Серия «Математика»
2023. Т. 46. С. 98—109

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского государственного университета

Research article

УДК 518.517
MSC 32-03,12-08,33C70
DOI https://doi.org/10.26516/1997-7670.2023.46.98

An Explanation of Mellin's 1921 Paper

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Abstract. In 1921 Mellin published a Comptes Rendu paper computing the principal solution of a polynomial using generalized hypergeometric functions of its coefficients. He used an integral transform nowadays bearing his name. Slightly over three pages, the paper is written in French in a terse style befitting the language. This article makes Mellin's landmark result accessible to people who are not experts in hypergeometric functions and complex analysis by deriving detailed proofs that were omitted in Mellin's paper.

Keywords: polynomial, principal solution, functions of hypergeometric type, Mellin–Barnes integral representation

Acknowledgements: This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of Regional Centers for Mathematics Research and Education (Agreement No. 075-02-2020-1534/1)

For citation: Lawton W. M. An Explanation of Mellin's 1921 Paper. *The Bulletin of Irkutsk State University. Series Mathematics*, 2023, vol. 46, pp. 98–109.
https://doi.org/10.26516/1997-7670.2023.46.98

Научная статья

Пояснения к статье Меллина 1921 года

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Аннотация. Приводятся подробные доказательства утверждений, которые отсутствуют в статье Меллина Comptes Rendu, что позволяет понять эту статью читателям, не являющимся экспертами в теории гипергеометрических функций и комплексном анализе.

Ключевые слова: многочлен, главное решение, функции гипергеометрического типа, интегральное представление Меллина – Барнса

Благодарности: Работа выполнена при поддержке Красноярского математического центра и финансировании Министерства науки и высшего образования. Российской Федерации в рамках создания и развития Региональных научно-образовательных математических центров (Договор № 075-02-2020-1534/1)

Ссылка для цитирования: Lawton W. M. An Explanation of Mellin's 1921 Paper // Известия Иркутского государственного университета. Серия Математика. 2023. Т. 46. С. 98–109.
https://doi.org/10.26516/1997-7670.2023.46.98

1. Principal Solution Z and Mellin Transform of Z^α

In the opening paragraph of his paper, Mellin says that it summarizes research he undertook years ago and was prompted by notes [10] of Richard Birkeland, a Norwegian mathematician known for his contributions to the theory of algebraic equations. The 7 numbered equations in Mellin's paper and this paper coincide. We prove each of them.

$$Z^n + x_1\, Z^{n_1} + \cdots + x_p\, Z^{n_p} - 1 = 0 \tag{1.1}$$

For integers $p \geq 1$, $0 < n_p < \cdots n_1 < n$, we define a principal solution to be an analytic function $Z(x_1, ..., x_p)$ on $[0, \infty)^p$ satisfying (1.1) and $Z(0, ..., 0) = 1$.

Lemma 1. *If a principal solution $Z(x_1, ..., x_p)$ of (1.1) exists, it is unique.*

Proof. For $r > 0$ define $D_r := \{z \in \mathbb{C} : |z| < r\}$. For sufficiently small r a principal solution $Z(x_1, \ldots, x_p)$ of (1.1) extends to give a holomorphic solution $\widetilde{Z}(x_1, \ldots, x_p)$ of (1.1) for $(x_1, ..., x_n)$ in the polydisc D_r^p. Therefore, for every n–th root of unity ϵ, the function $Z_\epsilon(x_1, ..., x_n) := \epsilon\, \widetilde{Z}(\epsilon^{n_1}\, x_1, ..., \epsilon^{n_p} x_p)$ is a holomorphic solution of (1.1) in D_r^p. Since a polynomial of degree n can have at most n distinct roots, these n distinct functions describe all holomorphic solutions of (1.1) on D_r^p. The conclusion follows since the restrictions of $\widetilde{Z}(x_1, ..., x_p)$ and $Z_1(x_1, ..., x_p)$ to $[0, r)^p$ are equal and analytic. □

Define $\Xi := \{\, \xi := (\xi_1, ..., \xi_p) \ : \ \xi_1 + \cdots + \xi_p \in (-\infty, -1]\,\}$, $W(\xi) := 1 + \xi_1 + \cdots + \xi_p$, and the holomorphic function $\Psi := (x_1, ..., x_n) : \mathbb{C}^p \backslash \Xi \to \mathbb{C}^p$ by

$$x_i(\xi) := \xi_i\, W(\xi_1, ..., \xi_p)^{\frac{n_i}{n} - 1},\ i = 1, ..., p \tag{1.2}$$

where $W^{\frac{1}{n}} : \mathbb{C}^p \backslash \Xi \to \mathbb{C}$ satisfies $W^{\frac{1}{n}}(0, ..., 0) = 1$.

Lemma 2. *If $x_1, ..., x_p$ are defined by (1.2), then $Z := W^{-\frac{1}{n}}$ satisfies (1.1).*

Proof. Follows by substitution. □

Lemma 3. *The restriction $\Psi : [0,\infty)^p \to [0,\infty)^p$ is a bijection. Furthermore, the Jacobian of $\Psi : \mathbb{C}^p \backslash \Xi \to \mathbb{C}^p$ satisfies*

$$\frac{\partial(x_1, ..., x_p)}{\partial(\xi_1, ..., \xi_p)} = \left(1 + \sum_{k=1}^{p} \xi_k\right)^{\frac{n_1+\cdots+n_p}{n} - p - 1} \left(1 + \frac{1}{n}\sum_{k=1}^{p} n_k\, \xi_k\right). \tag{1.3}$$

The restriction $\Psi : [0,\infty)^p \to [0,\infty)^p$ and its inverse $\Psi^{-1} : [0,\infty)^p \to [0,\infty)^p$ are analytic. The principal solution of (1.1) is $Z := W^{-\frac{1}{n}} \circ \Psi^{-1} : [0,\infty)^p \to [1,\infty)$.

Proof. The first assertion is Proposition 1. Define the Kronecker symbol

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Since the matrix

$$\frac{\partial \Psi_i}{\partial \xi_j} = W^{\frac{n_i}{n} - 1} \left(\delta_{i,j} + \xi_i \,(\frac{n_i}{n} - 1)\, W^{-1}\right),$$

is a product of two matrices, it follows that

$$\frac{\partial(x_1, ..., x_p)}{\partial(\xi_1, ..., \xi_p)} := \det \frac{\partial \Psi_i}{\partial \xi_j} = W^{\frac{\sum_{k=1}^{p} n_k}{n} - p} \det \left(\delta_{i,j} + \xi_i \,(\frac{n_i}{n} - 1)\, W^{-1}\right).$$

Proposition 2 implies that

$$\det\left(\delta_{i,j} + \xi_i\, W^{-1}\right) = 1 + \sum_{k=1}^{p} \xi_k \left(\frac{n_k}{n} - 1\right) W^{-1} = W^{-1}\left(1 + \frac{1}{n}\sum_{k=1}^{p} n_k\, \xi_k\right)$$

and concludes the proof of the second assertion. Since the Jacobian of Ψ is nozero and holomorphic on $\mathbb{C}^p \backslash \Xi$ and $[0,\infty)^p \subset \mathbb{C}^p \backslash \Xi$, assertion three follows from the implicit and inverse function theorems for holomorphic functions [17; 33]. The fourth assertion then follows from Lemma 2. □

Remark 1. Let $n = 2, p = 1$. Then $\Psi : \mathbb{C} \backslash (-\infty, -1] \to \mathbb{C} \backslash i((-\infty, -2] \cup [2, \infty))$ is a holomorphic bijection with inverse

$$\Psi^{-1}(z) = -1 + \left(\frac{z}{2} + \sqrt{1 + \left(\frac{z}{2}\right)^2}\,\right)^2.$$

Letting $\xi_1 = -1 + e^{s+it}$ with $s \in \mathbb{R}$ and $-\pi < t < \pi$ gives $\Psi(\xi_1) = 2\sinh\frac{s+it}{2} = u + iv$ where $u = 2\sinh\frac{s}{2}\cos t/2$ and $v = 2\cosh\frac{s}{2}\sin t/2$. For $t = 0$ this gives the curve $v = 0$. For fixed $t \neq 0$ this gives the branch of the hyperbola described by the equation $\left(\frac{v}{\sin t/2}\right)^2 - \left(\frac{u}{\cos t/2}\right)^2 = 4$ with $tv \geq 0$.

Question 1 For $n \geq 3$ and $p \geq 1$, is Ψ injective and what is its image?

Following (1.3) Mellin says that using the known formula, derived in Proposition 2, in combination with (1.2) and (1.3), one can deduce the following result:

Lemma 4. *The principal solution of (1.1) satisfies*

$$\int_0^\infty \cdots \int_0^\infty Z(x_1, ..., x_p)^\alpha \, x_1^{u_1-1} \cdots x_p^{u_p-1} \, dx_1 \cdots dx_p = \\ = \frac{\alpha}{n} \frac{\Gamma(u)\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(u + u_1 + \cdots + u_p + 1)}, \quad (1.4)$$

where $\alpha > 0$, *and the real parts of* $u := \frac{\alpha}{n} - \frac{n_1}{n}u_1 - \cdots - \frac{n_p}{n}u_p$, *and* $u_1, ..., u_p$ *are positive.*

Proof. Since $Z = W^{-\frac{1}{n}}$ and $\alpha > u_1 n_1 + \cdots + u_p n_p$, the integral I in (1.4) exists. Lemma 3 implies that $\Psi : [0,\infty)^p \to [0,\infty)^p$ is a bijective diffeomorphism, so the integral I above can be expressed as an integral over $[0,\infty)^p$ with respect to the variables $\xi_1, ...\xi_p$. Hence (1.2) and (1.3) give

$$I = \int_0^\infty \cdots \int_0^\infty \frac{(1 + \frac{n_1}{n}\xi_1 + \cdots + \frac{n_p}{n}\xi_p)\xi_1^{u_1-1} \cdots \xi_p^{u_p-1}}{W^\omega} \, d\xi_1 \cdots d\xi_p$$

where $\omega := u + u_1 + \cdots + u_p + 1$. Proposition 2 implies that $I = I_0 + I_1 + \cdots + I_p$ where

$$I_0 = \frac{\Gamma(\omega - u_1 - \cdots - u_p)\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(\omega)} = \frac{\Gamma(u+1)\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(\omega)},$$

and

$$I_i = \frac{n_i}{n} \frac{\Gamma(\omega - u_1 - \cdots - u_p - 1)\Gamma(u_1)\cdots\Gamma(u_i + 1)\cdots\Gamma(u_p)}{\Gamma(\omega)} = \frac{n_i u_i}{nu} I_0.$$

We conclude the proof by observing that

$$I = \left(1 + \frac{1}{u}\sum_{i=1}^p \frac{n_i u_i}{n}\right) I_0 = \frac{\alpha}{nu} I_0 = \frac{\alpha}{n} \frac{\Gamma(u+1)}{u} \frac{\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(\omega)}.$$

□

2. Computing Z^α from its Mellin Transform

Mellin continues: "The law of reciprocity relating to the integrals of this species, demonstrated by us in a previous work [23], allows us to invert as follows (1.4):"

$$Z(x_1,...,x_p)^\alpha = \frac{1}{(2\pi i)^p}\cdot$$
$$\int_{a_1-i\infty}^{a_1+i\infty}\cdots\int_{a_p-i\infty}^{a_p+i\infty}\frac{\alpha}{n}\,\frac{\Gamma(u)\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(u+u_1+\cdots+u_p+1)}\,x_1^{-u_1}\cdots x_p^{-u_p}\,du_1\cdots du_p, \tag{2.1}$$

$$\alpha - n_1a_1 - \cdots - n_pa_p > 0,\ a_s > 0,\ u := \frac{\alpha}{n} - \frac{n_1}{n}u_1 - \cdots - \frac{n_p}{n}u_p$$

$$u + u_1 + \cdots + u_p = \frac{\alpha}{n} + \frac{n_1^{'}}{n}u_1 + \cdots + \frac{n_p^{'}}{n}u_p,\quad n_s^{'} := n - n_s.$$

"This formula constitutes our solution of (1.1). It supposes that

$$-\frac{n_s\pi}{2n} < \arg(x_s) < \frac{n_s\pi}{2n}$$

but we can extend our domain of validity by suitably deforming the integration paths."

The right side of (1.4) is the Mellin transform of $Z(x_1,...,x_p)^\alpha$ and (2.1) represents $Z(x_1,...,x_p)^\alpha$ by the inverse Mellin transform. Lacking the luxury of accessing [23] we refer the reader to the derivation of the inverse Mellin transform via the Fourier transform by Debnath and Bhatta in section 8.1 of [11]. They address the univariate case, but the extension to the multivariate case is straightforward.

3. Generalized Hypergeometric Functions

Let n be a positive integer and $f_1,...,f_p,g_1,...,g_s : \mathbb{C}^p \to \mathbb{C}$ be entire functions and $F : \mathbb{C}^p \to \mathbb{C}$ satisfy the following system of functional equations

$$F(u_1,...,u_s+n,...,u_p) = \frac{f_s(u_1,...,u_p)}{g_s(u_1,...,u_p)}\,F(u_1,...,u_p),\quad s = 1,...,p. \tag{3.1}$$

and such that the following integral converges and does not change when we move the (vertical) integration paths for each u_s n units to the right:

$$y(x_1,...,x_p) = \frac{1}{(2\pi i)^p}\int_{u_1}\cdots\int_{u_p}F(u_1,...,u_p)\,x_1^{-u_1}\cdots x_p^{-u_p}\,du_1\cdots du_p.$$

Mellin cites a result [23] that y satisfies the following system of partial differential equations

$$f_s\left(-x_1\frac{\partial}{\partial x_1},...,-x_p\frac{\partial}{\partial x_p}\right)y=g_s\left(-x_1\frac{\partial}{\partial x_1},...,-x_p\frac{\partial}{\partial x_p}\right)x_s^n\,y,\ \ s=1,...,p. \tag{3.2}$$

Remark 2. (3.2) is a system of PDE's of finite order iff f_s and g_s are polynomials.

Mellin calls solutions of (3.1) hypergeometric type if the factors of f_s, g_s have the form

$$c_1u_1+\cdots+c_pu_p+a$$

where each c_i is a rational real number.

Remark 3. Polynomials are not hypergeometric functions or series as defined in [16;34] but they are of hypergeometric type since $\prod_{k=0}^{m-1}(x\frac{d}{dx}-k)=x^m\frac{d^m}{dx^m}$.

Formally (3.2) follows from (3.1) since the functions $x_s^{-u_s}$ are eigenfunctions with eigenvalue $-u_s$ of the Euler operator $x_s\frac{\partial}{\partial x_s}, s=1,...,p$. Clearly the function

$$F(u_1,...,u_p)=\frac{\alpha}{n}\,\frac{\Gamma(u)\Gamma(u_1)\cdots\Gamma(u_p)}{\Gamma(u+u_1+\cdots+u_p+1)}$$

satisfies (3.1) where f_s and g_s have the form above, hence $Z(u_1,...,u_p)^\alpha$ is of hypergeometric type whenever $\alpha > n_1n$.

Question 2 How large must α be to ensure that Z^α is a solution of (1.1) of hypergeometric type? The condition $\alpha > nn_1$ is sufficient but not necessary because for $n=1, p=1$ the root $Z(x_1):=-\frac{x_1}{2}+\sqrt{1+(x_1/2)^2}$ is of hypergeometric type. Semusheva and Tsikh [32] proved this fact directly by deriving the following Mellin–Barnes integral representation

$$Z(x_1)=\frac{1}{4\pi i}\int_{\Re z=\frac{1}{2}}\frac{\Gamma(z)\,\Gamma((1+z)/2)}{\Gamma((3+z)/2)}\,x_1^z\,dz.$$

Remark 4. For a comprehensive development of Mellin's solution of (1.3) and systems of differential equations of hypergeometric type see ([9], Chapter V).

4. Crucial Propositions

We prove results required to derive the equations in Mellin's paper.

Let $e_1 := (1, 0, ..., 0), e_2 := (0, 1, ..., 0), ..., e_p = (0, ..., 0, 1)$ be the standard basis for $\mathbb{R}^p$. For $s > 0$ let H_s denote the convex hull of $\{se_i : i = 1, ..., p\}$ and let K_s denote the convex hull of $\{s(1+s)^{\frac{n_i}{n}-1} e_i : i = 1, ..., p\}$.

Proposition 1. $\Psi : [0, \infty)^p \to [0, \infty)^p$ *is a bijection.*

Proof. For every $s > 0$ define the linear map $L_s : \mathbb{R}^p \to \mathbb{R}^p$ by

$$(L_s\, y)_i := y_i\, (1+s)^{\frac{n_i}{n}-1}, \quad i = 1, ..., p.$$

If $(\xi_1, ..., \xi_p) \in H_s$, then $W(\xi_1, ..., \xi_p) = 1 + s$ so Equation 1.2 implies that

$$\Psi(\xi_1, ..., \xi_p) = L_s(\xi_1, ..., \xi_p) = \sum_{i=1}^{p} \frac{\xi_i}{s}\, s(1+s)^{\frac{n_i}{n}-1} e_i \in K_s,$$

hence the restriction $\Psi : H_s \to K_s$ is a linear bijection. Clearly $\Psi(0, ..., 0) = (0, ..., 0)$ and $[0, \infty)^p$ is a disjoint union of $\{(0, ..., 0)\}$ and the sets $H_s, s > 0$. Since for $i = 1, ..., p$, the function $s(1+s)^{\frac{n_i}{n}-1}$ is increasing, it follows that $[0, \infty)^p$ is a disjoint union of $\{(0, ..., 0)\}$ and the sets $K_s, s > 0$. This concludes the proof. □

Proposition 2. *Let* $p \geq 1$, $y_1, ..., y_p$ *be indeterminates, and define the matrix*

$$M(y_1, ..., y_p) := \begin{bmatrix} 1+y_1 & y_1 & \cdots & y_1 & y_1 \\ y_2 & 1+y_2 & \cdots & y_2 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{p-1} & y_{p-1} & \cdots & 1+y_{p+1} & y_{p+1} \\ y_p & y_p & \cdots & y_p & 1+y_p \end{bmatrix}.$$

Then

$$\det M(y_1, ..., y_p) = 1 + y_1 + \cdots + y_p.$$

Proof. For $i = 1, ..., p$ let $\partial_i := \frac{\partial}{\partial y_i}$. Clearly $\det M(y_1, ..., y_p)$ is a polynomial with constant term $\det M(0, ..., 0) = \det I_p = 1$ and of degree at most 1 in each variable $y_1, ..., y_p$ so for $i = 1, ..., p$

$$\partial_i^2 \det M(y_1, ..., y_p) = 0.$$

Let $M_i(y_1, ..., y_p)$ denote the matrix obtained from $M(y_1, ..., y_p)$ by replacing each entry in its i–th row with 1 and for $j \neq i$ let $M_{i,j}(y_1, ..., y_p)$ denote the matrix obtained from $M(y_1, ..., y_p)$ by replacing each entry in its i–row and its j–row by 1. Since a determinant of a matrix is a linear function of each of its row vectors,

$$\partial_i \det M(y_1, ..., y_p) := \det M_i(y_1, ..., y_p)$$

and for $j \neq i$

$$\partial_j \partial_i \det M(y_1, ..., y_p) := \det M_{i,j}(y_1, ..., y_p) = 0$$

since $M_{i,j}(y_1, ..., y_p)$ has two identical rows. Therefore Taylor's expansion gives

$$\det M(y_1, ..., y_n) = 1 + \sum_{i=1}^{p} y_i \, \partial_i \det M(0, ..., 0) = 1 + \sum_{i=1}^{p} y_i \, \det M_i(0, ..., 0).$$

It suffices to prove $\det M_i(0, ..., 0) = 1$, $i = 1, ..., p$. This follows since

$$M_i(0, ...0) = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

□

Remark 5. Nilsson, Passare and Tsikh ([26], Example 1) used basic calculus to compute the following integrals for $\Re u_i \in (0, 1), i = 1, 2.$

$$\int_0^\infty \int_0^\infty \frac{\xi_1^{u_1-1}\, \xi_2^{u_2-1}}{1 + \xi_1 + \xi_2} d\xi_1 d\xi_2 = \frac{\Gamma(u_1)\, \Gamma(u_2)\, \Gamma(1 - u_1 - u_2)}{\Gamma(1)}$$

The following result uses exterior calculus to extend their computation.

Proposition 3. *For every positive integer* p, *complex* $u_1, ..., u_p$ *satisfying* $\Re\, u_i > 0$, *and* $\omega > \max\{\Re\, u_i\}$ *the integral below converges and satisfies the stated identity.*

$$\int_0^\infty \cdots \int_0^\infty \frac{\xi_1^{u_1-1} \cdots \xi_p^{u_p-1}}{(1 + \xi_1 + \cdots + \xi_p)^\omega}\, d\xi_1 \cdots d\xi_p =$$
$$= \frac{\Gamma(u_1) \cdots \Gamma(u_p)\Gamma(\omega - u_1 - \cdots - u_p)}{\Gamma(\omega)}.$$

Proof. Multiplying both sides of the asserted identity in the lemma by

$$\Gamma(\omega) = \int_0^\infty x^{\omega-1}\, e^{-x}\, dx$$

gives the equivalent identity $I_1(u_1,...,u_p,\omega) = \Gamma(u_1)\cdots\Gamma(u_p)\,\Gamma(\omega - u_1 - \cdots - u_p)$ where

$$I_1(u_1,...,u_p,\omega) := \int_0^\infty \cdots \int_0^\infty \frac{\xi_1^{u_1-1}\cdots\xi_p^{u_p-1}}{(1+\xi_1+\cdots+\xi_p)^\omega}\, x^{\omega-1}\, e^{-x}\, d\xi_1\wedge\cdots\wedge d\xi_p\wedge dx.$$

Here we express the volume form using exterior products to implement the following change of variables: $y := x/(1+\xi_1+\cdots+\xi_p)$ and $z_i := y\,\xi_i, i = 1,...,p$. Then $I_1(\xi_1,...,\xi_p,x) = I_2(z_1,...,z_p,y)$ where

$$I_2(z_1,...,z_p,y) := \int_0^\infty \cdots \int_0^\infty z_1^{u_1-1}e^{-z_1}\cdots z_p^{u_p-1}e^{-z_p}\, y^{\omega-u_1-\cdots-u_p-1}\, e^{-y}\,\theta$$

where

$$\theta = \frac{y^p}{1+\xi_1+\cdots+\xi_p}\, d\xi_1\wedge\cdots\wedge d\xi_p\wedge dx$$

In order to finish the proof it suffices to prove that $dz_1\wedge\cdots\wedge dz_p\wedge dy = \theta$. Since $dz_i = \xi_i dy + y\,d\xi_i$ and $dy\wedge dy = 0$ it follows that

$$dz_1\wedge\cdots\wedge dz_p\wedge dy = y^p\, d\xi_1\wedge\cdots\wedge d\xi_p\wedge dy.$$

Substitute

$$dy = \frac{1}{1+\xi_1+\cdots+\xi_p}\,dx - \frac{x}{(1+\xi_1+\cdots+\xi_p)^2}\left(\sum_{k=1}^p d\xi_k\right)$$

and use the fact that $d\xi_i\wedge d\xi_i = 0$ to conclude the proof by obtaining

$$dz_1\wedge\cdots\wedge dz_p\wedge dy = \frac{y^p}{1+\xi_1+\cdots+\xi_p}\, d\xi_1\wedge\cdots\wedge d\xi_p\wedge dx = \theta.$$

□

5. Historical Remarks

Slater ([34],p. 1) states that John Wallis [37] used the term hypergeometric (Greek for beyond) to denote series (beyond) the geometric series $1+x+x^2+\cdots$, in particular the series $1+a+a(a+1)+a(a+1)(a+2)+\cdots$. Over the next two centuries mathematicians including Euler, Gauss, Kummer, and Riemann developed the theory of hypergeometric functions and Euler [12] gave their first integral representations. Mainardi and Pagnini [19] describe how Pincherle [28–30] gave the first representations by contour integrals involving Gamma functions, now called Mellin–Barnes integrals after the two mathematicians who extensively developed their theory and applications [5–8; 20–24].

In recent decades numerous mathematicians have researched areas of multivariate complex analysis and algebraic geometry related to Mellin's 1921 paper. Antipova's 2007 paper [3] contains a proof of the Mellin-Barnes integral representation of the principal solution of a general algebraic equation with a refinement of the region of convergence proposed by Mellin. Related work by my colleagues in Siberian Federal University and their mentors include the development by Aizenberg, Tsikh and Yuzhakov [1; 2; 36] of multidimensional integral representations, some involving Grothendieck residues [14; 15]. Antipova, Kulikov, Mikhalkin, Sadykov, Semusheva, Stepanenko, and Tsikh have extended Mellin's work to systems of polynomial equations [4; 18; 31; 32; 35]. Their work used amoebas studied by Tsikh and his collaborators Cheshel, Forsberg, Passare, Nilsson and Zhdanov [13; 26; 27; 39].

Acknowledgments. We thank Elaine Wong, of the Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria, for her meticulous proofreading of and corrections to this article, to professor August Tsikh for explaining the development of the theory and application of Mellin–Barne's integrals in Krasnoyarsk, and to the referees for suggesting crucial revisions.

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Поступила в редакцию / Received 24.05.2023
Поступила после рецензирования / Revised 16.09.2023
Принята к публикации / Accepted 19.09.2023