations, starting in the 1970s, during three five-year plans (the entire 1970s and the first half of the 1980s of the XX century), the value of US domestic net investment was negative, which in the first halves of the 1970s and 1980s was offset by positive capital inflows from abroad. Capital outflow from the US economy has been observed since the 1970s, only in the second half of the 1970s and the first half of the 2000s of the XXI century. Thus, the US economy was almost constantly a net importer of capital; since the second half of the 1980s, this was accompanied by positive net domestic investment - i.e., the capital of the US economy was already growing for two (internal and external) reasons.

The direction of migration processes was determined in (3.3) by the values of two coefficients of the third and fourth equations: d_l and c. The parameter H in our particular task takes the value $L_1(K_2 - K_1)$ meaningfully according to the inequality (3.2) and is positive. Thus, when a positive value of the coefficient d_l and a negative value of the coefficient c are combined, positive migration to the USA is observed. When a negative value of the coefficient d_l and a positive value of the coefficient c are combined, positive migration is observed. If the coefficients d_l and c have equal signs, positive migration is observed when the condition is met:

$$d_l > c \cdot \frac{(K_2 - K_1)}{(L_2 - L_1)}$$

Since the early 1970s, positive migration occurred in four out of ten five-year periods, countered by significant natural population growth (exception: 2015-2019). The US economy saw diverse trends in capital and labor movement, impacting capital-to-weight production ratio. Real data underscores the importance of viewing capital as investment goods and production tech, not just financially. The model underestimates US migration and overestimates natural population growth due to the chosen quadratic function for population change.

In addition to the model presented above, other variants of its modifications were considered. So, in the equations describing the population change, the expression $a_iL_i + b_iL_i^2$ was considered in the following forms:

$$a_iL_i + b_iL_i^2 + c_iL_i^3, \tag{a}$$

$$a_iL_i + b_iL_i^2 + c_iL_i^3 + d_iL_i^4. \tag{b}$$

However, using the obtained models, the parameters found as a result of solving the inverse problem also accurately and correctly predict the dynamics of capital and population. Moreover, with an increase in the number of unknown parameters, a problem arises that must be avoided: the problem of the descriptive ability of each parameter. It is worth noting that the expression $a_iL_i + b_iL_i^2$ is based on the hypothesis of a parabolic type of labor force change [15]. The article also considered an expression in the form of an exponential change:

$$a_{0i} + e^{a_{1i}\frac{1}{L_i}}. \tag{c}$$

The set of parameters obtained as a result of solving the inverse problem with exponential variation showed low predictive ability. An additional modification of the model is based on changing the function $H(L_1, L_2, K_1, K_2)$. In the initial model, the function uses the values of capital and population at the predicted time. The modified function with a delay includes the values of capital and population at the predicted and previous time points.

$$H(L_1, L_2, \dot{K}_1, \dot{K}_2) = \begin{cases} L_1(1+\dot{K}_1), & \text{if } \dot{K}_1 \geq \dot{K}_2, \\ L_2(1+\dot{K}_2), & \text{if } \dot{K}_2 > \dot{K}_1. \end{cases} \tag{d}$$

$$\dot{K}_j = \frac{K_j(t_i) - K_j(t_{i-1})}{K_j(t_{i-1})}, \quad j = 1, 2,$$

where t_i is the predicted, and t_{i-1} is the previous point in time. As a result of numerical experiments of solving the inverse problem using a model (3.1) and a function (d), sets of parameters were obtained at which the projected dynamics of population and capital has a negative increase. That is, the modification of the model also did not allow us to obtain a satisfactory result of solving the inverse problem using the developed algorithms.

4. Direct problem

The system of ordinary differential equations (3.3) describes the change in the level of capital and population using the vector of parameters $\vec{q} \in \mathbb{R}^{13}$ (Table 1):

$$\vec{q} = (d_k, \lambda_1, \phi_1, \delta_1, \lambda_2, \phi_2, \delta_2, d_l, a_1, b_1, c, a_2, b_2)\,.$$

The direct problem is understood as the Cauchy problem of modeling the process of interaction between two regions of the world. In the system (3.3), it is necessary to find the values of the capitals $K_1(t, \vec{q})$, $K_2(t, \vec{q})$ and the population $L_1(t, \vec{q})$, $L_2(t, \vec{q})$ at each time $t \in [t_0, T]$. To solve a direct problem, add initial conditions to the (3.3) system:

$$L_1(t_0) = L_1^0, \quad L_2(t_0) = L_2^0, \quad K_1(t_0) = K_1^0, \quad K_2(t_0) = K_2^0.$$

To solve the direct problem, a one-step Rosenbrock method with a complex coefficient is used [3]. The choice of the method is due to the fact that when solving the inverse problem, the parameters of the mathematical model change, as a result of which rigid systems may appear.

Let's formulate the Cauchy problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t)), \quad y(0) = y_0. \tag{4.1}$$

The Rosenbrock scheme with complex coefficients is often used for the numerical solution of direct and conjugate problems.

The single–stage Rosenbrock method with complex coefficients is formulated as follows:

$$\begin{aligned} &y^{n+1} = y^n + \tau Re(\beta k), \\ &\left(E - \tau\gamma\frac{\partial y}{\partial t}(y^n)\right) k = f(y^n) \end{aligned} \tag{4.2}$$

By selecting the values of γ and β parameters, you can get a scheme with different properties: A-stability, L_1-stability, L_2-stability, etc. [3]. In this paper, the values $\beta = 1$ and $\gamma = \frac{1 \pm i}{2}$ were chosen to achieve L_2-stability, monotonicity and approximation of the second-order scheme [3]:

$$\begin{aligned} &y^{n+1} = y^n + \tau Re(k), \\ &\left(E - \tau\frac{1 \pm i}{2}\frac{\partial y}{\partial t}(y^n)\right) k = f(y^n) \end{aligned} \tag{4.3}$$

5. Sensitivity analysis

A sensitivity analysis of a mathematical model is conducted to assess how variations in input parameters or model assumptions impact the model's output or predictions. We apply the Sobol analysis [25] to understand the influence of input parameters on the output of the model. Sobol analysis measures parameter sensitivity, showing how changes affect the model's output. This guides resource allocation towards key factors during optimization. It also gauges parameter importance by assigning sensitivity indices. This aids in assessing model robustness to input variations, crucial for uncertain scenarios. Sobol sensitivity analysis dissects variance into first-order and total-order indices, revealing parameter impacts on model outcomes.

From the point of view of a black box for sensitivity analysis, the model (3.3) can be considered as a function:

$$Y = f(X_0, \vec{q}), \quad Y \in \mathbb{R},$$

where X_0 is a vector of the initial data of the model, and $\vec{q} \in \mathbb{R}^{13}$ (Table 1) is a vector of parameters of the model. Since it is assumed that the model has a scalar (a single real number) as its output, the function $f(X_0, \vec{q})$ transforms the solution of the model (3.3) as the sum of solutions for all variables at the discrete time points under consideration. Total-order indices were considered, which reflect the contribution to the output variance of $q_i \in \vec{q}$, as well as the entire variance caused by its interactions of any order with any other input parameters. As a result, the values of sensitivity indicators were obtained, which is shown on the Figure 1.

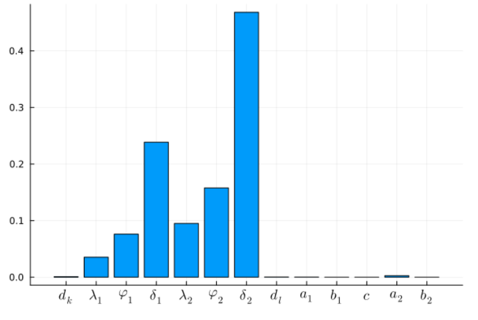


Figure 1. Total Order Indices

As a result, it was found that the change in parameters $d_k, d_l, a_1, b_1, c,$ a_2, b_2 has practically no effect on the change in the output signal of the model, whereas the parameters $\delta_2, \delta_1, \phi_2, \lambda_2, \phi_1, \lambda_1$ have the greatest influence (in order of decreasing influence) on the model.

The results obtained indicate that changes in the coefficients of the capital depreciation rate of the regions (δ_1, δ_2) under consideration have a high impact on changes in the dynamics of capital and population. This may be caused by the following factors:

– The rapid rate of retirement of fixed capital, on the one hand, with the limited capacity of the country's economy to conduct large-scale investments in fixed assets (as, in particular, was observed in the Russian economy in the 1990s – 2000s) leads to a slowdown in economic growth [5–7].

– On the other hand, the rapid rate of retirement of fixed capital in the economies of developed countries indicates, as a rule, high rates of scientific and technological progress, as a result of which there is a rapid moral, not physical obsolescence of fixed assets (which is well illustrated, for example, in the use of computer technology).

6. Inverse problem

The model assumes the absence of external flow in two closed regions. The USA and the rest of the world are considered as the regions, using GDP as an indicator for their capital level. GDP represents the market value of goods and services produced yearly in the region for consumption, export, and accumulation. The search for unknown parameters of the model is based on the use of available data on GDP and population in the USA and the rest of the world for the period from 1961 to 2019. The inverse problem is to determine the parameters $\vec{q}$ of the mathematical model (3.3) using the following known data [13]:

$$K_1(t_k) = F_{1k},\ \ K_2(t_k) = F_{2k},\ \ L_1(t_k) = F_{3k},\ \ L_2(t_k) = F_{4k},\ \ k = 1, 2, ..., N$$

The inverse problem is reduced to minimizing the target functional:

$$\begin{aligned} J(\vec{q}) = \sum_{k=1}^{N} (K_1(t_k;\vec{q}) - F_{1k})^2 + (K_2(t_k;\vec{q}) - F_{2k})^2 + \\ + (L_1(t_k;\vec{q}) - F_{3k})^2 + (L_2(t_k;\vec{q}) - F_{4k})^2 \to \min_{\vec{q}}. \end{aligned} \tag{6.1}$$

Here $K_i(t_k;\vec{q}),\ L_i(t_k;\vec{q})\ (i = 1, 2)$ — solution of a direct problem corresponding to a set of parameters $\vec{q}$.

To minimize the target functional, the stochastic method of global optimization is used — the method of differential evolution [27].

First, we investigate the stability of the proposed method on model data, for this:

1) We set the initial set of parameters $\vec{q}_{ex}$.

2) For a given $\vec{q}_{ex}$ we solve a direct problem and store model data:

$$K_1(t_k) = F_{1k}, K_2(t_k) = F_{2k}, L_1(t_k) = F_{3k}, L_2(t_k) = F_{4k}, k = 1, 2, ..., N$$

3) We solve the inverse problem from model data and compare the reconstructed set of parameters $\vec{q}_*$ with the exact solution $\vec{q}_{ex}$.

It is worth noting that the set of parameters $\vec{q}_{ex}$ selected for testing is artificial, that is, it does not characterize any region. In this case, the search for parameter values $\vec{q}_*$ occurs within the following limits: d_k, λ_1, δ_1, λ_2, δ_2, d_l, a_1, b_1, c, a_2, $b_2 \in (-10, 10)$; φ_1, $\varphi_2 \in (0, 1)$. The set values in the parameter set $\vec{q}_{ex}$ and the resulting values for $\vec{q}_*$ are displayed in the Table 2. Also, Figure 2 and Figure 3 graphically display the results obtained using the parameter sets $\vec{q}_{ex}$ and $\vec{q}_*$. Based on the above, we can draw conclusions about the high ability of the model to recover parameter values and the uniqueness of the parameters. The use of a priori information about the inverse problem solution significantly reduces the number of iterations [14].

Table 2

	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
$\vec{q}_{ex}$	0.2	0.1	0.7	0.6	0.1	0.7	0.1	0.1	-1	-1	0.1	0.1	0.1
$\vec{q}_*$	0.197	0.06	0.6	0.6	0.09	0.74	0.14	0.17	-1.8	-0.25	0.07	0.17	0.09

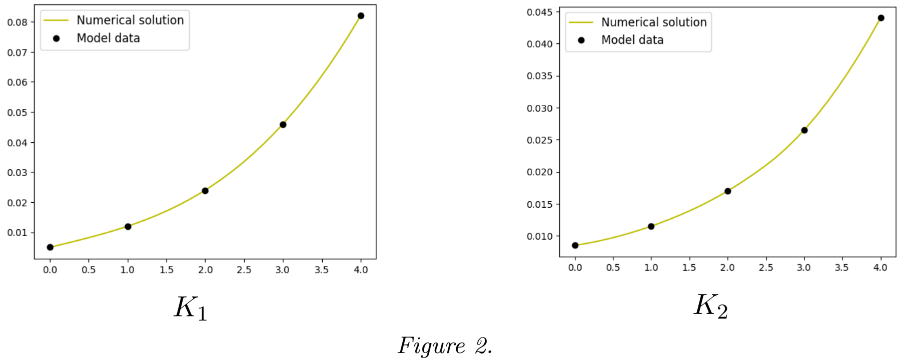
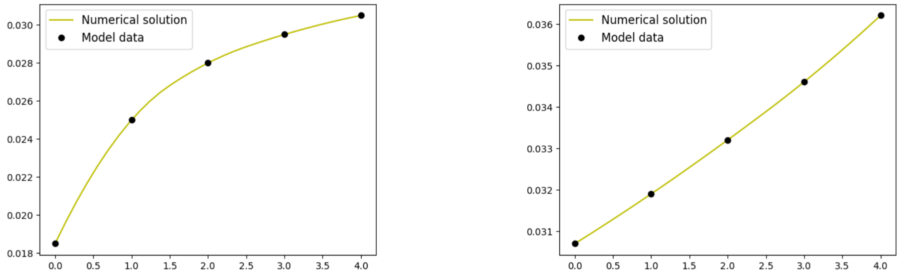


Figure 2.

As a result, data for the USA and the rest of the world from 1961 to 2019 were divided into 12 time intervals. Solutions of their own inverse problem were obtained at each of the intervals, the results are presented in Table 3.

It is known that each region has its own migration and demographic policy in different historical periods. For example, since 1980 there has been a trend towards an increase in the number of net migrants in the United States [23]. The decision to divide the time interval into intervals was made in order to study the degree of influence of the economy and the population on each other in different historical periods.



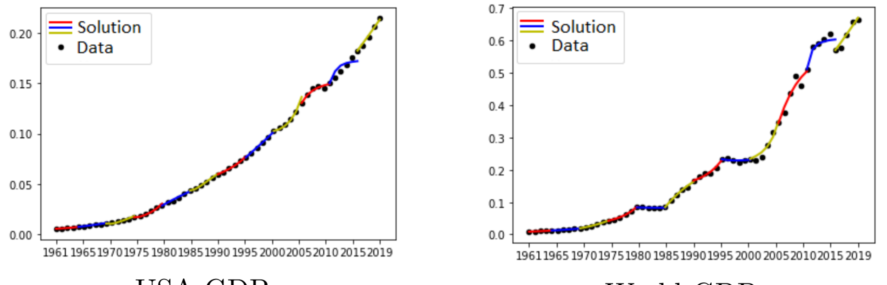
L_1 L_2

Figure 3.

Table 3

Parameters of the model: USA

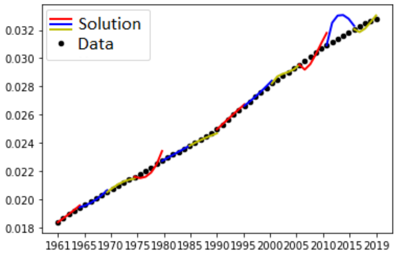
Years	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
1961–1965	1.34	0.53	0.71	0.65	0.02	0.5	2.98	-0.13	2.24	0.77	-1.31	-0.17	0.2
1965–1970	3.45	-0.54	0.79	0.98	0.42	0.84	2.91	-0.14	2.49	0.65	0.84	-0.13	0.06
1970–1975	2.038	-0.93	0.75	-0.72	-0.35	0.35	3.96	-0.16	2.71	2.55	-3.59	-0.107	-0.059
1975–1980	-0.93	0.72	0.81	3.38	0.29	0.48	1.23	-0.05	2.53	-1.16	1.4	-0.12	0.212
1980–1985	0.06	0.004	0.73	0.025	0.026	0.79	0	0.09	0.098	0.66	0.05	2.46	0.009
1985–1990	0.25	0.99	0.11	-0.67	1.04	0.67	-0.8	0.02	0.016	0.75	0.12	2.34	-0.001
1990–1995	0.07	0.25	0.67	-0.02	0.9	0.16	0.15	0.07	0.004	0.31	0.15	2.4	0.038
1995–2000	0.034	0.65	0.97	-0.56	1.28	0.16	-0.69	0.07	0.024	0.8	0.18	2.45	0.003
2000–2005	0.63	0.1	0.24	-0.84	-0.62	0.39	1.52	-0.134	2.91	0.53	0.68	1.27	-2.19
2005–2010	-0.007	3.68	0.46	-1.44	0.54	0.407	-0.59	-0.15	3.46	-0.24	-0.307	-2.06	3.059
2010–2015	3.078	-3.86	0.908	-4.4	2.39	0.003	-0.66	-0.11	3.82	-1.99	3.68	0.62	-1.11
2015–2019	1.55	1.21	0.34	-3.62	0.6	1	0.5	-0.13	2.53	-1.26	-2.08	-2.19	2.96



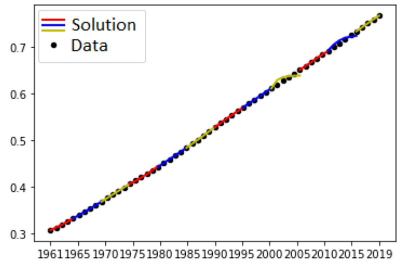
USA GDP World GDP

Figure 4.

The total error of the obtained piecewise solution (6.2) and the graphical interpretation of the obtained solution (Figure 4, Figure 5) allow us to



USA Population



World Population

Figure 5.

conclude that depending on the observed global population growth trend and the economic rhythm of the region, some parameters can be consider them constant, and some - time-dependent.

$$R(\vec{q}_k) = \sum_{k=1}^{12}\sum_{i=1}^{N}(K_1(t_{ki};\vec{q}_k) - F_{1ki})^2 + (K_2(t_{ki};\vec{q}_k) - F_{2ki})^2 + \\ +(L_1(t_{ki};\vec{q}_k) - F_{3ki})^2 + (L_2(t_{ki};\vec{q}_k) - F_{4ki})^2 = 0.0057 \qquad (6.2)$$

Next, in addition to the United States, we will consider countries such as China, India and Russia. For a more correct analysis, the time interval was divided into segments in accordance with the global economic crises that caused serious economic damage to the countries in question, namely:

– 1992–1998 (ruble crisis or the Russian flu [17]),

– 1999–2002 (2000s energy crisis),

– 2003–2009 (Subprime mortgage crisis [9]),

– 2010–2015 (Russian financial crisis [30]),

– 2016–2021. (COVID-19 recession).

At each of these intervals, for each country solved its own inverse problem, in which the first region was the country under study, and the second region was all the other countries of the world. As a result of repeated calculations, the parameter values for each country and the corresponding time period were determined, but they had different values, as a result of which their average values are displayed in the Table 4, Table 5, Table 6 and Table 7.

The greater volatility of the indicators presented in Tables 4–7 can be explained from the following substantive positions:

– A comparison of the calculated indicators presented in Tables 4-7 with real data provides a good meaningful explanation for the international flow of capital – with the important condition that capital in this study is understood not only capital in its natural form, but in the form of investment goods and production technologies (in kind, the amount of capital changes, as a rule, at a slow pace, while the flow of funds from industry to industry and from country to country is relatively easy – depending on the economic situation).

– Real economic data, on the basis of which calculations were made, are the results of stochastic, not deterministic processes – that is, they include both errors, interpreted in econometric studies as having the character of 1a normal distribution, and "outliers", i.e. one-time sharp deviations of a random nature.

– The results of solving the inverse problem, presented in Tables 4-7, determine the trend of the country's GDP; at the same time, as is known from the theory and practice of macroeconomics, there is a cyclical nature of this indicator, i.e. fluctuations in GDP of different duration around the trend. The crisis leads to a decline in production: at the same time, with a fixed value of the volume of capital and population in the country's economy, the level of use of production capacities decreases, and unemployment also increases (and this is often reflected not in the number of employed, but in the amount of working time); the stages of revival and recovery are accompanied by reverse processes. These processes in a somewhat simplified model – due to the complexity of calculating the coefficients in the inverse problem - are interpreted as an inflow and outflow of capital and labor, but not between countries, but within the country's economy (when production declines, they "go" into a temporarily unused reserve, and vice versa).

Table 4

Parameters of the model: China

Years	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
1992 – 1998	25.7	-34.9	0.4	33.7	-55.6	0.5	55.1	15.2	-34.4	-89.4	19.4	-31	93.1
1999 – 2002	-3.2	43.5	0.5	34.2	27	0.3	-50.8	-22.5	48.6	52.6	-23.5	-50.4	68.4
2003 – 2009	16.9	-33.5	0.3	-61.4	-58.5	0.5	67.7	22.8	-49.1	13.2	50.7	-7.4	30.3
2010 – 2015	7.1	-84.7	0.3	58.6	-30	0.1	29.4	-17.8	67.2	-6.9	12.5	-43	47.2
2016 – 2021	4	-34.7	0.1	3.1	-1.7	0.3	-4	-0.8	17.6	9.1	26.7	23.9	-45.6

Table 5

Parameters of the model: India

Years	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
1992 – 1998	-0.08	-32.8	0.5	-41.8	24.2	0.7	-14.4	-4.3	38.4	93.7	126.6	-8.04	-5.3
1999 – 2002	1.9	-54.7	0.6	-89.8	6.09	0.2	-24.8	21	-63.8	-85.4	25.4	-30.9	82.1
2003 – 2009	-4.3	88.1	0.6	37.5	-105.4	0.5	94.4	-4.5	-6.4	-44.2	-52.4	-63.8	119.2
2010 – 2015	9.5	-100.4	0.6	-75.3	-46	0.4	47.8	13.6	-4.1	49.2	70.1	10.3	-17.1
2016 – 2021	-5.3	14.5	0.4	5.05	17.9	0.4	-25.1	-15.9	45.1	1.3	-10.5	3.2	-23.3

Table 6

Parameters of the model: Russia

Years	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
1992 – 1998	0.7	-6.4	0.5	-41.4	34.4	0.7	-33.2	2.7	-95.9	12.5	21.5	-35.6	65.4
1999 – 2002	1.05	-68.2	0.5	-9.1	-6.5	0.3	-43.5	-1.3	37	-19.3	-65.5	-8.3	9.5
2003 – 2009	0.75	-42.2	0.2	15.9	18.7	0.7	-12.7	-2.6	72.8	-5.7	-78.4	-47	69
2010 – 2015	4.6	-130.8	0.3	-48.7	-39.9	0.3	40.2	-1.2	7	-29.2	-75.2	-38	53.6
2016 – 2021	-3.1	69.1	0.7	93.3	15.3	0.4	-20.4	-4.2	68.1	27.1	-167.2	-78.4	97.9

Table 7

Parameters of the model: USA

Years	d_k	λ_1	φ_1	δ_1	λ_2	φ_2	δ_2	d_l	a_1	b_1	c	a_2	b_2
1992 – 1998	-19.8	47.6	0.4	17.4	-21.4	0.6	5.9	-0.3	2.1	-31.8	-38.6	-22	38.5
1999 – 2002	99.3	-62.9	0.4	-80.8	76.3	0.2	-77.2	4.9	-90	33.1	99.3	-50	87.5
2003 – 2009	-74.3	33.8	0.4	87.3	-129.4	0.38	107	-2.7	61.1	-26.4	24.2	-24.3	32.4
2010 – 2015	-97.7	16.7	0.4	238	-297.2	0.03	270	13.5	-211.1	12.2	169.4	-70.9	115.9
2016 – 2021	28.3	-47.9	0.4	-37.2	-3.8	0.5	24.8	81.2	20.9	-24.7	-39.9	-15.4	-16.4

7. Conclusion

The article defines the parameters of a mathematical model of the relationship between population growth and changes in the level of capital in two regions of the world. Based on additional statistical data, the values of the parameters of the model under study were determined in the case of the United States and the rest of the world for the time period from 1961 to

2019. The intervals of economic crises were considered, in the conditions of which parameters were determined for countries such as Russia, the USA, India and China. The problem of determining parameters is reduced to the problem of minimizing the target functional and is solved by the method of differential evolution. The results of numerical calculations based on model data have shown the effectiveness of the methods used to solve the problem. The mathematical model extends to the largest economic regions of the world, if there are the data, it can be expanded the mathematical model to take into account all countries.

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