Серия «Математика»
2023. Т. 46. С. 130—144

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 004.827
MSC 68T27, 68T30
DOI https://doi.org/10.26516/1997-7670.2023.46.130

On the Local Coordination of Fuzzy Valuations

Gulnara E. Yakhyaeva[1]✉

[1] Novosibirsk State Technical University, Novosibirsk, Russian Federation
✉ gul_nara@mail.ru

Abstract. The paper is devoted to the model-theoretic formalization of the semantic model of the object domain. The article discusses the concept of a fuzzy model, which is a model where the truth function exhibits properties of a fuzzy measure. We demonstrate that a fuzzy model is a generalization of the concept of fuzzification of a precedent (semantic) model to include a countable number of precedents.

In practice, it is common to have partial expert knowledge about the set of events in the object domain, making it difficult to immediately describe the fuzzy model. Additionally, since expert valuations are subjective, they may be incorrect and inconsistent with any fuzzy model. In the article, we introduce the concepts of coordinated and locally coordinated valuation of a set of sentences, and provide proofs for interval theorems and an analogue of the compactness theorem.

Keywords: fuzzy model, theory of fuzzy models, fuzzy measure, coordinated valuation, locally coordinated valuation

For citation: Yakhyaeva G. E. On the Local Coordination of Fuzzy Valuations. *The Bulletin of Irkutsk State University. Series Mathematics*, 2023, vol. 46, pp. 130–144. https://doi.org/10.26516/1997-7670.2023.46.130

Научная статья

О локальной согласованности нечетких означиваний

Г. Э. Яхъяева[1]✉

[1] Новосибирский государственный технический университет, Новосибирск, Российская Федерация
✉ gul_nara@mail.ru

Аннотация. Исследуется теоретико-модельная формализация семантической модели предметной области. Рассматривается понятие нечеткой модели, т. е. модели, на которой истинностная функция обладает свойствами нечеткой меры. Показывается, что нечеткая модель является обобщением понятия фазификации прецедентной (семантической) модели на случай счетного числа прецедентов. Вводятся понятия согласованного и локально согласованного означивания множества предложений, доказываются теоремы об интервалах и аналог теоремы компактности.

Ключевые слова: нечеткая модель, теория нечетких моделей, нечеткая мера, согласованное означивание, локально согласованное означивание

Ссылка для цитирования: Yakhyaeva G. E. On the Local Coordination of Fuzzy Valuations // Известия Иркутского государственного университета. Серия Математика. 2023. Т. 46. С. 130–144.
https://doi.org/10.26516/1997-7670.2023.46.130

1. Introduction

Artificial intelligence (AI) plays a crucial role in various fields of activity that have adopted new information technologies. The importance of intelligent machines that can learn, reason, and adapt is widely recognized. These capabilities enable AI methods to achieve remarkable performance and solve increasingly complex computational problems [12]. However, the complexity of AI-based systems has reached a point where human intervention is almost unnecessary for their development and deployment. This raises concerns, especially when these systems make decisions that impact people's lives, such as in the field of medicine [7].

As the use of black box models for critical predictions becomes more prevalent, there is a growing demand for transparency in AI systems. The risk lies in using solutions that lack justification, legality, or detailed explanations of their behavior [2].

It is now understood that focusing solely on performance can lead to increasingly opaque systems. There is a trade-off between model performance and transparency. It is essential to interpret the system to understand its mechanisms, generated predictions, visualize its rules, and identify potential pitfalls that could undermine the model.

One approach to developing trustworthy AI is to employ logical-semantic methods, particularly semantic models. A four-level semantic model of knowledge representation has been proposed in previous work [8]. The modeling process of any object domain starts with describing the concepts of the subject area at the first level of the semantic model [10]. Selecting an appropriate set of concepts for domain description is a crucial step in conceptual modeling. It should be easily understood by domain experts and ideally accessible to a broader group of specialists for describing domain constituents [5; 6].

However, the further description can continue by starting with the formalization of any of the remaining levels. For example, when describing mathematical object domains, it is customary to start from the second level of formalization, which involves describing the set of axioms for a given object domain.

Another empirical approach is to form a database of precedents in the object domain and statistically process this database [3]. The analysis of formal concepts methodology is striking example of such an approach to knowledge representation [4]. Knowledge is formalized in the form of a formal context and then subjected to various statistical processing.

From a model-theoretic perspective, the empirical approach to constructing a semantic model is described using the language of precedent and Boolean valued models [13; 14]. This paper continues research in this direction and introduces the concept of a fuzzy model, where the truth function has the properties of a fuzzy measure [1]. It is shown that a fuzzy model is a generalization of the concept of fuzzification of a case model to a countable number of cases.

In practice, we often have partial expert knowledge about the set of events in the object domain, making it challenging to immediately describe the fuzzy model. Additionally, since expert valuations are subjective, they may be incorrect or inconsistent with any fuzzy model. The article introduces the concepts of coordinated and locally coordinated valuation of a set of sentences, proving interval theorems and an analogue of the compactness theorem.

Let us introduce the main notation. We will consider the fuzzy algebraic system $\mathfrak{A}$ of the signature σ with base set A. In this paper we consider signatures that do not contain function symbols. We denote by $F(\sigma)$ the set of formulas of signature σ, and by $S(\sigma)$ the set of sentences of the same signature.

For convenience, in order to talk not about the truth of arbitrary formulas in $\mathfrak{A}$, but only about the truth of sentences, we extend the signature σ with new constants. We will use signature $\sigma_A \rightleftharpoons \sigma \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\} \cap \sigma = \emptyset$. Moreover, $c_a^{\mathfrak{A}} = a$ on the model $\mathfrak{A}$.

2. Submodels of Fuzzy Models

When analyzing complex systems under uncertainty, methods of probability theory and mathematical statistics are widely used. These methods involve a probabilistic interpretation of the processed data and the statistical conclusions derived from them. Recently, there has been an increasing need for new approaches to the mathematical description of information characterized by a high level of uncertainty. We propose one of the possible ways to solve this problem.

Definition 1. *[9] A triple $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$ is called a **fuzzy model** if the valuation $\mu : S(\sigma_A) \to [0,1]$ is a probability measure defined on the Lidenbaum-Tarski algebra $S(\sigma_A)_{/\sim}$, i.e. the following conditions are satisfied:*

(A1) If the sentence $\varphi \in S(\sigma_A)$ is identically true (identically false), then $\mu(\varphi) = 1 \quad \big(\mu(\varphi) = 0\big)$.

(A2) For any countable sequence of sentences $\{\varphi_i \in S(\sigma_A) | i \in N\}$ such that $\mu(\varphi_i \& \varphi_j) = 0$ for any $i, j \in N$ we have

$$\mu(\bigvee_{i\in N} \varphi_i) = \sum_{i\in N} \mu(\varphi_i).$$

(A3) For any sentences $\varphi, \psi \in S(\sigma_A)$ we have

$$\varphi \sim \psi \;\Rightarrow\; \mu(\varphi) = \mu(\psi).$$

(A4) For any formula $\varphi(x) \in F(\sigma)$ with one free variable we have

$$\mu(\forall x \varphi(x)) = \mu(\bigwedge_{a\in A} \varphi(a)), \quad \mu(\exists x \varphi(x)) = \mu(\bigvee_{a\in A} \varphi(a)).$$

Note that the property (A3) guarantees that all sentences that are true on all fuzzy models of the signature σ_A are also true on the classical theory of models of the signature. This means that fuzzy models can be considered as a natural extension of the classical theory of models.

We introduce an order relation on the set of sentences $S(\sigma_A)$ in a standard way. For sentences $\varphi, \psi \in S(\sigma_A)$ we denote $\varphi \preceq \psi$ if $\varphi \sim (\varphi \& \psi)$.

Lemma 1. *Consider the fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$. For any sentences $\varphi, \psi \in S(\sigma_A)$ the property **monotonicity** is satisfied, i.e.*

$$\varphi \preceq \psi \;\Rightarrow\; \mu(\varphi) \le \mu(\psi).$$

Proof. For any sentences $\varphi, \psi \in S(\sigma_A)$ we have

$$\psi \sim \big(\varphi \& \psi\big) \vee \big(\neg\varphi \& \psi\big).$$

Then from (A2) and (A3) it follows that

$$\mu(\psi) = \mu\Big(\big(\varphi \& \psi\big) \vee \big(\neg\varphi \& \psi\big)\Big) = \mu\big(\varphi \& \psi\big) + \mu\big(\neg\varphi \& \psi\big).$$

Thus, from $\varphi \sin \varphi \& \psi$ the assertion of the Lemma follows. □

Lemma 2. *Consider a countable fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$, i.e. such that $A = \{a_i | i \in N\}$. Then for any formula with one free variable $\varphi(x) \in F(\sigma)$ we have*

$$\mu\Big(\forall x \varphi(x)\Big) = \lim_{n\to\infty} \mu\Big(\bigwedge_{i=1}^{n} \varphi(a_i)\Big) \;\; \text{and} \;\; \mu\Big(\exists x \varphi(x)\Big) = \lim_{n\to\infty} \mu\Big(\bigvee_{i=1}^{n} \varphi(a_i)\Big).$$

Proof. Let us show the proof for the existence quantifier. The proof for the universal quantifier will be similar.

Lemma 1 implies that

$$\mu\big(\varphi(a_1)\big) \leq \mu\big(\varphi(a_1) \vee \varphi(a_2)\big) \leq \mu\big(\varphi(a_1) \vee \varphi(a_2) \vee \varphi(a_3)\big) \leq ...$$

Since this sequence is bounded, it has an limit, i.e.

$$\mu\Big(\bigvee_{a\in A} \varphi(a)\Big) = \lim_{n\to\infty} \mu\Big(\bigvee_{i=1}^{n} \varphi(a_i)\Big).$$

Thus, property (A5) implies the assertion of the Lemma. □

Constructing the theory of fuzzy models as a conservative extension of the classical theory of models, we have to abandon the principle of functionality of the logic under consideration. This means, in particular, that on any fuzzy model, the truth values of quantifier-free formulas cannot be unambiguously derived from the truth values of atomic formulas. In this regard, by introducing the concept of a submodel, we are forced to explicitly specify the coincidence of truth values on all quantifier-free formulas.

Definition 2. *Let $\mathfrak{A}_{\mu_1} = \langle A_1, \sigma, \mu_1\rangle$ and $\mathfrak{A}_{\mu_2} = \langle A_2, \sigma, \mu_2\rangle$ be fuzzy models of identic signature σ. We will say that the fuzzy model $\mathfrak{A}_{\mu_1}$ is a* ***submodel*** *of the fuzzy model $\mathfrak{A}_{\mu_2}$ (and will denoted $\mathfrak{A}_{\mu_1} \subseteq \mathfrak{A}_{\mu_2}$), if the following conditions are satisfied:*

1) $A_1 \subseteq A_2$;

2) For any constant $c \in \sigma$ we have $c^{\mathfrak{A}_{\mu_1}} = c^{\mathfrak{A}_{\mu_2}}$.

3) For any quantifier-free sentence $\varphi \in S(\sigma_A)$ we have $\mu_1(\varphi) = \mu_2(\varphi)$.

Proposition 1. *Let $\mathfrak{A}_{\mu_1} \subseteq \mathfrak{A}_{\mu_1}$ and $\varphi(x_1, \dots, x_n)$ be quantifier-free formula. Then*

$$\mu_1(\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)) \geq \mu_2(\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n));$$

$$\mu_1(\exists x_1 \dots \exists x_n \varphi(x_1, \dots, x_n)) \leq \mu_2(\exists x_1 \dots \exists x_n \varphi(x_1, \dots, x_n)).$$

The proof of the Proposition follows directly from Lemma 2.

Just as in classical model theory, we introduce the concept of an elementary submodel.

Definition 3. *Let $\mathfrak{A}_{\mu_1} = \langle A_1, \sigma, \mu_1\rangle$ and $\mathfrak{A}_{\mu_2} = \langle A_2, \sigma, \mu_2\rangle$ be fuzzy models of the same signature σ. We will say that the fuzzy model $\mathfrak{A}_{\mu_1}$ is an* ***elementary submodel*** *of the fuzzy model $\mathfrak{A}_{\mu_2}$ (and will denoted $\mathfrak{A}_{\mu_1} \leq \mathfrak{A}_{\mu_2}$), if the following conditions are satisfied:*

1) $A_1 \subseteq A_2$;

2) For any constant $c \in \sigma$ we have $c^{\mathfrak{A}_{\mu_1}} = c^{\mathfrak{A}_{\mu_2}}$.

3) For any sentence $\varphi \in S(\sigma_A)$ we have $\mu_1(\varphi) = \mu_2(\varphi)$.

Lemma 3. *For any fuzzy model* $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$ *and for any set* A' *such that* $A \subseteq A'$ *it is possible to construct a fuzzy model* $\mathfrak{A}_{\mu'} = \langle A', \sigma, \mu'\rangle$ *such that* $\mathfrak{A}_\mu \leq \mathfrak{A}_{\mu'}$.

Proof. Let us arbitrarily choose the element $a_0 \in A$. Let us define the mapping $h : A' \to A$ as follows:

$$h(a) = \begin{cases} a, & a \in A; \\ a_0, & a \in A' \setminus A. \end{cases}$$

For any formula $\varphi(x_1, ..., x_n) \in F(\sigma)$ and for any tuple of elements

$$(a_1, ..., a_n) \in (A')^n$$

we define

$$\mu'\big(\varphi(a_1, ..., a_n)\big) = \mu\big(\varphi(h(a_1), ..., h(a_n))\big).$$

Obviously, the mapping μ' will have properties (A1)-(A4) from Definition 1, i.e. will define a fuzzy model$\mathfrak{A}_{\mu'}$. □

3. Coordinated Significations

Defining a fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$ as a formal representation of the object domain, it is necessary to possess knowledge regarding the truth values of all quantifier-free formulas in the model $\mathfrak{A}_\mu$ with signature σ. However, it is possible that the expert (or even a group of experts) may not possess such comprehensive knowledge. In this context, the task arises of utilizing the set of subjective valuations of events in the object domain obtained from experts to reconstruct as much complete knowledge as possible about the object domain [16].

Thus, at the input, we have a certain set of sentences S and the valuation $\eta_S : S \to [0, 1]$. We will denote by σ_S the signature of the set of sentences S. As we agreed above, we will consider the case when the signature σ_S does not contain any functional symbols.

The primary task is to check the logical correctness of the valuation η_S received from experts. To do this, we introduce the concept of coordinacy of the valuation η_S.

Definition 4. *Consider the set of sentences* S *and the mapping* $\eta_S : S \to [0, 1]$. *The valuation* η_S ***is coordinated with the fuzzy model*** $\mathfrak{A}_\mu = \langle A, \sigma_S, \mu\rangle$ *if for any sentence* $\varphi \in S$ *it holds equality* $\eta_S(\varphi) = \mu(\varphi)$. *The valuation will be called* ***coordinated*** *if there is a fuzzy model of signature* σ_S *with which the given valuation is coordinated.*

Note that the concepts of coordination can be considered as a generalization of the concept of satisfiability from classical model theory. We will call a valuation η_S **trivial** if it is a mapping into the singleton set $\{1\}$. Clearly, a set of propositions is satisfiable (in classical model theory) if and only if its trivial valuation is coordinated.

Next we present some properties of coordinated valuations, the proof of which can be found in the papers [11; 16]

Proposition 2. *For any set of sentences S there is always a coordinated valuation.*

Proposition 3. *If the set S contains at least one satisfiable-falsifiable sentence, then the set S has a non-trivial coordinated valuation.*

Proposition 4. *Let S be a set of atomic sentences, then any valuation $\eta_S : S \to [0,1]$ is coordinated.*

Let us denote by $K(\eta_S)$ the class of all fuzzy models coordinated with the valuation η_S. According to Lemma 3 and to simplify the presentation, we will assume that all fuzzy models from the class $K(\eta_S)$ are defined on the same basic set, i.e. we will assume that

$$K(\eta_S) = \{\mathfrak{A}_\mu = \langle A, \sigma_S, \mu\rangle | \forall\varphi \in S : \mu(\varphi) = \eta_S(\varphi)\}.$$

Let $\psi \in S(\sigma_S)$. Let us denote

$$\xi_{\eta_S}(\psi) = \{\mu(\psi) | \mathfrak{A}_\mu \in K(\eta_S)\}.$$

The set $\xi_{\eta_S}(\psi)$ will be called the **generalized truth value** of the sentence ψ on the class of fuzzy models $K(\eta_S)$. Obviously, if $\psi \in S$, then $\xi_{\eta_S}(\psi)$ is a singleton set, i.e. $\xi_{\eta_S}(\psi) = \{\eta_S(\psi)\}$.

Definition 5. *Consider the sets of sentences S_1 and S_2 such that $S_1 \subseteq S_2$. We will say that the valuation η_{S_2} is an* ***extension*** *of the valuation η_{S_1} if for any sentence $\varphi \in S_1$ we have $\eta_{S_1}(\varphi) = \eta_{S_2}(\varphi)$.*

In this case, we will call the valuation η_{S_1} a ***narrowing*** *of the valuation η_{S_2}.*

Lemma 4. *Consider the sets of sentences S_1 and S_2 such that $S_1 \subseteq S_2$. Then, if the valuation η_{S_2} is an extension of the valuation η_{S_1}, then for any sentence ψ of signature σ_{S_1} we have*

$$\xi_{\eta_{S_2}}(\psi) \subseteq \xi_{\eta_{S_1}}(\psi).$$

Proof. Obviously, if the valuation η_{S_2} is an extension of the valuation η_{S_1}, then $K(\eta_{S_2}) \subseteq K(\eta_{S_1})$. And the statement of the Lemma directly follows from this. □

Theorem 1. *Consider the set of sentences S and the coordinated valuation $\eta_S : S \to [0,1]$. For any sentence $\psi \in S(\sigma_S)$ the generalized truth value $\xi_{\eta_S}(\psi)$ is an interval.*

Proof. Let $\mathfrak{A}_{\mu_1} = \langle A, \sigma_S, \mu_1\rangle, \mathfrak{A}_{\mu_2} = \langle A, \sigma_S, \mu_2\rangle \in K(\eta_S)$. Consider the sentence ψ of the signature σ_S such that $\psi \notin S$. Without loss of generality, we will assume that $\mu_1(\psi) < \mu_2(\psi)$. Let us show that for any $\alpha \in [\mu_1(\psi); \mu_2(\psi)]$ there is such a fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma_S, \mu\rangle \in K(\eta_S)$ that $\mu(\psi) = \alpha$.

Let $k_1 = \frac{\alpha - \mu_1(\psi)}{\mu_2(\psi) - \mu_1(\psi)}$ and $k_2 = \frac{\mu_2(\psi) - \alpha}{\mu_2(\psi) - \mu_1(\psi)}$. Let's define the mapping $\mu = k_2 \cdot \mu_1 + k_1 \cdot \mu_2$.

It is not difficult to verify that $\mu(\psi) = \alpha$. Let us now show that the mapping μ has properties (A1)-(A4) from Definition 1.

(A1). Let $\varphi \in S(\sigma_A)$ be identically true. Then $\mu_1(\varphi) = \mu_2(\varphi) = 1$. Therefore, $\mu(\varphi) = k_2 + k_1 = 1$.

(A2). Consider the sequence of sentences $\{\varphi_i \in S(\sigma_A) | i \in \mathsf{N}\}$ such that $\mu(\varphi_i \& \varphi_j) = 0$ for any $i, j \in \mathsf{N}$. Since $k_1 \geq 0$ and $k_2 \geq 0$ then $\mu_1(\varphi_i \& \varphi_j) = \mu_2(\varphi_i \& \varphi_j) = 0$ Then

$$\mu\left(\bigvee_{i\in\mathsf{N}} \varphi_i\right) = k_2 \cdot \mu_1\left(\bigvee_{i\in\mathsf{N}} \varphi_i\right) + k_1 \cdot \mu_2\left(\bigvee_{i\in\mathsf{N}} \varphi_i\right) =$$

$$= k_2 \cdot \sum_{i\in\mathsf{N}} \mu_1(\varphi_i) + k_1 \cdot \sum_{i\in\mathsf{N}} \mu_2(\varphi_i) =$$

$$= \sum_{i\in\mathsf{N}} \left(k_2 \cdot \mu_1(\varphi_i) + k_1 \cdot \mu_2(\varphi_i)\right) = \sum_{i\in\mathsf{N}} \mu(\varphi_i).$$

(A3). Let the sentences $\varphi, \psi \in S(\sigma_A)$ be equivalent, i.e. $\varphi \sim \psi$. Then

$$\mu(\varphi) = k_2 \cdot \mu_1(\varphi) + k_1 \cdot \mu_2(\varphi) = k_2 \cdot \mu_1(\psi) + k_1 \cdot \mu_2(\psi) = \mu(\psi).$$

(A4) Proved by analogy with (A2).

Consequently, the valuation μ defines a fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma_S, \mu\rangle$.

Let us now consider the sentence $\varphi \in S$. We have

$$\mu(\varphi) = k_2 \cdot \mu_1(\varphi) + k_1 \cdot \mu_2(\varphi) = k_2 \cdot \eta_S(\varphi) + k_1 \cdot \eta_S(\varphi) = \eta_S(\varphi).$$

Thus, we get that $\mathfrak{A}_\mu = \langle A, \sigma_S, \mu\rangle \in K(\eta_S)$. □

4. Relationship Between Fuzzy Models and Classes of Classical Models

Consider the fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$. Let us denote by $\mathbb{K}$ the class of all (classical) models of signature σ defined on the set A, i.e.

$$\mathbb{K} = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, \sigma\rangle\}.$$

For any sentence $\varphi \in S(\sigma_A)$ we define

$$\mathbb{K}(\varphi) = \{\mathfrak{A} \in \mathbb{K} \mid \mathfrak{A} \models \varphi\}.$$

Definition 6. *Mapping $\nu : \mathbb{K} \cup \{\emptyset\} \to [0; 1]$ we will call* ***distribution*** *defined on the class of models $\mathbb{K}$ if the following properties are satisfied:*
(B1) $\nu(\emptyset) = 0$;
(B2) $\sum_{\mathfrak{A}\in\mathbb{K}} \nu(\mathfrak{A}) = 1$.

Proposition 5. *Let on the class of models $\mathbb{K}$ be given the distribution $\nu : \mathbb{K} \cup \{\emptyset\} \to [0; 1]$. Then the mapping $\mu : S(\sigma_A) \to [0; 1]$ such that for any sentence $\varphi \in S(\sigma_A)$ it holds*

$$\mu(\varphi) = \sum_{\mathfrak{A}\in\mathbb{K}(\varphi)} \nu(\mathfrak{A}),$$

is the valuation of some fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$.
We will say that the distribution ν ***generates*** *a fuzzy model $\mathfrak{A}_\mu$.*

Proof. To prove the Proposition, it is enough to check that properties (A1)-(A4) from Definition 1 are satisfied. □

Proposition 6. *For any fuzzy model $\mathfrak{A}_\mu = \langle A, \sigma, \mu\rangle$ there is a generating distribution.*

Proof. Consider the set $S_a(\sigma_A)$ of all atomic sentences of signature σ_A. Let $S_a(\sigma_A) = \{\varphi_i \mid i \in I\}$.

For each model $\mathfrak{A} \in \mathbb{K}$ we define an atomic diagram $D(\mathfrak{A})$ defined as follows:

for any $i \in I$ if $\mathfrak{A} \models \varphi_i$ then $\varphi_i \in D(\mathfrak{A})$, otherwise $\neg\varphi_i \in D(\mathfrak{A})$.

Thus, the atomic diagram of any model consists of all atomic sentences of signature σ_A, taken with or without negation. We will denote $D(\mathfrak{A}) = \{\varphi_i^{\mathfrak{A}} \mid i \in I\}$.

Let us define the mapping $\nu : \mathbb{K} \cup \{\emptyset\} \to [0; 1]$ as follows:

$$\nu(\emptyset) = 0; \qquad \nu(\mathfrak{A}) = \mu\Big(\bigwedge_{i\in I} \varphi_i^{\mathfrak{A}}\Big).$$

Note that if the set $S_a(\sigma_A)$ is countable, then, reasoning in the same way as in Lemma 2 we obtain

$$\nu(\mathfrak{A}) = \lim_{n\to\infty} \mu\Big(\bigwedge_{i=1}^{n} \varphi_i^{\mathfrak{A}}\Big).$$

Let us show that the mapping ν defined in this way has property (B2) from Definition 6. Let us give a proof for the case when the set $S_a(\sigma_A)$ is countable.

Indeed, from properties (A1), (A2) and (A3) of the Definition 1 it follows

$$\sum_{\mathfrak{A}\in\mathbb{K}} \nu(\mathfrak{A}) = \sum_{\mathfrak{A}\in\mathbb{K}} \mu\Big(\bigwedge_{i\in I} \varphi_i^{\mathfrak{A}}\Big) = \sum_{\mathfrak{A}\in\mathbb{K}} \lim_{n\to\infty} \mu\Big(\bigwedge_{i=1}^{n} \varphi_i^{\mathfrak{A}}\Big) =$$
$$= \lim_{n\to\infty}\left(\sum_{\mathfrak{A}\in\mathbb{K}} \mu\Big(\bigwedge_{i=1}^{n} \varphi_i^{\mathfrak{A}}\Big)\right) = \lim_{n\to\infty} \mu\left(\bigvee_{\mathfrak{A}\in\mathbb{K}} \Big(\bigwedge_{i=1}^{n} \varphi_i^{\mathfrak{A}}\Big)\right) = \lim_{n\to\infty} \mu(1) = 1.$$

□

From Propositions 5 and 6 it follows that the concept of a fuzzy model can be considered as a generalization of the concept of fuzzification of a precedent model to the case of a countable number of precedents. (Definitions of precedent models and their fuzzification can be found, for example, in [9;15]. In these works, only finite sets of precedents of the object domain were studied.)

Theorem 2. *Consider the set of sentences S and the coordinated valuation $\eta_S : S \to [0,1]$. Then for any sentence $\psi \in S(\sigma_S)$ the generalized truth value $\xi_{\eta_S}(\psi)$ is a segment.*

Proof. In Theorem 1 we showed that for any sentence $\psi \in S(\sigma_S)$ the generalized truth value $\xi_{\eta_S}(\psi)$ is an interval. Now we will show that this interval contains its limit points.

Let us consider the case when the set of sentences S is finite, i.e. $S = \{\varphi_1, ..., \varphi_n\}$. By the conditions of the Theorem, the value of η_S is coordinated.

Let's index the class $\mathbb{K}$, that is, assume that $\mathbb{K} = \{\mathfrak{A}_i \mid i \in I\}$. Let us introduce into consideration the set of variables $X = \{x_i \mid i \in I\}$ and compose a system of linear equations:

$$\begin{cases} \sum_{i\in I} x_i = 1, \\ \sum_{i\in I} k_{ij} x_i = \eta_S(\varphi_j), (j = 1, ..., n), \end{cases} \tag{4.1}$$

where

$$k_{ij} = \begin{cases} 1, & \mathfrak{A}_i \models \varphi_j; \\ 0, & \mathfrak{A}_i \not\models \varphi_j. \end{cases}$$

Since the valuation η_S is coordinated, there is at least one fuzzy model $\mathfrak{A}_\mu$ coordinated with the valuation η_S. Therefore, according to Proposition 6, there is at least one distribution ν that is a solution to the system (4.1) under the constraints

$$0 \leq x_i \leq 1, i \in I. \tag{4.2}$$

Thus, the problem of finding limit points of the set $\xi_{\eta_S}(\psi)$ is reduced to the linear programming problem of finding the optimal solution:

$$\Psi = \sum_{i \in I} p_i x_i \to \max(\min),$$

where

$$p_i = \begin{cases} 1, & \mathfrak{A}_i \models \psi; \\ 0, & \mathfrak{A}_i \not\models \psi; \end{cases}$$

under restrictions (4.1) and (4.2).

And since the objective function Ψ is limited by the segment $[0, 1]$, the optimal solution always exists.

Thus, we have shown that in the case when the set of sentences S is finite, then for any sentence $\psi \in S(\sigma_S)$ the generalized truth value $\xi_{\eta_S}(\psi)$ is an segment (i.e. closed interval).

Let us now show that the Theorem will also be true in the case when the set S is countable.

Let $S = \{\varphi_i \mid i \in \mathbb{N}\}$. Let's define

$$S_0 = \{\varphi_0\},$$

$$S_i = S_{i-1} \cup \{\varphi_i\}, \quad (i \in \mathbb{N}).$$

It is obvious that all S_i are finite, and also $S_0 \subset S_1 \subset S_2 \subset ....$

Therefore, by Lemma 4, we get

$$\xi_{\eta_{S_0}}(\psi) \supseteq \xi_{\eta_{S_1}}(\psi) \supseteq \xi_{\eta_{S_2}}(\psi) \supseteq ....$$

Thus we obtain a sequence of nested segments. And since all the segments are closed, then (according to the Cauchy-Cantor principle) there is their limit, i.e.

$$\xi_{\eta_S}(\psi) = \lim_{n \to \infty} \xi_{\eta_{S_i}}(\psi),$$

which is also a segment.

□

4.1. Compactness Theorem

We call a valuation $\eta_S : S \to [0, 1]$ **locally coordinated** if any of its restrictions to a finite subset is coordinated.

Theorem 3. *Any valuation of a countable set of sentences S is coordinated if and only if it is locally coordinated.*

Proof. Consider the set of sentences S and the valuation η_S. Obviously, if the valuation of η_S is coordinated, then it is locally coordinated.

Let the valuation η_S be locally coordinated. Since the set S is countable, it is possible to construct a countable sequence of finite sets $S_1, S_2, , ...$ such that

$$S_1 \subset S_2 \subset ... \quad and \quad S = \bigcup_{i \in \mathsf{N}} S_i.$$

For each $i \in \mathsf{N}$ we define the valuation η_{S_i} as the restriction of the valuation η_S to the finite set S_i. Since the valuation of η_S is locally coordinated, then for each $i \in \mathsf{N}$ we have $K(\eta_{S_i}) \neq \emptyset$.

Therefore, by Theorem 2, for any sentence ψ of signature σ_S, the generalized truth value $\xi_{\eta_{S_i}}(\psi)$ is a segment. And by Lemma 4 the sequence $\xi_{\eta_{S_1}}(\psi), \xi_{\eta_{S_2}}(\psi), ...$ forms a sequence of nested segments.

Thus, the set $K(\eta_S)$ is also not empty, i.e. there is at least one fuzzy model with which the valuation η_S is coordinated. □

5. Conclusion

This work is devoted to the study of model-theoretic properties of fuzzy models. The concept of a fuzzy model is a conservative extension of the concept of a model in classical model theory. In other words, the set of formulas that are true on all fuzzy models coincides with the set of identically true formulas. This is the main advantage of the developed approach over fuzzy logic in the style of multi-valued logics, such as J. Lukasevich or L. Zade. This approach allows us to work with incomplete and inaccurate knowledge about the object domain while still complying with all the laws of Aristotelian logic. This is achieved by abandoning the principle of functionality of the logic in question, meaning that the truth values of complex formulas are functionally independent of the truth values of atomic formulas. This assumption inevitably complicates the algorithms for calculating the truth values of various sentences. However, instead of using point truth values, we can consider interval truth values (Theorem 1 and Theorem 2), which will enable us to utilize the full power of interval analysis in further research. In this article, this approach allowed us to prove an analogue of the compactness theorem (Theorem 3).

References

1. Beliakov G., James S., Wu J.-Z. *Discrete fuzzy measures: computational aspects.* Springer Cham Publ., 2020, 260 p. https://doi.org/10.1007/978-3-030-15305-2

2. Castelvecchi D. Can we open the black box of AI? *Nature News*, 2016, vol. 538, no. 7623, pp. 21-23. https://doi.org/10.1038/538020a
3. Hajek A. Arguments For—Or Against—Probabilism? *British Journal for the Philosophy of Science*, 2009, vol. 59, no. 4, pp. 229–251. https://doi.org/10.1093/bjps/axn045
4. Kuznetsov S.O., Poelmans J. Knowledge representation and processing with formal concept analysis. *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery*, 2013, vol. 3, no. 3, pp. 200–215. https://doi.org/10.1002/widm.1088
5. Mantsivoda A.V., Ponomaryov D.K. A Formalization of Document Models with Semantic Modelling. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 27, pp. 36–54. https://doi.org/10.26516/1997-7670.2019.27.36
6. Mantsivoda A.V., Ponomaryov D.K. Towards Semantic Document Modelling of Business Processes. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 29, pp. 52–67. https://doi.org/10.26516/1997-7670.2019.29.52
7. Miller T. Explanation in artificial intelligence: Insights from the social sciences. *Artif. Intell.*, 2019, vol. 267, pp. 1–38. https://doi.org/10.1016/j.artint.2018.07.007
8. Naydanov C., Palchunov D., Sazonova P. Development of automated methods for the critical condition risk prevention, based on the analysis of the knowledge obtained from patient medical records. *Proceedings International Conference on Biomedical Engineering and Computational Technologies, SIBIRCON 2015*, Novosibirsk, 2015, pp. 33–38.
9. Palchunov D., Yakhyaeva G. Fuzzy logics and fuzzy model theory. *Algebra and Logic*, 2015, vol. 54, no. 1, pp. 74–80. https://doi.org/10.1007/s10469-015-9326-9
10. Palchunov D.E., Tishkovsky D.E., Tishkovskaya S.V., Yakhyaeva G.E. Combining logical and statistical rule reasoning and verification for medical applications. it Proceedings – 2017 International Multi-Conference on Engineering, Computer and Information Sciences, SIBIRCON 2017, Novosibirsk, 2017, pp. 309–313. https://doi.org/10.1109/SIBIRCON.2017.8109895
11. Palchunov D., Yakhyaeva G. Application of Boolean-valued models and FCA for the development of ontological model. *CEUR Workshop Proceedings*, 2017, vol. 1921, pp. 77–87.
12. Sokolov I.A. Theory and Practice of Application of Artificial Intelligence Methods. *Herald of the Russian Academy of Sciences*, 2019, vol. 89, no. 2, pp. 115-–119. https://doi.org/10.1134/S1019331619020205
13. Yakhyaeva G. Fuzzy model truth values. *Proceedings of the 6-th International Conference Aplimat.* Bratislava, Slovak Republic, 2007, pp. 423–431.
14. Yakhyaeva G., Ershov A. Knowledge Base System for Risk Analysis of the Multi-Step Computer Attacks. *Proceedings of the 18th International Conference on Enterprise Information Systems, ICEIS 2016*, vol. 2, pp. 143–150. https://doi.org/10.5220/0005772401430150
15. Yakhyaeva G., Karmanova A., Ershov A. Application of the Fuzzy Model Theory for Modeling QA-Systems. *Computing and Informatics*, 2021, vol. 40, no 6, pp. 1197-–1216. https://doi.org/10.31577/cai_2021_6_1197
16. Yakhyaeva G., Skokova V. Subjective Expert Evaluations in the Model-Theoretic Representation of Object Domain Knowledge. *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, 2021, vol. 12948 LNAI, pp. 152—165. https://doi.org/10.1007/978-3-030-86855-0_11

Список источников

1. Beliakov G., James S., Wu J.-Z. Discrete fuzzy measures: computational aspects. Springer, 2020. 260 p. https://doi.org/10.1007/978-3-030-15305-2
2. Castelvecchi D. Can we open the black box of AI? // Nature News. 2016. Vol. 538, N 7623. P. 21–23. https://doi.org/10.1038/538020a
3. Hajek A. Arguments For—Or Against—Probabilism? // British Journal for the Philosophy of Science. 2009. Vol. 59, N 4. P. 229–251. https://doi.org/10.1093/bjps/axn045
4. Kuznetsov S.O., Poelmans J. Knowledge representation and processing with formal concept analysis // Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery. 2013. Vol. 3, N 3. P. 200–215. https://doi.org/10.1002/widm.1088
5. Манцивода А. В., Пономарев Д. К. Формализация документных моделей средствами семантического моделирования // Известия Иркутского государственного университета. Серия Математика. 2019. Т. 27. С. 36–54. https://doi.org/10.26516/1997-7670.2019.27.36
6. Манцивода А. В. , Пономарев Д. К. К семантическому документному моделированию бизнес-процессов // Известия Иркутского государственного университета. Серия Математика. 2019. Т. 29. С. 52–67. https://doi.org/10.26516/1997-7670.2019.29.52
7. Miller T. Explanation in artificial intelligence: Insights from the social sciences. // Artif. Intell. 2019. Vol. 267. P. 1—38. https://doi.org/10.1016/j.artint.2018.07.007
8. Naydanov C., Palchunov D., Sazonova P. Development of automated methods for the critical condition risk prevention, based on the analysis of the knowledge obtained from patient medical records // Proceedings International Conference on Biomedical Engineering and Computational Technologies, SIBIRCON 2015. Novosibirsk, 2015. P 33–38.
9. Palchunov D., Yakhyaeva G. Fuzzy logics and fuzzy model theory // Algebra and Logic. 2015. Vol. 54, N 1. P. 74—80. https://doi.org/10.1007/s10469-015-9326-9
10. Palchunov D.E., Tishkovsky D.E., Tishkovskaya S.V., Yakhyaeva G.E. Combining logical and statistical rule reasoning and verification for medical applications // Proceedings - 2017 International Multi-Conference on Engineering, Computer and Information Sciences (SIBIRCON 2017), 18-22 September 2017, Novosibirsk, Russia. Novosibirsk, 2017. P. 309–313. https://doi.org/10.1109/SIBIRCON.2017.8109895
11. Palchunov D., Yakhyaeva G. Application of Boolean-valued models and FCA for the development of ontological model // CEUR Workshop Proceedings. 2017. Vol. 1921. P. 77–87.
12. Соколов И. А. Теория и практика применения методов искусственного интеллекта // Вестник Российской академии наук. 2019. Т. 89, № 4. С. 365–370. https://doi.org/10.31857/S0869-5873894365-370
13. Yakhyaeva G. Fuzzy model truth values // Proceedings of the 6-th International Conference Aplimat. February 6-9. Bratislava, Slovak Republic, 2007. P. 423–431.
14. Yakhyaeva G., Ershov A. Knowledge Base System for Risk Analysis of the Multi-Step Computer Attacks // Proceedings of the 18th International Conference on Enterprise Information Systems (ICEIS 2016). Vol. 2. P. 143–150. https://doi.org/10.5220/0005772401430150
15. Yakhyaeva G., Karmanova A., Ershov A. Application of the Fuzzy Model Theory for Modeling QA-Systems // Computing and Informatics. Vol. 40, N 6. 2021. P. 1197–1216. https://doi.org/10.31577/cai_2021_6_1197
16. Yakhyaeva G., Skokova V. Subjective Expert Evaluations in the Model-Theoretic Representation of Object Domain Knowledge // Lecture Notes in

Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics). 12948 LNAI. 2021. P. 152–165. https://doi.org/10.1007/978-3-030-86855-0_11

Об авторах

Яхъяева Гульнара Эркиновна, канд. физ.-мат. наук, доц., Новосибирский государственный технический университет, Новосибирск, 630073, Российская Федерация, gul_nara@mail.ru

About the authors

Gulnara E. Yakhyaeva, Cand. Sci. (Phys.–Math.), Assoc. Prof., Novosibirsk State Technical University, Novosibirsk, 630073, Russian Federation, gul_nara@mail.ru

Поступила в редакцию / Received 05.06.2023
Поступила после рецензирования / Revised 01.09.2023
Принята к публикации / Accepted 04.09.2023