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Formulas Expressing Totally Nonstable Truth Values of Formulas

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Abstract. In this short paper we consider problem of nonstable truth values of formulas. We investigate temporal-modal logic for description of reliability information. The logic itself is generated by rather standard models on linear time. We consider intervals of nonstable truth values of formulas. This mean that the formula permanently and for a reasonably big time change its truth from true to false and vice versa. We aim to construct some technique and to find an algorithm which may recognize permanently nonstable formulas.

Keywords: modal logic, frame and model Kripke, satisfiability problem, decidability of logic

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Научная статья

Формулы, выражающие тотальную нестабильность истинностных значений формул

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Аннотация. Рассматривается проблема нестабильности истинностных значений формул. Исследуется временная модальная логика на предмет описания надёжности информации. Логика сама по себе порождена стандартными моделями линейного времени. Рассматриваются интервалы нестабильности истинности формул в этих моделях. Это означает, что формула перманентно и в разумно большое время меняет свою истинность с истины на ложь и наоборот. Создается специальная техника и находится алгоритм, который позволит распознавать перманентную нестабильность истинности формул.

Ключевые слова: модальная логика, фрейм и модель Крипке, выполнимость формул, разрешимость логики

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1. Introduction

Usage of modal, temporal and other non-classical logics has a long and fruitful history. Origin of such many-valued logics may be dated to Lukasiewicz (1917) and his three–valued and many–valued propositional calculi, as well as to Goedel (1932), who refuted the finite — validness of intuitionistic logic. Later A. Tarski (1951) and S. Kripke (1960th) suggested semantical models for the studies of modal and temporal logics such as topological boolean algebras and relational models (Kripke–Hintikka models).

Conception of knowledge may be dated to the end of 1950. At 1962 Hintikka printed the manuscript: *Knowledge and Belief*, that was about the first book-length work involving modalities to represent the semantics of knowledge. That book contributed a good first addendum for the subject area, but a great deal of research has taken place since that time. One of logics in that line of research was temporal logic (cf. for historical outlook for reasonably close days Gabbay, Hodkinson, Reynolds [2; 3], Goldblat [4], Goranko [5], van Benthem [16], Yde Venema [19]).

Among various extensions of temporal logic, the linear temporal logic $\mathcal{LTL}$ with operation **U** - until – introduced by Amir Pnueli – was especially popular for applications and due to interesting mathematical base. Besides, it is a good idea to mention automaton technique for solution satisfiability in this logic developed by Vardi [17; 18]. From reasonably modern results concerning this logic we would mention the solution for admissibility problem for $\mathcal{LTL}$ in Rybakov [6; 7], the basis for admissible rules of $\mathcal{LTL}$ was obtained in Babenyshev and Rybakov [1]. The unification problem for $\mathcal{LTL}$

was solved in [11]. Concerning applications of logical methods in AI and CS, the tools around temporal logic work well for analysis in multi-agent environment (cf. eg. [8; 9]).

Another popular direction in Information Sciences studies representation of knowledge via multi-agent environment using usually many agent modal logics. That concerns diverse subjects of multi-agent environment – interaction and autonomy, effects of cooperation etc. For example tools for representation agents' interaction for the logic $\mathcal{LTL}$ of linear time were developed in Rybakov [8; 9]. In current time this logic was investigated from many viewpoints, in particular extensions of $\mathcal{LTL}$ for the case of non-transitive models, were studied in Rybakov [12; 15] for the case of the interval versions of the logic. Also modelling multi-agent reasoning via temporal models was applied in Rybakov [10; 13; 14] for the versions of liner logic.

In this our short paper we consider reasonably new logical problem in information sciences. We attempt to formalize what means that information (presumably written by modal formulas) is not stable, permanently not stable. We suggest in approach using Kripke-like linear models and technique of realizers sets of formulas. The aim is to find an algorithm which would recognize permanently nonstable formulas. The algorithm is constructed and we prove that it solves the pointed task.

2. Satisfiability in Logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$

Formulas of our logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ will be defined as the set of special formulas, which are true at states of certain relational Kripke-like model.

Alphabet for the language of our logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ is defined in a standard way and consists of denumerable set of propositional letters (variables), parenthesises, logical Boolean operations, and modal operations $\Box$ and $\Diamond$ and also special time operation $Next$ (unary operator $<< NextTime >>$).

We remind, that every modal operation $\Box$ can be defined by means of modal operation $\Diamond$ as follows $\Box = \neg\Diamond\neg$. Now we give inductive definition of the formulas in the language of our logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$.

1) Any propositional variable $p \in Prop$ is formula.

2) If A is formula, then $\neg A$ is formula also.

3) If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are formulas as well.

4) If A is formula, then $\Box A$ is a formula also.

5) If A is formula, then $\mathcal{N}A$ is formula as well.

There is no other formulas in the language of logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$. Now we turn to the definition of our Kripke-like models. Let we have not empty set W, binary relation R on this set, $R \subseteq W^2$, and the set $Prop$ of propositional variables.

Relational model is a model:

$$\mathcal{M} = \langle W,\, R,\, V \rangle,$$

where the valuation V of any propositional variable p from a fixed chosen set of variables is defined as a some chosen subset $V(p)$ from W.

We will first consider models where W is the set of all natural numbers $\mathbb{N}$. So, the relation $Next$ is binary relation where $a\, Next\, b$ if and only if $b = a + 1$. It is convenient to write $Next(a) = b$. Now we precisely define the truth value of formulas in arbitrary model $\mathcal{M}$ as follows.

For any a, b, $c \in \mathcal{M}$ the truth relations are as follows, let $\leqslant$ is the standard linear order on $\mathbb{N}$ and

$$\begin{gathered}
\forall\, p \in Prop:\; a \Vdash_V p \iff a \in V(p),\\
a \Vdash_V \neg\varphi \iff a \nVdash_V \varphi,\\
a \Vdash_V (\varphi \wedge \psi) \iff a \Vdash_V \varphi \quad \text{and} \quad a \Vdash_V \psi,\\
a \Vdash_V \mathcal{N}\,\varphi \iff \forall\, b\,[(a\, Next\, b) \Rightarrow b \Vdash_V \varphi]\,,\\
a \Vdash_V \Box\,\varphi \iff \forall\, b\,[(a \leqslant b) \Rightarrow (b \Vdash_V \varphi)]\,,\\
a \Vdash_V \Diamond\,\varphi \iff \exists\, b\,[(a \leqslant b) \wedge (b \Vdash_V \varphi)]\,.
\end{gathered}$$

So, in accordance with our notation, a linear relational model $\mathcal{M}_{\mathbb{N}}$ is a model:

$$\mathcal{M}_{\mathbb{N}} = \langle \mathbb{N},\, \leqslant,\, Next,\, V \rangle\,.$$

The set of all formulas written in the language of model $\mathcal{M}_{\mathbb{N}}$ and which are true in it, is called a logic, generated by model $\mathcal{M}_{\mathbb{N}}$. Notice that this definition differs from standard definition of logics, because the our one not to be compulsory closed w.r.t. substitutions.

Formula φ is said to be satisfiable in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$, if there can be find a state $a \in \mathcal{M}_{\mathbb{N}}$ such as $a \Vdash_V \varphi$. Formula φ is called not satisfiable in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$, if there is no $a \in \mathcal{M}_{\mathbb{N}}$ such as $a \Vdash_V \varphi$. Formula φ is said to be refutable in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$, if there exist a state $a \in \mathcal{M}_{\mathbb{N}}$ such as $a \nVdash_V \varphi$. Respectively a formula φ is said to be true in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ if it is true at any state a from $\mathcal{M}_{\mathbb{N}}$. Recall that formula φ is a theorem of the logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ if and if only $\neg\varphi$ is not satisfiable formula in that logic.

Construction of the model $\mathcal{M}_1$

We need now some axillary technique. Let us consider the formula

$$A(\alpha) = \Box\Diamond\varphi(\alpha) \wedge \Box\Diamond\varphi(\neg\alpha) \quad , \tag{1}$$

where
$\varphi(\alpha) = \Big(\alpha \wedge N(\alpha \wedge N(\alpha \wedge \cdots \wedge N(\alpha)...))\Big)$,
$\varphi(\neg\alpha) = \Big(\neg\alpha \wedge N(\neg\alpha \wedge N(\neg\alpha \wedge \cdots \wedge N(\neg\alpha)...))\Big)$.
Here every big parentheses contain exactly k formulas α. In what follows, always the formula α does not contain temporal operation $\mathcal{N}$ that is α is pure modal formula (we will speak about permanent non-stability only such formulas).

Let for some $x \in \mathbb{N}$ formula $A(\alpha)$ is true, so it is satisfiable in the state x.

Common picture representing this event, may be depicted like this:

$$\ldots x \ldots [\ldots \alpha \ldots] \ldots [\ldots \neg\alpha \ldots] \ldots [\ldots \alpha \ldots] \ldots$$

Here every interval include exactly k states where formula α or $\neg\alpha$ to be true.

Without loss of generality, keeping intuition, we can express such α-intervals precisely in these sequences. If formula $A(\alpha)$ is satisfiable in a state of a model $\mathcal{M}_{\mathbb{N}}$, then formula α is called *permanently unstable formula (or information).*

To solve the question of the satisfiability of formula $A(\alpha)$ we need to construct some finite models $\mathcal{M}_2$ and show, that this formula is satisfiable in some model $\mathcal{M}_{\mathbb{N}}$ if and only if it is satisfiable in some this model $\mathcal{M}_2$ with effectively computable number of states.

At first we shell consider some auxiliary considerations. So, let we assume that the formula $A(\alpha) = \Box\Diamond\varphi(\alpha) \wedge \Box\Diamond\varphi(\neg\alpha)$ is true at some sate b from our model.

Lemma 1.

$$\forall b, \forall c, \Big((c \geqslant b) \Rightarrow \exists S(S \subseteq Sub(\alpha)) \wedge \forall\beta[(\beta \in S) \Rightarrow (c \Vdash_V \beta)] \wedge \wedge \forall\beta \in Sub(\alpha)((b \Vdash_V \beta) \Longrightarrow (\beta \in S))\Big). \quad (2)$$

Proof follows by simple direct evaluation of possible truth values of formulas from $Sub(\alpha)$.

The state b is called the realiser of the subset S, and S is realised in the state b. Equivalence (2) does mean that for any given natural number b only some realizers from some finite set of subsets S of the set of all sub-formulas from $Sub\,(A(\alpha))$ are accessible.

In ascending order from any b, the variety of subsets realizers S decreases and therefore there exists the state c such that in all states $y \geqslant c$ only some subsets S from the fixed finite set $\{S_1, S_2, \ldots, S_n\}$ are exactly to be all possible realizers (starting from the state c) and these realizers to be infinitely many times repeated in the future. That means that for any $x \geqslant c$, for any $S \in \{S_1, S_2, \ldots, S_n\}$ there is some $y \geq x$ such that y is realizer for S. Also it is clear that S as a realizer is unique. We need now to construct certain intermediate model $\mathcal{M}_1$ with some desirable properties.

Basic set of the new model M_1 will begin from the state $c_0 \geqslant c$, where c_0 the earliest state after c where formula $A(\alpha)$ is true, that is $c_0 \Vdash_V A(\alpha)$.

Then, in the basic set, we leave finite increasing set of all fixed states-realizers for all sets from $\{S_1, S_2, \ldots, S_n\}$ situated between first interval

$$Int_1 := [\ldots \alpha \ldots]$$

after c_0 where α is true and next interval

$$Int_2 := [\ldots \neg\alpha \ldots],$$

where $\neg\alpha$ is true; then we take the first interval

$$Int_3 := [\ldots \alpha \ldots]$$

after $Int_2 = [\ldots \neg\alpha \ldots]$ where α is true; next we delete all other states starting from c_0 and direct the state x the final one in Int_3 to the state z which is next one in the obtained model after the final state w in the interval Int_1. In the resulting model the relation $\leqslant$ and valuation V remain to be as before. We can then observe now following picture:

$$\ldots [c_0 \ldots \alpha \ldots] \ldots [\ldots \neg\alpha \ldots] \ldots [\ldots \alpha \ldots] \ldots$$

Lemma 2. *For any natural number $b \geqslant c_0$, where $b \in \mathcal{M}_1$, and any subformulas $\beta \in Sub(\alpha)$ the next statement holds:*

$$(\mathcal{M}_{\mathbb{N}}\, b) \Vdash_V \beta \Longleftrightarrow (\mathcal{M}_1,\, b), \Vdash_V \beta$$

Proof. The proof follows by induction on the length ℓ of subformulas $\beta \in Sub(\alpha)$.

1. $\ell = 0$. For the propositional variables it is evident.
2. Let the statement is true for $\forall \ell < r$. We shell prove for $\ell = r$.

Inductive steps for the Boolean logical operations are evident. In the case, when $\beta = \Diamond\gamma$ or $\beta = \Box\gamma$, the inductive steps easy follow from the presence of all possible realizers. Lemma is proved. □

If a formula

$$A(\alpha) = \Box\Diamond\varphi(\alpha) \wedge \Box\Diamond\varphi(\neg\alpha)$$

is satisfiable in a model $\mathcal{M}_1$, it is also satisfiable in some usual model (it easy follows by standard unravelling technique).

As we may see, the model $\mathcal{M}_1$ has a finite size computable from the length of the formulas

$$A(\alpha) = \Box\Diamond\varphi(\alpha) \wedge \Box\Diamond\varphi(\neg\alpha).$$

Therefore from Lemma 1 and Lemma 2 we obtain our main result:

Theorem 1. The problem of satisfiability formulas $A(\alpha)$ is decidable and therefore the problem of recognizing permanent unstable formulas is solvable.

3. Conclusion

In this paper we considered problem of nonstable truth values of formulas. We investigated temporal modal logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ for description of reliability information. The logic $\mathcal{L}(\mathcal{M}_{\mathbb{N}})$ itself is generated by standard model $\mathcal{M}_{\mathbb{N}} = \langle \mathbb{N}, \leqslant, Next, V \rangle$ on linear time. We considered intervals of nonstable truth values of formulas, when formula permanently changes its truth from true to false and vice versa. We constructed certain technique and offered algorithm which may recognize permanently unstable formulas.

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