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# Numerical Solution of Fractional Order Fredholm Integro-differential Equations by Spectral Method with Fractional Basis Functions

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**Abstract.** This paper introduces a new numerical technique based on the implicit spectral collocation method and the fractional Chelyshkov basis functions for solving the fractional Fredholm integro-differential equations. The framework of the proposed method is to reduce the problem into a nonlinear system of equations utilizing the spectral collocation method along with the fractional operational integration matrix. The obtained algebraic system is solved using Newton's iterative method. Convergence analysis of the method is studied. The numerical examples show the efficiency of the method on the problems with non-smooth solutions.

**Keywords:** fractional integro-differential equations, fractional order Chelyshkov polynomials, spectral collocation method, convergence analysis

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Научная статья

# Численное решение интегро-дифференциальных уравнений Фредгольма дробного порядка спектральным методом с дробными базисными функциями

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**Аннотация**. Представлен эффективный спектральный метод решения дробных интегро-дифференциальных уравнений Фредгольма. Вводится неявный метод спектральной коллокации, основанный на дробных базисных функциях Челышкова. Суть метода заключается в сведении задачи к нелинейной системе уравнений с использованием метода спектральной коллокации наряду с матрицей дробного операторного интегрирования. Полученная алгебраическая система решается с использованием итерационного метода Ньютона. Исследуется анализ сходимости метода. На численных примерах показана эффективность метода на задачах с негладкими решениями.

**Ключевые слова:** дробные интегро-дифференциальные уравнения, полиномы Челышкова дробного порядка, метод спектральной коллокации, анализ сходимости

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## 1. Introduction

Fredholm integro-differential equations appear in modeling various physical processes such as neutron transport problems [12], neural networks [10], population model [24], filtering and scattering problems [7]. Fractional differential equations are essential tools in the mathematical modeling of some real-phenomena problems with memory [8]. Theoretical and numerical analysis of fractional differential equations has been considered by many researchers [6]. Also, many applications of fractional differential equations can be found in bio-mathematical modeling [9; 15–18]. Consider the fractional Fredholm integro-differential equations of the form

$$\begin{cases} D^{\alpha}_{*0}y(x) = g(x) + \int_0^1 k(x,t)f(t,y(t))dt, \quad 0<\alpha<1, \ x\in\Omega=[0,1], \\ y(0)=c, \quad c\in\mathbb{R}, \end{cases} \tag{1.1}$$

where the function $g \in C(\Omega)$ and $k \in C(\Omega\times\Omega)$ are known. Assume that $f$ is continuous and satisfies the following Lipschitz condition argument; i.e.,

$$|f(x,y_2)-f(x,y_2)| \le L_f|y_2-y_1|, \quad x\in\Omega, \qquad (L_f>0). \tag{1.2}$$

The operator $D^{\alpha}_{*0}$ denotes the Caputo fractional differential operator [6] with $0 < \alpha < 1$ (see Definition 2 in Section 2).

Orthogonal basis polynomials are the main tool in constructing approximate solutions in spectral methods. Recently, some numerical methods have been introduced by modifying the standard basis polynomials by using the change of $x$ to $x^{\nu}$, $(0 < \nu < 1)$ [1; 3; 22; 23]. The existence, uniqueness, and smoothness of the solution of (1.1) are investigated. The framework of this paper is to convert the problem into a fractional nonlinear integral equation and present a high-order implicit collocation method for its numerical solution. Because of the non-smooth behavior of the solutions of (1.1), we utilize the fractional Chelyshkov polynomials of the form

$$\widetilde{C}_{N,i,\nu}(x) := C_{N,i}(x^{\nu}), \quad 0 < \nu \leq 1. \tag{1.3}$$

where $\{C_{N,i}(x)\}_{i=0}^{N}$ is a set of orthogonal Chelyshkov polynomials on $[0, 1]$ [5]. The Chelyshkov polynomials have been used based on the spectral method to solve various types of differential and integral equations [20; 21]. The advantage of the method is to reduce the given problem into an algebraic system of equations. Simplicity in calculating the fractional operational matrix and high accuracy of the method in solving problems with non-smooth solutions by selecting a smaller number of fractional Chelyshkov polynomials are the other main advantage of our method.

## 2. Fractional Chelyshkov polynomials

The fractional Chelyshkov polynomials [5] defined as follows

$$\widetilde{C}_{N,n,\nu}(x) = \sum_{j=n}^{N} (-1)^{j-n} \binom{N-n}{j-n} \binom{N+j+1}{N-n} x^{j\nu}, \quad n = 0, 1, ..., N, \tag{2.1}$$

in which the notation $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ stands for binomial coefficient.

**Lemma 1.** *[21] Let $x_i$ be the roots of $\widetilde{C}_{N+1,0,1}(x)$. Therefore, the fractional Chelyshkov polynomial $\widetilde{C}_{N+1,0,\nu}(x)$ has $N+1$ roots as $x_i^{\frac{1}{\nu}}$ for $i = 1, ..., N+1$.*

**Theorem 1.** *Let $\mathbf{\Phi}(x) := [\widetilde{C}_{N,0,\nu}(x), \widetilde{C}_{N,1,\nu}(x), ..., \widetilde{C}_{N,N,\nu}(x)]^T$ be the fractional Chelyshkov polynomials vector. Then, $\int_0^x (x-s)^{\alpha-1}\mathbf{\Phi}(s)ds \simeq \mathcal{P}\mathbf{\Phi}(x)$, where* $\mathcal{P} = \begin{pmatrix} \Theta(0,0) & \Theta(0,1) & \ldots & \Theta(0,0) \\ \Theta(1,0) & \Theta(1,1) & \cdots & \Theta(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \Theta(N,0) & \Theta(N,1) & \ldots & \Theta(N,N) \end{pmatrix}$, *with*

$$\Theta(n,k) = \sum_{j=n}^{N} (-1)^{j-n} \binom{N-n}{j-n} \binom{N+j+1}{N-n} B(\alpha, j\nu+1)\xi_{k,j}, \tag{2.2}$$

*is called the fractional operational matrix of integration. Here, $B(\cdot,\cdot)$ denotes the Beta function.*

*Proof.* Integrating of $\widetilde{C}_{N,n,\nu}(x)$ from 0 to $x$ yields

$$\int_0^x (x-s)^{\alpha-1}\widetilde{C}_{N,n,\nu}(s)ds = \\ = \sum_{j=n}^{N}(-1)^{j-n}\binom{N-n}{j-n}\binom{N+j+1}{N-n}B(\alpha, j\nu+1)x^{j\nu+\alpha}. \quad (2.3)$$

Approximating $x^{j\nu+\alpha}$ in terms of fractional Chelyshkov polynomials, we get

$$x^{j\nu+\alpha} \simeq \sum_{k=0}^{N}\xi_{k,j}\widetilde{C}_{N,k,\nu}(x), \quad (2.4)$$

where the coefficients $\xi_{k,j}$ can be computed as follows

$$\xi_{k,j} = \nu(2k+1)\int_0^1 x^{j\nu+\alpha}\widetilde{C}_{N,k,\nu}(x)\varpi(x)dx, \\ = \nu(2k+1)\sum_{l=k}^{N}\frac{(-1)^{l-k}}{(j+l+1)\nu+\alpha}\binom{N-k}{l-k}\binom{N+l+1}{N-k}. \quad (2.5)$$

where $\varpi(x) = x^{\nu-1}$. By substituting (2.4), (2.5) in (2.3), we have

$$\int_0^x (x-s)^{\alpha-1}\widetilde{C}_{N,n,\nu}(s)ds \simeq \sum_{k=0}^{N}\Theta(n,k)\widetilde{C}_{N,k,\nu}(x). \quad (2.6)$$

□

Now, we study the existence, uniqueness and smoothness of the solution to the problem (1.1):

**Definition 1.** *[6] The Riemann–Liouville fractional integral of order $\alpha$ for any $u \in L_1[a,b]$ is defined as $J_a^\alpha u(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)dt$. For $\alpha = 0$, we have $J_a^0 := I$ the identity operator.*

**Definition 2.** *[6] The operator $D_{*a}^\alpha$ defined by*

$$D_{*a}^\alpha u(x) := J_a^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}u(x) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_a^x (x-t)^{\lceil\alpha\rceil-\alpha-1}u^{\lceil\alpha\rceil}(t)dt \quad (2.7)$$

*is called the Caputo differential operator of order $\alpha \in \mathbb{R}_+$.*

To study the regularity properties of the exact solution of the problem (1.1), we introduce the weighted space $C^{m,\lambda}(0,1]$:

**Definition 3.** *[4] For given $m \in \mathbb{N}$ and $-\infty < \lambda < 1$, the space $C^{m,\lambda}(0,1]$ is a set of all $m$ times continuously differentiable functions $u : (0,1] \to \mathbb{R}$ such that for all $x \in (0,1]$ and $i = 0, ..., m$, the following estimate holds*

$$|u^{(i)}(x)| \leq c \begin{cases} 1, & i < 1-\lambda, \\ 1 + |\log(x)|, & i = 1-\lambda, \\ x^{1-\lambda-i}, & i > 1-\lambda. \end{cases}$$

*Here c for is a positive constant.*

By applying fractional integral operator $J_0^\alpha$ on both sides of the Eq. (1.1), we obtain

$$y(x) = \widetilde{g}(x) + \int_0^x (x-t)^{\alpha-1} \widetilde{K}(y(t))dt, \tag{2.8}$$

where

$$\widetilde{K}y(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 k(x,z) f(z, y(z)) dz, \quad \widetilde{g}(x) = c + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t) dt. \tag{2.9}$$

Therefore, the problem (1.1) is equivalent to a fractional nonlinear Volterra integral equation of the form Eq. (2.8). In the following theorems, we consider the existence, uniqueness and smoothness of the solution of Eq. (2.8).

**Theorem 2.** *Assume that the function $f$ satisfies the Lipschitz condition (1.2) and $\mathcal{M}_k := \max_{x,t \in \Omega} |k(x,t)|$. If $L_f \mathcal{M}_k < \Gamma(\alpha+1)$, then, the Eq. (2.8) has a unique continuous solution on $\Omega$.*

**Theorem 3.** *Let $\widetilde{g} \in C^{m,1-\alpha}(\Omega)$, $\widetilde{K}y \in C^m(\mathbb{R})$, $m \in \mathbb{N}$ and $0 < \alpha < 1$. Then, the solution of Eq. (2.8) satisfy the smoothness properties as follows $|y^{(i)}(x)| = O(x^{\alpha-i}), \quad x \in (0,1]$, for $i = 1, ..., m$.*

*Proof.* For the proof see Theorem 2.1 in [4]. □

The result of Theorem 3 implies that the solution of Eq. (2.8) has a singularity at the origin as $x \to 0^+$, which indicates deterioration of the accuracy of the existing spectral methods based on standard basis functions such as Chebyshev, Legendre, etc. To overcome such drawbacks, we utilize the fractional Chelyshkov polynomials as basis functions to obtain a consistency between the approximate and exact solutions.

## 3. Numerical method

According to the implicit version of the collocation method in [11], we consider the transformation

$$w(x) = \widetilde{K}y(x), \tag{3.1}$$

to approximate the solutions of Eq. (2.8). By using (2.8) and (2.9), the following equation holds

$$w(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 k(x,z)f\left(z, \widetilde{g}(z) + \int_0^z (z-t)^{\alpha-1}w(t)dt\right)dz. \tag{3.2}$$

After determining the unknown function $w$, then we can obtain a solution of the Eq. (2.8) by

$$y(x) = \widetilde{g}(x) + \int_0^x (x-t)^{\alpha-1}w(t)dt. \tag{3.3}$$

Now, we focus on the implementation of a spectral collocation method to solve Eq. (3.2). By approximating $w(x)$ in terms of fractional Chelyshkov polynomials we have

$$w(x) \simeq w_N(x) = \sum_{i=0}^{N} w_i \widetilde{C}_{N,i,\nu}(x) = \mathbf{W}_N\mathbf{\Phi}(x), \tag{3.4}$$

where, $\mathbf{W}_N = [w_0, ..., w_N]$ is an unknown vector.

**Theorem 4.** *Let $w_N$ be the approximation defined in (3.4) and $\mathcal{P}$ be the fractional operational matrix defined in Theorem 1, then we have*

*(A)* $\displaystyle\int_0^x (x-t)^{\alpha-1}w_N(t)dt \simeq \widehat{\mathbf{W}}_N\mathbf{\Phi}(x); \quad \widehat{\mathbf{W}}_N = \mathbf{W}_N\mathcal{P}.$

*(B)* $\widetilde{g}(x) \simeq (\mathcal{C}+\mathcal{G})\mathbf{\Phi}(x)$, *in which* $\mathcal{C} := [\mathcal{C}_0, ..., \mathcal{C}_N]$, $\widetilde{\mathbf{G}} := [g_0, ..., g_N]$, *with*

$$\mathcal{C}_i = \frac{c(2i+1)}{N+1}, \quad g_i = \frac{\nu(2i+1)}{\Gamma(\alpha)}\int_0^1 \widetilde{C}_{N,i,\nu}(x)g(x)\varpi(x)dx, \quad \mathcal{G} = \widetilde{\mathbf{G}}\mathcal{P}.$$

*Proof.* Part (A): From Theorem 1, we have

$$\int_0^z (z-t)^{\alpha-1}w_N(t)dt = \mathbf{W}_N\int_0^z (z-t)^{\alpha-1}\mathbf{\Phi}(t)dt \simeq \mathbf{W}_N\mathcal{P}\mathbf{\Phi}(x) = \widehat{\mathbf{W}}_N\mathbf{\Phi}(z).$$

Part (B): Considering (2.9) and letting $\frac{1}{\Gamma(\alpha)}g(x) \simeq \widetilde{\mathbf{G}}\mathbf{\Phi}(x)$, $c = \mathcal{C}\mathbf{\Phi}(x)$ we get

$$\mathcal{C}_i = \nu(2i+1)\int_0^1 c\, \widetilde{C}_{N,i,\nu}(x)\varpi(x)dx = \frac{c(2i+1)}{N+1},$$

and

$$g_i = \frac{\nu(2i+1)}{\Gamma(\alpha)} \int_0^1 \widetilde{C}_{N,i,\nu}(x) g(x) \varpi(x) dx.$$

By using Theorem 1, we can write

$$\widetilde{g}(x) = \simeq \mathcal{C}\boldsymbol{\Phi}(x) + \widetilde{\mathbf{G}} \int_0^x (x-t)^{\alpha-1} \boldsymbol{\Phi}(t) dt = \left(\mathcal{C} + \widetilde{\mathbf{G}}\mathcal{P}\right) \boldsymbol{\Phi}(x) = (\mathcal{C} + \mathcal{G})\, \boldsymbol{\Phi}(x),$$

which completes the proof. □

Assume that $\ell_i^{\nu}(x)$ be the fractional Lagrange basis function associated with the points $\widehat{x}_i = x_i^{\frac{1}{\nu}}$, $i = 0, ..., N$, the roots of the polynomial $\widetilde{C}_{N+1,0}(x)$, and define the fractional interpolation operator as

$$\mathcal{I}_N u(x) = \sum_{i=0}^{N} u(\widehat{x}_i) \ell_i^{\nu}(x); \quad \ell_i^{\nu}(x) = \prod_{j=0,\ j \neq i}^{N} \frac{x - \widehat{x}_j}{\widehat{x}_i - \widehat{x}_j}. \tag{3.5}$$

By substituting $w_N(x)$ in Eq. (3.2) and applying $\mathcal{I}_N$ we obtain

$$\mathcal{I}_N w_N(x) = \mathcal{I}_N \left( \frac{1}{\Gamma(\alpha)} \int_0^1 k(x,z) f\left(z, \widetilde{g}(z) + \int_0^z (z-t)^{\alpha-1} w_N(t) dt\right) dz \right), \tag{3.6}$$

consequently,

$$w_N(\widehat{x}_i) = \frac{1}{\Gamma(\alpha)} \int_0^1 k(\widehat{x}_i, z) f\left(z, \widetilde{g}(z) + \int_0^z (z-t)^{\alpha-1} w_N(t) dt\right) dz. \tag{3.7}$$

By using Theorem 4 in Eq. (3.7), we obtain

$$\mathbf{W}_N \boldsymbol{\Phi}(\widehat{x}_i) = \frac{1}{\Gamma(\alpha)} \int_0^1 k(\widehat{x}_i, z) f\left(z, \left(\mathcal{C} + \mathcal{G} + \widehat{\mathbf{W}}_N\right) \boldsymbol{\Phi}(z)\right) dz. \tag{3.8}$$

The integral term in (3.8) is approximated by the Gauss-Legendre quadrature formula on $[0,1]$ with the weights and nodes $(z_\ell, \omega_\ell)_{\ell=0}^{N}$,

$$\int_0^1 k(\widehat{x}_i, z) \mathcal{H}(z) dz \simeq$$

$$\simeq \sum_{\ell=0}^{N} \omega_\ell k(\widehat{x}_i, z_\ell) \mathcal{H}(z_\ell); \quad \mathcal{H}(z) = \frac{1}{\Gamma(\alpha)} f\left(z, \left(\mathcal{C} + \mathcal{G} + \widehat{\mathbf{W}}_N\right) \boldsymbol{\Phi}(z)\right) \tag{3.9}$$

By substituting Eq. (3.9) in Eq. (3.8), we obtain

$$\mathbf{f}_i(\mathbf{W}_N) = \mathbf{W}_N \boldsymbol{\Phi}(\widehat{x}_i) - \sum_{\ell=0}^{N} \omega_\ell k(\widehat{x}_i, z_\ell) \mathcal{H}(z_\ell) = 0. \tag{3.10}$$

Therefore,

$$\mathbb{F}_N(\mathbf{W}_N) = \left[\mathbf{f}_0(\mathbf{W}_N), ..., \mathbf{f}_N(\mathbf{W}_N)\right] \equiv \mathbf{0}, \tag{3.11}$$

which gives a nonlinear algebraic system that can be solved by Newton's iterative method. The approximate solution for Eq. (2.8) is obtained in the following form

$$y_N(x) = (\mathcal{C} + \mathcal{G} + \mathbf{W}_N\mathcal{P})\, \mathbf{\Phi}(x). \tag{3.12}$$

Newton's method reads as follows:

$$\begin{cases} \mathbb{J}(\mathbf{W}_{N,i})\delta_{N,i} = -\mathbb{F}_N(\mathbf{W}_{N,i}); \\ \mathbf{W}_{N,i+1} \leftarrow \mathbf{W}_{N,i} + \delta_{N,i}, \\ i \leftarrow i+1, \end{cases} \tag{3.13}$$

in which $\delta_{N,i} = \mathbf{W}_{N,i+1} - \mathbf{W}_{N,i}$ with initial guess $\mathbf{W}_{N,0}$ and end condition $\|\mathbb{F}_N(\mathbf{W}_{N,i})\|_{\infty,N} \leq \epsilon$, where $\epsilon > 0$ be a small enough number. The norm $\|\cdot\|_{\infty,N}$ is a vector norm defined by $\|U\|_{\infty,N} := \max\limits_{0\leq i\leq N}\{|U_i|\}$ where $U = [U_0, ..., U_N]$. The Jacobian matrix $\mathbb{J}$ is defined as $\mathbb{J}_{i,j} = \frac{\partial \mathbf{f}_i}{\partial w_j}$. By applying the iterative process (3.13), a sequence of approximate solutions $w_{N,i}(x) = \mathbf{W}_{N,i}\mathbf{\Phi}(x)$, $i = 0, 1, 2, ...$, is generated. It can be seen that for $\|w_{N,i} - w_N\|_\infty \to 0$, the Jacobian matrix should be nonsingular. In the next section, we state the convergence results for Newton's method. To select a proper initial guess for Newton's method, by using the initial condition $y_N(0) = c = \mathcal{C}\mathbf{\Phi}(0)$ and Eq. (3.12), we choose the initial guess such that $y_N(0) = (\mathcal{C} + \mathcal{G} + \mathbf{W}_{N,0}\mathcal{P})\, \mathbf{\Phi}(0) = \mathcal{C}\mathbf{\Phi}(0)$. Since $\mathcal{G} = \widetilde{\mathbf{G}}\mathcal{P}$, we conclude that $\mathbf{W}_{N,0} = -\widetilde{\mathbf{G}}$. Now we obtain an upper bound for the error vector of the fractional integration operational matrix and analyze the convergence of the method.

**Theorem 5.** *(Generalized Taylor series [19]) Let* $D^{i\nu}_{*,0}u(x) \in C(0,1]$, $i = 0, ..., N+1$, *where* $0 < \theta < 1$. *Then,*

$$u(x) = \sum_{i=0}^{N} \frac{D^{i\nu}_{*,0}u(0)}{\Gamma(i\nu+1)}x^{i\nu} + \frac{x^{(N+1)\nu}}{\Gamma((N+1)\nu+1)}D^{(N+1)\nu}_{*,0}u(x)|_{x=\xi}, \tag{3.14}$$

*with* $0 < \xi \leq x$, $\forall x \in (0,1]$.

**Theorem 6.** *Let* $D^{i\nu}_{*,0}u(x) \in C(0,1]$, $i = 0, ..., N+1$, $0 < \nu < 1$ *and* $\widehat{u}_N(x) = \sum_{n=0}^{N} a_n\widetilde{C}_{N,n,\nu}(x)$ *be the best approximation to* $u(x)$ *out of* $M_N$. *Then,*

$$\|u - \widehat{u}_N\|_\infty \leq \frac{\mathcal{N}_\nu}{\Gamma((N+1)\nu+1)}; \quad \mathcal{N}_\nu := \max_{x\in[0,1]}|D^{(N+1)\nu}_{*,0}u(x)| \tag{3.15}$$

*in which* $\|\cdot\|_\infty$ *stands for the* $L^\infty$*-norm*

$$\|u\|_\infty := \max\{|u(x)| : x \in [0,1]\}. \tag{3.16}$$

*Proof.* From Theorem 5, we have

$$\|u-\widehat{u}_N\|_\infty \leq$$
$$\leq \|u-\sum_{i=0}^{N}\frac{D_{*,0}^{i\nu}u(0)}{\Gamma(i\nu+1)}x^{i\nu}\|_\infty \leq \|\frac{x^{(N+1)\nu}\mathcal{N}_\nu}{\Gamma((N+1)\nu+1)}\|_\infty \leq \frac{\mathcal{N}_\nu}{\Gamma((N+1)\nu+1)}. \tag{3.17}$$

□

**Theorem 7.** *Let $\mathcal{E}(x)=[e_0(x),...,e_N(x)]:=\int_0^x(x-s)^{\alpha-1}\mathbf{\Phi}(s)ds-\mathcal{P}\mathbf{\Phi}(x)$ be the error vector related to $\mathcal{P}$. Then, $\|e_n\|_\infty\to 0$ as $N\to\infty$ for $n=0,...,N$.*

*Proof.* From relations (2.2)-(2.4), we have

$$e_n(x)=\int_0^x(x-s)^{\alpha-1}\widetilde{C}_{N,n,\nu}(s)ds-\sum_{k=0}^{N}\Theta(n,k)\widetilde{C}_{N,k,\nu}(x)=$$
$$=\sum_{j=n}^{N}(-1)^{j-n}\binom{N-n}{j-n}\binom{N+j+1}{N-n}B(\alpha,j\nu+1)\left(x^{j\nu+\alpha}-\sum_{k=0}^{N}\xi_{k,j}\widetilde{C}_{N,k,\nu}(x)\right), \tag{3.18}$$

for $n=0,1,...,N$. On the other hand, from Theorem 6, we have

$$\|x^{j\nu+\alpha}-\sum_{k=0}^{N}\xi_{k,j}\widetilde{C}_{N,k,\nu}(x)\|_\infty\to 0,\quad N\to\infty. \tag{3.19}$$

Therefore, we can conclude that $\|e_n\|_\infty\to 0$ as $N\to\infty$ for $n=0,...,N$. □

**Theorem 8.** *Assume that $y(x)$ and $y_N(x)$ are the exact solution and approximate solution of (1.1), respectively. Then, $\|y-y_N\|_\infty\to 0,\quad N\to\infty$.*

*Proof.* From (3.12) and (3.3), we can get

$$\|y-y_N\|_\infty\leq\|\widetilde{g}(x)-\big(\mathcal{C}+\mathcal{G}\big)\mathbf{\Phi}(x)\|_\infty+\|\int_0^x(x-t)^{\alpha-1}\big(w(t)-w_N(t)\big)dt\|_\infty. \tag{3.20}$$

On the other hand,

$$w(x)-w_N(x)=\frac{1}{\Gamma(\alpha)}\int_0^1k(x,z)f\left(z,\widetilde{g}(z)+\int_0^z(z-t)^{\alpha-1}w(t)dt\right)dz$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^1k(x,z)f\Big(z,\big(\mathcal{C}+\mathcal{G}+\widehat{\mathbf{W}}_N\big)\mathbf{\Phi}(z)\Big)dz, \tag{3.21}$$

and from (1.2), we can write

$$|w(x)-w_N(x)| \leq$$
$$\leq \frac{L_f\mathcal{M}_k}{\Gamma(\alpha)}\bigg(\int_0^1 |\widetilde{g}(z)-\big(\mathcal{C}+\mathcal{G}\big)\mathbf{\Phi}(z)|dz+\int_0^1 |\int_0^z (z-t)^{\alpha-1}w(t)dt-\widehat{\mathbf{W}}_N\mathbf{\Phi}(z)|dz\bigg). \tag{3.22}$$

So, by using Theorems 6 and 7 in (3.22), we can obtain the desired result. □

Now, we discuss the conditions under which Newton's iterative method (3.13) is convergent. Consider the operator form of Eq. (3.11) as follows

$$\mathcal{F}_N(w_N) = w_N - \mathcal{I}_N\mathcal{K}_N w_N \equiv 0, \tag{3.23}$$

where $\mathcal{K}_N$ is an approximate quadrature of integral operator $\mathcal{K}$ defined as

$$\mathcal{K}w(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 k(x,z)f\bigg(z,\widetilde{g}(z)+\int_0^z (z-t)^{\alpha-1}w(t)dt\bigg)dz.$$

The Frechet derivative of $\mathcal{F}_N$ at $w_N$ is defined as $\mathcal{F}'_N(w_N)(v) = v - \mathcal{I}_N\mathcal{K}'_N(w_N)(v)$, in which $\mathcal{K}'_N(w)(v) = \frac{1}{\Gamma(\alpha)}\sum_{\ell=0}^{N}\omega_\ell k(x,z_\ell)f'\bigg(z,\big(\mathcal{C}+\mathcal{G}+\widehat{\mathbf{W}}_N\big)\mathbf{\Phi}(z_\ell)\bigg)v(z_\ell)$, and $f' := f_y(x,y) \in C(\Omega)$. From Lemma 2.2 in [14], we can conclude that if $\|\mathcal{I}_N\mathcal{K}'_N w_N - \mathcal{K}'w\|_\infty \to 0$ as $N\to\infty$ and $\mathcal{K}'w$ has no eigenvalue equal to 1, then $[I-\mathcal{I}_N\mathcal{K}'_N w_N]$ is invertible. To this end, assume that the following conditions hold: (R1)$|f_y(x,y)-f_y(x',y)| \leq C_1|x-x'|^\beta$, (R2) $|f_y(x,u)-f_y(x,v)| \leq C_2|u-v|$, where $C_i$ are positive constants. According to triangular inequality, we get

$$\begin{aligned}\|\mathcal{I}_N\mathcal{K}'_N w_N - \mathcal{K}'w\|_\infty &\leq \|\mathcal{I}_N\mathcal{K}'w - \mathcal{K}'w\|_\infty + \|\mathcal{I}_N\mathcal{K}'_N w_N - \mathcal{I}_N\mathcal{K}'_N w\|_\infty \\ &+ \|\mathcal{I}_N\mathcal{K}'_N w - \mathcal{I}_N\mathcal{K}'w\|_\infty.\end{aligned} \tag{3.24}$$

From Theorem(6) and that $\mathcal{I}_N\mathcal{K}'w \in M_N$ is the best approximation to $\mathcal{K}'w$, under condition (R1), we conclude that $\|\mathcal{I}_N\mathcal{K}'w-\mathcal{K}'w\|_\infty \to 0, \quad N\to\infty$. Using conditions (R2) and (3.22), we have $\|\mathcal{I}_N\mathcal{K}'_N w_N - \mathcal{I}_N\mathcal{K}'_N w\|_\infty \leq \|\mathcal{I}_N\|_\infty\|w_N-w\|_\infty \to 0$ as $N\to\infty$. From Theorem 1 in [13] and integration error estimation from the Gauss-Legendre quadrature role one can show that $\|\mathcal{I}_N\mathcal{K}'_N w - \mathcal{I}_N\mathcal{K}'w\|_\infty \to 0$ as $N\to\infty$. In the following theorem, we deal with the local convergence of Newton's method:

**Theorem 9.** *Assume that $w_N$ is the solution of Eq. (3.23) and $[I-\mathcal{K}'w]^{-1}$ exists. Assume further that the conditions (R1)-(R2) be held. If there exist a $\epsilon > 0$ such that $\|w_{N,0}-w_N\|_\infty \leq \epsilon$, then Newton's method (3.13) is converges. Furthermore, $\|w_{N,i}-w_N\|_\infty \leq \frac{(r\epsilon)^{2^i}}{r}$, provided that $r\epsilon < 1$ for some constant $r$.*

*Proof.* If 1 is not the eigenvalue of $\mathcal{K}'w$, then $[I - \mathcal{K}'w]$ is invertible. The proof is straightforward from Theorem 5.4.1 in [2] and the above discussion. □

## 4. Numerical examples

In this section, we intend to show the accuracy of the proposed method to solve the problem (1.1) with non-smooth solutions. The $L^\infty$-norm of error function $E_N(x) = |y(x) - y_N(x)|$ is computed in all examples that is defined in (3.16). In these examples, $m$ denotes the number of Newton's iterations with initial value $W_{N,0} = -\widetilde{\mathbf{G}}$.

The steps of the numerical method can be summarized as follows:

**Input:** Input $N$, $\alpha$, $\nu$, $k$, $f$ and $g$.

**Output:** The approximate solution $y_N(x) = \widetilde{g}(x) + \mathbf{W}\mathcal{P}\mathbf{\Phi}(x)$.

**Step 1.** Construct the vector basis $\mathbf{\Phi}(t)$.

**Step 2.** Compute the vectors $\mathcal{C}, \mathcal{G}, \widehat{\mathbf{W}}$ from Theorem 4.

**Step 3.** Construct the nonlinear algebraic system (3.11).

**Step 4.** Solve the system (3.11) using Newton's iterative method.

**Example 1.** Consider the problem $\begin{cases} D_{*0}^{\frac{1}{2}} y(x) = \dfrac{\sqrt{\pi}}{2} - \dfrac{1}{4} + \dfrac{1}{2}\displaystyle\int_0^1 y^2(t)dt, \\ y(0) = 0. \end{cases}$

with the non-smooth solution $y(x) = \sqrt{x}$. Using relations (2.8)-(2.9), we obtain the equivalent nonlinear integral equation

$$y(x) = \frac{\sqrt{x}(\sqrt{\pi} - \frac{1}{2})}{\sqrt{\pi}} + \int_0^x (x-t)^{\frac{-1}{2}} \widetilde{K}(y(t))dt; \quad \widetilde{K}y(x) = \frac{1}{\sqrt{\pi}} \int_0^1 y^2(z)dz \tag{4.1}$$

From (3.2), we have $w(x) = \frac{1}{\sqrt{\pi}} \int_0^1 \left( \dfrac{\sqrt{z}(\sqrt{\pi} - \frac{1}{2})}{\sqrt{\pi}} + \displaystyle\int_0^z (z-t)^{-\frac{1}{2}} w(t)dt \right) dz$.

Now, we apply our method with $N = 1$. From **Step 1.** let

$$\begin{aligned} w_1(x) = w_0 \widetilde{C}_{1,0,1/2}(x) + w_1 \widetilde{C}_{1,1,1/2}(x) = \mathbf{W}_1 \mathbf{\Phi}(x); \\ \mathbf{W}_1 = [w_0, w_1], \ \mathbf{\Phi}(x) = [2 - 3\sqrt{x}, \sqrt{x}]^T. \end{aligned} \tag{4.2}$$

From **Step 2.**,

$$\mathcal{P} = \begin{bmatrix} \dfrac{\pi}{8} & 4 - \dfrac{9\pi}{8} \\ \dfrac{-\pi}{24} & \dfrac{3\pi}{8} \end{bmatrix}, \quad \widehat{\mathbf{W}}_1 = \mathbf{W}_1 \mathcal{P} = [\frac{\pi}{8} w_0 - \frac{\pi}{24} w_1, (4 - \frac{9\pi}{8}) w_0 + \frac{3\pi}{8} w_1],$$

$$\mathcal{C}=[0,0],\quad \widetilde{\mathbf{G}}=[\frac{2\sqrt{\pi}-1}{8\sqrt{\pi}},\frac{6\sqrt{\pi}-3}{8\sqrt{\pi}}],\quad \mathcal{G}=\widetilde{\mathbf{G}}\mathcal{P}=[0,1-\frac{1}{2\sqrt{\pi}}],$$

$$\mathcal{H}(z)=\frac{1}{\sqrt{\pi}}\Big((4\sqrt{x}-\frac{3\pi\sqrt{x}}{2}+\frac{\pi}{4})w_1+(\frac{\pi\sqrt{x}}{2}-\frac{\pi}{12})w_2+\sqrt{x}-\frac{\sqrt{x}}{2\sqrt{\pi}}\Big)^2.$$

From **Step 3.**, we obtain we obtain $\mathbf{f}_i(\mathbf{W}_1)=\mathbf{W}_1\mathbf{\Phi}(\widehat{x}_i)-\int_0^1\mathcal{H}(z)dz=0$, consequently, $\mathbb{F}_1(\mathbf{W}_1)=\big[\mathbf{f}_0(\mathbf{W}_1),\mathbf{f}_1(\mathbf{W}_1)\big]\equiv 0$, with the collocation points $\widehat{x}_0=\frac{3}{5}+\frac{\sqrt{6}}{10},\widehat{x}_1=\frac{3}{5}-\frac{\sqrt{6}}{10}$. From **Step 4.**, Newton's iterative method with the initial guess $\mathbf{W}_{1,0}=-\widetilde{\mathbf{G}}$ gives

$$\mathbf{W}_{1,0}=\begin{bmatrix}-1.79476\times10^{-1}\\-5.38429\times10^{-1}\end{bmatrix},...,\mathbf{W}_{1,7}=\begin{bmatrix}7.052370\times10^{-2}\\2.115711\times10^{-1}\end{bmatrix},\qquad(4.3)$$

where $\|\mathbb{F}_1(\mathbf{W}_{1,7})\|_{\infty,7}\simeq10^{-40}$. Finally, we obtain an approximate solution as $y_1(x)=\widetilde{g}(x)+\mathbf{W}_1\mathcal{P}\mathbf{\Phi}(x)=\sqrt{x}+1.70\times10^{-40}$.

**Example 2.** Consider the problem

$$\begin{cases}D_{*0}^{\frac{1}{2}}y(x)=g(x)+\displaystyle\int_0^1 sin(x+t)y^2(t)dt,\\ y(0)=0,\end{cases}$$

with the non-smooth solution $y(x)=x^{\frac{1}{2}}-\frac{1}{3!}x^{\frac{3}{2}}+\frac{1}{5!}x^{\frac{5}{2}}$. Table 1, illustrates the $L^2$-errors obtained by our method for different values of $N$ and $\nu=1/4,1/2,3/4,1$, the order of fractional Chelyshkov polynomials denoted in (2.1) with $m=10$. The semi-log representation in Fig. 1 shows the linear variations of the errors versus the degree of approximation in case of $\nu=\frac{1}{2}$. This is so-called exponential convergence or spectral accuracy of the collocation methods that have been recovered in the proposed method for the problems with non-smooth solutions.

## 5. Conclusions

In this paper, a new fractional version of the collocation method has been introduced to solve a class of nonlinear fractional integro-differential equations. Numerical examples illustrate that the obtained results are significant. All calculations are computed by Maple 2018 with Digits=40. The proposed method is computationally simple and the approximate solutions converge to the exact solution of the problem as the number of basis functions increases.

Table 1

The $L^{\infty}$-errors for different values of $\nu$ and $N$ for Example 2

| N | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $\nu = 1/4$ | 6.4516e-02 | 2.7042e-03 | 5.9642e-05 | 1.5752e-07 | 2.2561e-07 |
| $\nu = 1/2$ | 1.2869e-02 | 3.1695e-05 | 2.1710e-06 | 3.8550e-10 | 4.8739e-10 |
| $\nu = 3/4$ | 7.9259e-02 | 3.9917e-02 | 2.5327e-02 | 1.8072e-02 | 1.3812e-02 |
| $\nu = 1$ | 1.7435e-01 | 1.0157e-01 | 7.1987e-02 | 5.5816e-02 | 4.5596e-02 |
| N | 12 | 14 | 16 | 18 | 20 |
| $\nu = 1/4$ | 8.5604e-09 | 3.0335e-10 | 4.1066e-11 | 1.1823e-12 | 3.6870e-14 |
| $\nu = 1/2$ | 1.5453e-12 | 4.7476e-14 | 2.2760e-16 | 2.5102e-18 | 1.5224e-20 |
| $\nu = 3/4$ | 1.1046e-02 | 9.1230e-03 | 7.7183e-03 | 6.6530e-03 | 5.8209e-03 |
| $\nu = 1$ | 3.8547e-02 | 3.3389e-02 | 2.9450e-02 | 2.6343e-02 | 2.3830e-02 |

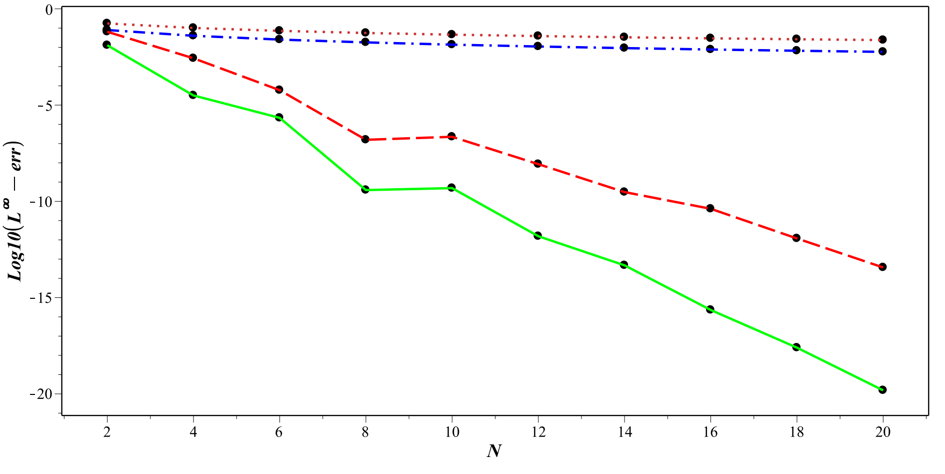


*Figure 1.* The $L^{\infty}$-error for different values of $N$ with $\nu$=1/4 (dashed-lines), $\nu = 1/2$ (solid-line), $\nu = 3/4$ (dashed-dotted-lines) and $\nu = 1$ (dotted-lines) for Example 2.

## References

1. Ardabili J.S., Talaei T. Chelyshkov collocation method for solving the two-dimensional Fredholm-Volterra integral equations. *Int. J. Appl. Comput. Math.*, 2018, vol. 4, art. no. 25. https://doi.org/10.1007/s40819-017-0433-2
2. Atkinson K.E., Han W. *Theoretical numerical analysis; a functional analysis framework*. Springer, New York, 2001.
3. Azizipour G. Shahmorad S. A new Tau-collocation method with fractional basis for solving weakly singular delay Volterra integro-differential equations. *J. Appl. Math. Comput.*, 2022, vol. 68, pp. 2435–2469.

4. Brunner H., Pedas A., Vainikko G. The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. *Math. Comp.*, 1999, vol. 68, pp. 1079–1095.
5. Chelyshkov V.S. Alternative orthogonal polynomials and quadratures. *Electron. Trans. Numer. Anal.*, 2006, vol. 25, pp. 17–26.
6. Diethelm K. *The analysis of fractional differential equations, Lectures notes in mathematics*. Springer, Berlin, 2010.
7. Hale N. An ultraspherical spectral method for linear Fredholm and Volterra integro-differential equations of convolution type. *IMA J. Numer. Anal.*, 2018, pp. 1–20.
8. He J.H. Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Technol.*, 1999, vol. 15, no. 2, pp. 86–90.
9. Hedayati M., Ezzati R., Noeiaghdam S. New Procedures of a Fractional Order Model of Novel coronavirus (COVID-19) Outbreak via Wavelets Method. *Axioms*, 2021, vol. 10, art. no. 122. https://doi.org/10.3390/axioms10020122
10. Jackiewicz Z., Rahman M., Welfert B. D. Numerical solution of a Fredholm integro-differential equation modelling $\theta$-neural networks. *Appl. Math. Comput.*, 2008, vol. 195, pp. 523–536.
11. Kumar S., Sloan I. A new collocation-type method for Hammerstein integral equations. *Math. of Comp.*, 1987, vol. 48, pp. 585–593.
12. Martin O. On the homotopy analysis method for solving a particle transport equation. *Appl. Math. Model.*, 2013, vol. 37, no. 6, pp. 3959–3967.
13. Nevai P. Mean Convergence of Lagrange Interpolation. III. *Trans. Math. Soc.*, 1984, vol. 282, no. 2, pp. 669–988.
14. Anselone P.M. *Collectively compact operator approximation theory.* Printice-Hall, Inc., Engelwood Cliffs, N. J., 1971.
15. Noeiaghdam S., Dreglea A., Isik H., Suleman M. Comparative Study between Discrete Stochastic Arithmetic and Floating-Point Arithmetic to Validate the Results of Fractional Order Model of Malaria Infection. *Mathematics*, 2021, vol. 9, art. no. 1435. https://doi.org/10.3390/math9121435
16. Noeiaghdam S., Micula S., Nieto J.J. Novel Technique to Control the Accuracy of a Nonlinear Fractional Order Model of COVID-19: Application of the CESTAC Method and the CADNA Library. *Mathematics*, 2021, vol. 9, art. no. 1321. https://doi.org/10.3390/math9121321
17. Noeiaghdam S., Micula S. Dynamical Strategy to Control the Accuracy of the Nonlinear Bio-mathematical Model of Malaria Infection. *Mathematics*, 2021, vol. 9, no. 9, art. no. 1031. https://doi.org/10.3390/math9091031
18. Noeiaghdam S., Sidorov D. Application of the stochastic arithmetic to validate the results of nonlinear fractional model of HIV infection for CD8+T-cells. *Mathematical Analysis of Infectious Diseases,* Elsevier, 2022, pp. 259–285. https://doi.org/10.1016/B978-0-32-390504-6.00020-6
19. Odibat Z.M., Shawagfeh N T. Generalized Taylor's formula. *Appl. Math. Comput.*, 2007, vol. 186, pp. 286–293.
20. Talaei Y., Asgari M. An operational matrix based on Chelyshkov polynomials for solving multi-order fractional differential equations. *Neural. Comput. Appl.*, 2018, vol. 30, pp. 1369–1376.
21. Y. Talaei, Chelyshkov collocation approach for solving linear weakly singular Volterra integral equations. *J. Appl. Math. Comput.*, 2019, vol. 60, pp. 201–222.
22. Talaei Y., Micula S., Hosseinzadeh H., Noeiaghdam S. A novel algorithm to solve nonlinear fractional quadratic integral equations. *AIMS Mathematics*, 2022, vol. 7, no. 7, pp. 13237–13257. doi: 10.3934/math.2022730

23. Talaei Y., Shahmorad S., Mokhtary P. A fractional version of the recursive Tau method for solving a general class of Abel-Volterra integral equations systems. *Fract. Calc. Appl. Anal.*, 2022, vol. 25, pp. 1553–1584. https://doi.org/10.1007/s13540-022-00070-y
24. Yuzbas? S., Sezer M., Kemanci B. Numerical solutions of integro-differential equations and application of a population model with an improved Legendre method. *Appl. Math. Model.*, 2013, vol. 37, no. 4, pp. 2086–2101.
25. Zhu L., Fan Q. Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. *Commun Nonlinear Sci Numer Simulat*, 2012, vol. 17, no. 3, pp. 2333–2341.

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