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On an Approach to Finding Sums of Multiple Numerical Series

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Abstract. An approach to calculating sums of some types of multiple numerical series is presented. This approach is based on using the formula for the resultant of a polynomial (or an entire function with a finite number of zeros) and an entire function obtained earlier by A.M. Kytmanov and E.K. Myshkina. This formula does not require values of the roots of the functions under study and is a combinatorial expression. By calculating the resultant of a polynomial and an entire function in two different ways, it is possible to obtain a relation for multiple numerical series. For the second way to find the resultant, we use the product of one function at the roots of another. In this article, the sums of some types of multiple numerical series that were previously absent in known reference books are found. They are expressed in terms of well-known special functions such as the Bessel function. This approach to calculating the sums of multiple numerical series differs significantly from the method based on the use of residue integrals associated with a system of equations. The relevance of this problem is determined by the fact that in applied problems, for example, in the equations of chemical kinetics, there are functions and systems of equations consisting of exponential polynomials.

Keywords: sum of a multiple numerical series, resultant, entire function

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Научная статья

Об одном подходе к нахождению сумм кратных числовых рядов

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Аннотация. Представлен подход к вычислению сумм некоторых типов кратных числовых рядов. Данный подход основан на использовании формулы для результанта многочлена (или целой функции с конечным числом нулей) и целой функции, полученной А. М. Кытмановым и Е. К. Мышкиной ранее. Данная формула не требует значения корней исследуемых функций и представляет собой некоторое комбинаторное выражение. Вычисляя результант многочлена и целой функции двумя разными способами, удается получить соотношение для кратных числовых рядов. В качестве второго способа нахождения результанта выбирается формула для произведения одной функции в корнях другой. Найдены суммы некоторых типов кратных числовых рядов, ранее отсутствовавших в известных справочниках. Они выражаются через известные специальные функции, такие как функция Бесселя. Данный подход к вычислению сумм кратных числовых рядов существенно отличается от способа, основанного на использовании вычетных интегралов, связанных с системой уравнений. Актуальность данной задачи определяется тем, что в прикладных задачах, например в уравнениях химической кинетики, возникают функции и системы уравнений, состоящие из экспоненциальных многочленов.

Ключевые слова: сумма кратного числового ряда, результант, целая функция

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1. Introduction

An important tool that opened a way to developing algorithmic methods for studying and solving systems of algebraic equations is the concept of the Gröbner basis of the ideal of the ring of polynomials. However, the classical methods for eliminating unknowns from systems of algebraic equations, based on the method of Gröbner bases, cannot be applied for the analysis of essentially nonalgebraic equations (i. e., the equations that cannot be reduced to algebraic equations by changing variables).

However, systems of nonalgebraic equations occur in many areas of knowledge. In particular, in processes described by systems of differential equations with the right-hand sides that can be expanded in Taylor series, it is important to determine the number of stationary states (and localize them) in sets of a certain type. This problem leads to problems of constructing algorithms for determining the number of roots of the given system of equations in various sets, to finding the roots themselves, and to eliminating a part of unknowns. In particular, in [2; 3] one can find numerous examples from chemical kinetics in which algorithms for eliminating unknowns are required. It is important to apply the developed methods for the qualitative and numerical analysis of mathematical models of thermokinetics of the processes of combustion and catalysis with the aim of obtaining conditions of ignition, explosion, and critical phenomena in chemically reacting systems. For applications, in particular, for chemical kinetics equations, an important problem is to explore the dependence of solutions to systems of nonlinear (including nonalgebraic) equations on parameters. This problem is computationally costly. The degree of its complexity strongly depends on the dimension of the space of unknowns. Therefore, the reduction of dimension by eliminating some variables can simplify the original problem.

One of the methods for the elimination of unknowns is based on constructing the resultant of two entire functions. There is the well-known classical Sylvester's resultant of two polynomials and the elimination method based on it. For nonalgebraic functions, no similar concept was earlier studied. Only recently, an approach to finding the resultant of two entire functions based on Newton's recurrent formulas has been discussed in [15].

If we talk about finding the sums of multiple numerical series, then in [10; 13; 17] power sums of roots of systems of nonalgebraic equations consisting of entire or meromorphic functions of finite order of growth were studied. These power sums are associated with certain residue integrals that are not Grothendieck residues. As an application, the sums of some multiple series were found. Note that our method for calculating multiple numerical series differs significantly from the method of residue integrals considered in [10; 13; 17]. In this article we will rely on the formula for the resultant from [14].

2. Sylvester's classical resultant and its generalizations

Recall that for the given polynomials

$$\begin{cases} f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \ldots + \alpha_{n-1} x + \alpha_n, \\ g(x) = \beta_0 x^s + \beta_1 x^{s-1} + \beta_2 x^{s-2} + \ldots + \beta_{s-1} x + \beta_s \end{cases}$$

the classical resultant $R(f,g)$ can be defined in various ways using

a) Sylvester's determinant (e. g., see [1]):

$$R(f,g) = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \ldots & 0 \\ 0 & \alpha_0 & \alpha_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \alpha_0 & \ldots & \alpha_n \\ \beta_0 & \beta_1 & \beta_2 & \ldots & 0 \\ 0 & \beta_0 & \beta_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \beta_0 & \ldots & \beta_s \end{vmatrix}.$$

b) formulas for the product (e. g., see [1]):

$$R(f,g) = \alpha_0^s \prod_{\{x\colon f(x)=0\}} g(x);$$

c) the Bezout – Caley method (the name of the method is symbolic; in addition to Bezout and Caley, Euler, Cauchy and Hermite took part in its development; e. g., see [9]) using the formula

$$R(f,g) = \det B \cdot \alpha_0^s / (-1)^{\frac{n(n-1)}{2}},$$

where $B = [\beta_{kj}]_{k,j=0}^{n-1}$ is a matrix of coefficients of

$$g_k(x) = \beta_{k0} x^{n-1} + \beta_{k1} x^{n-2} + \ldots + \beta_{k,n-1}$$

of remainders from dividing $x^k g(x)$ by $f(x)$ for $k = 0, \ldots, n-1$.

The coefficients of polynomial $g_k(x)$ can be expressed in terms of the coefficients of polynomial $g_{k-1}(x)$ using the formulas

$$\beta_{kj} = \beta_{k-1,j+1} - \beta_{k-1,0}\alpha_{j+1}/\alpha_0 \quad (j = 0, \ldots, n-1).$$

In a number of papers (e. g., see [5–7]), various researchers proposed generalizations of the concept of resultant for analytic functions in the ring of matrix-valued functions. B. Gustafsson and V.G. Tkachev in [8] gave several integral formulas for the resultant, connected it with potential theory, and gave explicit formulas for the algebraic dependence between two meromorphic functions on a compact Riemann surface. In [16] A.Yu. Morozov and Sh.R. Shakirov described several areas of research in the theory of resultants, including formulas of the Sylvester-Bezout (determinant) and Schur (analytic) type for the resultant, invariant formulas and symmetric reduction for the discriminant, Ward identities and representations in the form of hypergeometric series for partition functions (integral discriminants and roots of algebraic equations). In these studies, it was assumed that the number of roots or poles is finite. In the case considered in this paper, the entire functions may have an infinite number of roots. Therefore, for finding the resultant, the passage to the limit is required.

The first step in finding the resultant of two entire functions was the work [15], in which the case when one of the functions is entire and another one is a polynomial (or an entire function with a finite number of zeros) was considered. For entire functions, the question of localization of real zeros was considered in the classical works of N.G. Chebotarev [4, p. 28-56], as well as in [12] (we refer to the collected works of N.G. Chebotarev, since his original works are inaccessible). In [11], the results of [15] are generalized for the case when one of the entire functions satisfies certain severe constraints but may have an infinite number of zeros.

The work [14] presents a construction of the resultant of two entire functions on the complex plane. The formula for the product was chosen as the main definition of the resultant. This choice is explained by the fact that entire functions are a natural generalization of polynomials in complex analysis. The advantage of this approach is that it makes possible to answer the question whether or not entire functions have common zeros without calculating the zeros themselves. The final formula for the resultant includes power sums of the roots that can be calculated using Newton's formulas without finding the zeros themselves. When obtaining relations for the sums of multiple numerical series, we will rely on the formula for the resultant from [14].

3. Resultant of polynomial and entire function

Recall the classical Newton's recurrent formulas for polynomials. They relate the coefficients of the polynomial to the power sums of its roots.

Let

$$P(z) = z^m + c_1 z^{m-1} + \ldots + c_{m-1} z + c_m.$$

Denote its roots by $z_1, z_2, \ldots, z_m$ (there may be multiple roots among them). Define the power sum of the roots

$$S_k = z_1^k + \ldots + z_m^k, \quad k \in \mathbb{N}, \quad S_0 = m.$$

The power sums S_k and the coefficients c_j are connected by Newton's recurrent formulas:

$$\begin{cases} S_j + \sum\limits_{i=1}^{j-1} S_{j-i}c_i + jc_j = 0, \quad 1 \leq j \leq m, \\ S_j + \sum\limits_{i=1}^{m} S_{j-i}c_i = 0, \quad j > m. \end{cases} \tag{3.1}$$

Consider the system of equations consisting of a quadratic polynomial $f(z)$ and a polynomial $g(z)$ of degree n:

$$\begin{cases} f(z) = a_0 + a_1 z + z^2, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases} \tag{3.2}$$

Using the definition of the resultant as a product, we obtain the following result.

Theorem 1 ([14])**.** *The resultant $R(f,g)$ of the system of polynomials* (3.2) *is found by the formula*

$$R(f,g) = \sum_{k=0}^{n} b_k^2 a_0^k + \sum_{t=0}^{n-1} \sum_{s=t+1}^{n} b_t b_s a_0^t S_{s-t}, \tag{3.3}$$

where the power sums S_j of the roots of the polynomial $f(z)$ are determined by Newton's recurrent formulas.

By passing to the limit in (3.3) over n, we obtain the following theorem concerning the resultant of an entire function and a polynomial (or an entire function with a finite number of zeros).

Theorem 2 ([14])**.** *Let $g(z)$ be an entire function on the complex plane* $\mathbb{C}$

$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots,$$

and let $f(z)$ be a polynomial of form (3.2)*. Then, the resultant $R(f,g)$ is*

$$R(f,g) = \sum_{k=0}^{\infty} b_k^2 a_0^k + \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} b_t b_s a_0^t S_{s-t},$$

where the series on the right-hand side are absolutely convergent.

Let us demonstrate formula (3.3) for two polynomials using the following example.

Example 1. Consider the system of equations ($n = 20$)

$$\begin{cases} f(z) = (z-1)(z+2) = z^2 + z - 2, \\ g(z) = \left(z^{14} - 25z^{12} + 144\right) z^6 = z^{20} - 25z^{18} + 144z^6. \end{cases}$$

In this case

$$a_0 = -2,\ a_1 = 1,\ b_6 = 144,\ b_{18} = -25,\ b_{20} = 1,$$
$$b_0 = b_1 = \ldots = b_5 = b_7 = \ldots = b_{17} = b_{19} = 0.$$

Then formula (3.3) has the form

$$R\,(f, g) = \sum_{k=0}^{20} b_k^2 a_0^k + \sum_{t=0}^{19} \sum_{s=t+1}^{20} b_t b_s a_0^t S_{s-t}.$$

Due to the zero coefficients b_j only the terms corresponding to $k = 6$, $k = 18$, $k = 20$, $t = 6$, $t = 18$, $s = 18$ and $s = 20$ will remain in these sums. Thus,

$$R\,(f, g) = b_6^2 a_0^6 + b_{18}^2 a_0^{18} + b_{20}^2 a_0^{20} + b_6 b_{18} a_0^6 S_{12} + b_6 b_{20} a_0^6 S_{14} + b_{18} b_{20} a_0^{18} S_2.$$

Let us find the power sums of the roots S_2, S_{12}, S_{14}, without using the values of the roots themselves, namely, using Newton's recurrent formulas. For S_1 and S_2 we use the first equality of the system (3.1). For S_3, S_4, …, S_{14} we use the second equality of the system (3.1). Here $c_1 = 1$, $c_2 = -2$. We have

$$\begin{cases} S_1 + 1c_1 = 0, \\ S_2 + S_1 c_1 + 2c_2 = 0, \\ S_3 + S_2 c_1 + S_1 c_2 = 0, \\ S_4 + S_3 c_1 + S_2 c_2 = 0, \\ S_5 + S_4 c_1 + S_3 c_2 = 0, \\ S_6 + S_5 c_1 + S_4 c_2 = 0, \\ S_7 + S_6 c_1 + S_5 c_2 = 0, \\ S_8 + S_7 c_1 + S_6 c_2 = 0, \\ S_9 + S_8 c_1 + S_7 c_2 = 0, \\ S_{10} + S_9 c_1 + S_8 c_2 = 0, \\ S_{11} + S_{10} c_1 + S_9 c_2 = 0, \\ S_{12} + S_{11} c_1 + S_{10} c_2 = 0, \\ S_{13} + S_{12} c_1 + S_{11} c_2 = 0, \\ S_{14} + S_{13} c_1 + S_{12} c_2 = 0. \end{cases}$$

Thus, from this system of equations we successively find power sums:
$S_1 = -c_1 = -1$,
$S_2 = -S_1 c_1 - 2c_2 = 1 \cdot 1 - 2 \cdot (-2) = 1 + 4 = 5$,
$S_3 = -S_2 c_1 - S_1 c_2 = -5 \cdot 1 + 1 \cdot (-2) = -5 - 2 = -7$,
$S_4 = -S_3 c_1 - S_2 c_2 = 7 \cdot 1 - 5 \cdot (-2) = 7 + 10 = 17$,
$S_5 = -S_4 c_1 - S_3 c_2 = -17 \cdot 1 + 7 \cdot (-2) = -17 - 14 = -31$,
$S_6 = -S_5 c_1 - S_4 c_2 = 31 \cdot 1 - 17 \cdot (-2) = 31 + 34 = 65$,

$S_7 = -S_6c_1 - S_5c_2 = -65 \cdot 1 + 31 \cdot (-2) = -65 - 62 = -127,$
$S_8 = -S_7c_1 - S_6c_2 = 127 \cdot 1 - 65 \cdot (-2) = 127 + 130 = 257,$
$S_9 = -S_8c_1 - S_7c_2 = -257 \cdot 1 + 127 \cdot (-2) = -257 - 254 = -511,$
$S_{10} = -S_9c_1 - S_8c_2 = 511 \cdot 1 - 257 \cdot (-2) = 511 + 514 = 1025,$
$S_{11} = -S_{10}c_1 - S_9c_2 = -1025 \cdot 1 + 511 \cdot (-2) = -1025 - 1022 = -2047,$
$S_{12} = -S_{11}c_1 - S_{10}c_2 = 2047 \cdot 1 - 1025 \cdot (-2) = 2047 + 2050 = 4097,$
$S_{13} = -S_{12}c_1 - S_{11}c_2 = -4097 \cdot 1 + 2047 \cdot (-2) = -4097 - 4094 = -8191,$
$S_{14} = -S_{13}c_1 - S_{12}c_2 = 8191 \cdot 1 - 4097 \cdot (-2) = 8191 + 8194 = 16385.$

Thus,

$$\begin{aligned} R(f,g) = {} & 144^2 \cdot (-2)^6 + (-25)^2 \cdot (-2)^{18} + 1^2 \cdot (-2)^{20} + \\ & + 144 \cdot (-25) \cdot (-2)^6 \cdot 4097 + 144 \cdot 1 \cdot (-2)^6 \cdot 16385 + \\ & + (-25) \cdot 1 \cdot (-2)^{18} \cdot 5 = 1327104 + 163840000 + 1048576 - \\ & - 943948800 + 151004160 - 32768000 = -659496960 \neq 0, \end{aligned}$$

therefore, the polynomials $f(z)$ and $g(z)$ do not have common roots.

4. The main result

These methods lead to the calculation of the sums of some multiple numerical series that were previously absent in well-known reference books. An essential point in obtaining these relations is the fact that the coefficient a_0 in expansion (3.2) of the function $f(z)$ is different from zero.

Problem 1. Consider the system of equations

$$\begin{cases} f(z) = (z-1)(z+4) = z^2 + 3z - 4, \\ g(z) = e^z - 1 = z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots . \end{cases}$$

Calculating the resultant $R(f,g)$, using Theorem 2 on the one hand, and using the definition of the resultant in the form of a formula for the product on the other hand, we obtain the relation:

$$\sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^2 (-4)^k - 1 + \sum_{t=1}^{\infty} \sum_{s=t+1}^{\infty} \left(\frac{1}{t!}\right)\left(\frac{1}{s!}\right)(-4)^t S_{s-t} = \left(e^1 - 1\right)\left(e^{-4} - 1\right).$$

The -1 term in the last relation is due to the fact that $b_0 = 0$. Here

$$S_{s-t} = z_1^{s-t} + z_2^{s-t} = 1^{s-t} + (-4)^{s-t} = 1 + (-4)^{s-t}.$$

Thus,

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(k!)^2} - 1 + \sum_{t=1}^{\infty} \sum_{s=t+1}^{\infty} \frac{(-4)^t + (-4)^s}{t!\, s!} = 1 - e + e^{-3} - e^{-4}.$$

In this case, the first sum is [18, formula 5.2.10.1] a well-known special function. This is a Bessel function of the first kind, namely $J_0(4)$. Therefore, we can express the sum of a multiple numerical series as follows:

$$\sum_{t=1}^{\infty}\sum_{s=t+1}^{\infty}\frac{(-4)^t+(-4)^s}{t!\,s!}=2-e+e^{-3}-e^{-4}-J_0(4).$$

Problem 2. Consider the system of equations

$$\begin{cases} f(z)=(z-2\pi)(z-\pi)=z^2-3\pi z+2\pi^2, \\ g(z)=\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots+\frac{(-1)^n z^{2n}}{(2n)!}+\ldots. \end{cases}$$

Since in this problem the expansion of the entire function $g(z)$ contains only coefficients of even powers, the formula for the resultant from Theorem 2 takes the form:

$$R(f,g)=\sum_{k=0}^{\infty}b_{2k}^2a_0^{2k}+\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty}b_{2t}b_{2s}a_0^{2t}S_{2s-2t}.$$

Then, similarly to the previous problem, we obtain the following relation:

$$\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k)!}\right)^2(2\pi^2)^{2k}+\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty}\left(\frac{(-1)^t}{(2t)!}\right)\left(\frac{(-1)^s}{(2s)!}\right)\left(2\pi^2\right)^{2t}S_{2s-2t}= \\ =\cos 2\pi\cos\pi,$$

where

$$S_{2s-2t}=z_1^{2s-2t}+z_2^{2s-2t}=(2\pi)^{2s-2t}+\pi^{2s-2t}.$$

Consider

$$\left(2\pi^2\right)^{2t}S_{2s-2t}=2^{2t}\pi^{4t}\left(2^{2s-2t}\pi^{2s-2t}+\pi^{2s-2t}\right)=\pi^{2t+2s}\left(2^{2s}+2^{2t}\right).$$

Thus,

$$\sum_{k=0}^{\infty}\frac{\left(2\pi^2\right)^{2k}}{[(2k)!]^2}+\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty}\frac{(-1)^{t+s}\pi^{2t+2s}(2^{2t}+2^{2s})}{(2t)!\,(2s)!}=-1.$$

Notice [18, formula 5.2.10.3], that

$$\sum_{k=0}^{\infty}\frac{\left(2\pi^2\right)^{2k}}{[(2k)!]^2}=\frac{1}{2}\left[J_0\left(2\sqrt{2\pi^2}\right)+I_0\left(2\sqrt{2\pi^2}\right)\right],$$

where $J_0\left(2\sqrt{2\pi^2}\right)$ is the Bessel function of the first kind and $I_0\left(2\sqrt{2\pi^2}\right)$ is the modified Bessel function of the first kind (Bessel function of the imaginary argument). Thus, we have

$$\sum_{t=0}^{\infty}\sum_{s=t+1}^{\infty}\frac{(-1)^{t+s}\pi^{2t+2s}\left(2^{2t}+2^{2s}\right)}{(2t)!\,(2s)!}=-1-\frac{1}{2}\left[J_0\left(2\sqrt{2\pi^2}\right)+I_0\left(2\sqrt{2\pi^2}\right)\right].$$

Problem 3. Consider the system of equations

$$\begin{cases} f(z) = \left(z - \frac{\pi}{2}\right)\left(z + \frac{\pi}{2}\right) = z^2 - \frac{\pi^2}{4}, \\ g(z) = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots. \end{cases}$$

Since in this problem the expansion of the entire function $g(z)$ contains only coefficients of odd powers, the formula for the resultant from Theorem 2 takes the form:

$$R(f,g) = \sum_{k=0}^{\infty} b_{2k+1}^2 a_0^{2k+1} + \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} b_{2t+1} b_{2s+1} a_0^{2t+1} S_{2s+1-(2t+1)}.$$

Using this relation and the definition of the resultant in the form of a formula for the product, we obtain

$$\sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k+1)!}\right)^2 \left(-\frac{\pi^2}{4}\right)^{2k+1} +$$
$$+ \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} \frac{(-1)^t}{(2t+1)!} \frac{(-1)^s}{(2s+1)!} \left(-\frac{\pi^2}{4}\right)^{2t+1} S_{2s-2t} = \sin\left(\frac{\pi}{2}\right) \sin\left(-\frac{\pi}{2}\right).$$

Here

$$S_{2s-2t} = z_1^{2s-2t} + z_2^{2s-2t} = \left(\frac{\pi}{2}\right)^{2s-2t} + \left(-\frac{\pi}{2}\right)^{2s-2t} = 2\left(\frac{\pi}{2}\right)^{2s-2t}.$$

Thus,

$$-\sum_{k=0}^{\infty} \frac{1}{[(2k+1)!]^2} \left(\frac{\pi}{2}\right)^{4k+2} - 2\sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} \frac{(-1)^{t+s}}{(2t+1)!\,(2s+1)!} \left(\frac{\pi}{2}\right)^{2t+2s+2} = -1.$$

It is known [18, formula 5.2.10.5] that

$$\sum_{k=0}^{\infty} \frac{1}{[(2k+1)!]^2} \left(\frac{\pi}{2}\right)^{4k+2} = \frac{1}{2}\left[I_0(\pi) - J_0(\pi)\right],$$

where $J_0(\pi)$ is the Bessel function of the first kind and $I_0(\pi)$ is the modified Bessel function of the first kind (Bessel function of the imaginary argument). Therefore,

$$2\sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} \frac{(-1)^{t+s}}{(2t+1)!\,(2s+1)!} \left(\frac{\pi}{2}\right)^{2t+2s+2} = 1 - \frac{1}{2}\left[I_0(\pi) - J_0(\pi)\right].$$

5. Conclusion

This article provides an approach to finding sums of multiple numerical series. The main method for obtaining these relations is the formula for the resultant from [14].

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