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On Anti-endomorphisms of Groupoids

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Abstract. In this paper, we study the problem of element-by-element description of the set of all anti-endomorphisms of an arbitrary groupoid. In particular, the structure of the set of all anti-automorphisms of a groupoid is studied. It turned out that the set of all anti-endomorphisms of an arbitrary groupoid decomposes into a union of pairwise disjoint sets of transformations of a special form. These sets of transformations are referred to in this paper as basic sets of anti-endomorphisms. Each base set of anti-endomorphisms is parametrized by some mapping of the support set of the groupoid into a fixed set of two elements. These mappings are called the bipolar anti-endomorphism type. Since the base sets of anti-endomorphisms of various types have an empty intersection, each anti-endomorphism can uniquely be associated with its bipolar type. This assignment leads to a bipolar classification of anti-endomorphisms of an arbitrary groupoid. In this paper, we study the semiheap (3-groupoid of a special form) of all anti-endomorphisms. A subsemiheap of anti-endomorphisms of the first type and a subsemiheap of anti-endomorphisms of the second type are constructed. These monotypic semiheaps can be expressed into empty sets for specific groupoids. A conjecture is made about a subsemiheap of a special type of anti-endomorphisms of mixed type. The main research method in this work is the use of internal left and right translations of the groupoid (left and right translations). Since an arbitrary groupoid is considered, the set of all left translations (similarly to right translations) need not be closed with respect to the composition of transformations of the support set of the groupoid.

Keywords: groupoid endomorphism, anti-endomorphism, groupoid anti-automorphism, bipolar type of groupoid anti-endomorphism, groupoid, anti-endomorphism semiheap, monotypic anti-endomorphism semiheap

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Научная статья

Об антиэндоморфизмах группоидов

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Аннотация. Исследуется проблема поэлементного описания множества всех антиэндоморфизмов произвольного группоида. В частности, исследуется строение множества всех антиавтоморфизмов группоида. Выяснилось, что множество всех антиэндоморфизмов произвольного группоида раскладывается в объединение попарно непересекающихся множеств преобразований специального вида. Данные множества преобразований получают название базовых множеств антиэндоморфизмов. Каждое базовое множество антиэндоморфизмов параметризуется некоторым отображением множества носителя группоида в фиксированное множество из двух элементов. Эти отображения получают название биполярного типа антиэндоморфизма. Поскольку базовые множества антиэндоморфизмов различных типов имеют пустое пересечение, то каждому антиэндоморфизму можно единственным образом сопоставить его биполярный тип. Данное присвоение приводит к биполярной классификации антиэндоморфизмов произвольного группоида. Изучается полугруда (3-группоид специального вида) всех антиэндоморфизмов. Строится подполугруда антиэндоморфизмов первого типа и подполугруда антиэндоморфизмов второго типа. Данные монотипные полугруды могут выраждаться в пустые множества для конкретных группоидов. Делается гипотеза о подполугруде специального вида антиэндоморфизмов смешанного типа. Основным методом исследования в данной работе является использование внутреннего левого и правого сдвигов группоида (левое и правое умножение). Поскольку рассматривается произвольный группоид, то множество всех левых сдвигов (аналогично правых сдвигов) не обязано быть замкнуто относительно композиции преобразований множества носителя группоида.

Ключевые слова: эндоморфизм группоида, антиэндоморфизм, антиавтоморфизм группоида, биполярный тип антиэндоморфизма группоида, группоид, полугруда антиэндоморфизмов, монотипная полугруда антиэндоморфизмов

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1. Introduction

An *anti-endomorphism* of a groupoid $G = (G, *)$ is any transformation α of a set G such that for any $x, y \in G$ the equality $\alpha(x * y) = \alpha(y) * \alpha(x)$

holds. Reversible anti-endomorphisms are called *anti-automorphisms*. The set of all anti-endomorphisms $\mathrm{Aend}(G)$ of the groupoid G need not be closed under the composition of transformations. For commutative groupoids, the set of all anti-endomorphisms coincides with the set of all endomorphisms of the groupoid.

There are many examples in the literature of the study of the following general problem.

Problem 1. *Given a groupoid G, give an element-by-element description of the monoid of all endomorphisms and the group of all automorphisms.*

This problem is most often studied for semigroups and quasigroups (in particular, groups). There are examples of studying these problems for various groupoids (see, for example, [11; 12; 14]). An element-by-element description naturally means a description of endomorphisms as transformations of the set G.

In the context of Problem 1, there naturally arises an interest in studying the general problem

Problem 2. *Given a groupoid G, give an element-by-element description of the set of all anti-endomorphisms and the set of all anti-automorphisms.*

There is a simple connection between the concepts of endomorphism and anti-endomorphism of some groupoid G. The composition of any two anti-endomorphisms of a groupoid is an endomorphism of that groupoid. Thus, knowledge of specific anti-endomorphisms of a groupoid allows us to construct specific endomorphisms of this groupoid.

Anti-endomorphisms can be used for other studies as well. So in the work [5] reveals a connection between a semi-endomorphism of a finite simple group with order greater than two and its anti-automorphisms.

In the papers [1–3] solutions of various functional equations over various groupoids (semigroups, monoids, and groups) with anti-endomorphisms are considered. These studies emphasize the relevance of identifying cases where the groupoid G has an anti-endomorphism. The latter emphasizes the relevance of problem 2.

Uses of anti-endomorphisms in applications can be found in [9]. Anti-endomorphisms and anti-automorphisms can be used as a tool for making calculations and formulating assertions (see, for example, [6]).

An example of the study of the properties of anti-endomorphisms of some matrix semigroups is the paper [15].

In work [10], anti-endomorphisms of groupoids that are not semigroups and quasigroups are studied.

Anti-endomorphisms of other algebraic systems are studied in, for example, [4; 7; 8].

The above studies point to the importance of developing methods for studying the set of all anti-endomorphisms and anti-automorphisms of a groupoid. The main results of this paper include Theorem 1, which gives a decomposition of the set of all anti-endomorphisms of an arbitrary groupoid into a union of pairwise disjoint sets of a special form, which in this paper are called *base sets of anti-endomorphisms* of a groupoid (see Definition 5). Thus, the set of all endomorphisms of a groupoid has been calculated up to the calculation of all basic sets of anti-endomorphisms of a given groupoid (the latter is a non-trivial problem in the general case). This result can be used as a method for investigating Problem 2 for specific groupoids (or classes of groupoids). In particular, Theorem 1 and its Corollary 1 can be used as a tool for an element-by-element description of the group of all anti-automorphisms of some groupoid.

Each base set of anti-endomorphisms $T(\gamma)$ of the groupoid G is defined by a suitable mapping $\gamma : G \to \{1, 2\}$, this mapping is called *bipolar type* anti-endomorphisms or simply *type* of anti-endomorphisms (see definitions 1, 2, 3). It turned out that the base sets of anti-endomorphisms of various types have an empty intersection (see Theorem 1). Therefore, each anti-endomorphism $\alpha \in T(\gamma)$ can be assigned its own type γ (see Definitions 7). This type assignment leads to a classification of anti-endomorphisms of the given groupoid, which is called the *bipolar classification of all anti-endomorphisms* of the groupoid.

The set of all anti-endomorphisms of a groupoid is not (generally) closed under composition. Moreover, this set is closed under the ternary operation q, which consists in the composition of three anti-endomorphisms:

$$q(\alpha, \beta, \tau) := \alpha \cdot \beta \cdot \tau \quad (\alpha, \beta, \tau \in \mathrm{Aend}(G)).$$

In addition, this ternary operation will have the skew-associativity property. Thus, one can consider the *semiheap* of all anti-endomorphisms $\mathrm{Aend}(G) = (\mathrm{Aend}(G), q)$ (see Section 3 for details). The term *semiheap* is common, see, for example, [16]. Thus, it becomes possible to consider the set of all anti-endomorphisms as an algebraic system and study its properties. In particular, select subsystems.

The main results of this paper are Theorems 2 and 3, which introduce subsemiheaps of anti-endomorphisms $\mathrm{Mantiend}(A, G)$ and $\mathrm{Mantiend}(\Omega, G)$. The subsemiheap $\mathrm{Mantiend}(A, G)$ consists of anti-endomorphisms of the first type, and the subsemiheap $\mathrm{Mantiend}(\Omega, G)$ consists of the second type anti-endomorphisms. These hemiheaps of anti-endomorphisms can be called *monotypic* semiheaps of anti-endomorphisms. Depending on the groupoid G, the constructed monotypic semiheaps of anti-endomorphisms can degenerate into the empty set. Computational experiments show that this is not the case in general.

The bipolar classification of anti-endomorphisms of a groupoid is constructed in a similar way to the bipolar classification of endomorphisms of a groupoid (see [13]). At the same time, differences are found in the proofs of similar theorems, which lead to the need to build these classifications separately. Thus, in order to construct a bipolar classification of endomorphisms of a groupoid, one could manage only with left translations. For anti-endomorphisms, one has to consider left and right translations. And for anti-endomorphisms, one has to consider subsemiheaps.

2. Anti-endomorphisms

Notation related to the symmetric semigroup and anti-endomorphisms. The symmetric semigroup of all transformations of the set G will be denoted by $I(G)$. If α_1, α_2 is transformations from $I(G)$, then their composition $(\cdot)$ will be defined by the equality

$$(\alpha_1 \cdot \alpha_2)(x) = \alpha_1(\alpha_2(x))$$

for any $x \in G$. In this paper, anti-endomorphisms and their compositions have no special notation and are considered in the notation of the symmetric semigroup.

Let $G = (G, *)$ be some groupoid. For any $x \in G$ we denote by l_x and r_x transformations of the set G such that for any $y \in G$ the equalities are true:

$$l_x(y) = x * y, \quad r_x(y) = y * x.$$

The transformations l_x and r_x belongs to the symmetric semigroup $I(G)$.

The transformation l_x (r_x) is an left (right) translation of the groupoid G (hereinafter, we will simply say the left or right translation corresponding to the element x).

Definition 1. *Denote by* $\mathrm{Bte}(G)$ *the set of all possible mappings of the set G into the set $\{1, 2\}$. Mappings from this set will be called bipolar types of anti-endomorphisms of the groupoid G (or simply types).*

Definition 2. *If $\gamma \in \mathrm{Bte}(G)$ and for any element $g \in G$ equality $\gamma(g) = 1$ (reciprocally $\gamma(g) = 2$) is true, then the mapping γ will be called first type (reciprocally second type).*

Definition 3. *If the mapping $\gamma \in \mathrm{Bte}(G)$ is not constant on elements of G, then the mapping γ will be called the mixed type.*

The connection with anti-endomorphisms will be revealed in Theorem 1. The first type will be denoted by A, and the second type denoted by Ω.

As usual, the centralizer of the transformation α in a symmetric semigroup $I(G)$ will be denoted by

$$C(\alpha) := \{\beta \in I(X) \mid \alpha \cdot \beta = \beta \cdot \alpha\}.$$

Definition 4. *For every $s \in G$ we define the sets*

$$U^{(1)}(s) := \{\alpha \in C(l_s) \mid l_s = r_{\alpha(s)}\},$$

$$U^{(2)}(s) := \{\alpha \in I(G) \mid l_s \neq r_{\alpha(s)}, \quad \alpha \cdot l_s = r_{\alpha(s)} \cdot \alpha\}.$$

For any $s \in G$ the sets $U^{(1)}(s)$ and $U^{(2)}(s)$ have an empty intersection (it is clear from the definition of these sets).

Definition 5. *For every $\gamma \in \mathrm{Bte}(G)$ we define the set*

$$T(\gamma) := \bigcap_{s \in G} U^{(\gamma(s))}(s).$$

The set $T(\gamma)$ will be called the base set of anti-endomorphisms of type γ (the fact that these sets consist of anti-endomorphisms will be proved in Theorem 1).

Let us formulate and prove the main theorem.

Theorem 1. *For any groupoid G the equality*

$$\mathrm{Aend}(G) = \bigcup_{\gamma \in \mathrm{Bte}(G)} T(\gamma) \tag{2.1}$$

is true. Moreover, if ξ and ω are two different types from $\mathrm{Bte}(G)$, *then the sets $T(\xi)$ and $T(\omega)$ are disjoint.*

Proof. 1. We assume that α is some anti-endomorphism of the groupoid G. By the definition of a groupoid anti-endomorphism, this is possible if and only if for any $x, y \in G$ the following equality holds:

$$\alpha(x * y) = \alpha(y) * \alpha(x). \tag{2.2}$$

Let's write the equality (2.2) using left and right translations:

$$\alpha(x * y) = \alpha(l_x(y)) = (\alpha \cdot l_x)(y),$$

$$\alpha(y) * \alpha(x) = r_{\alpha(x)}(\alpha(y)) = (r_{\alpha(x)} \cdot \alpha)(y), \quad (\alpha \cdot l_x)(y) = (r_{\alpha(x)} \cdot \alpha)(y).$$

Thus, the transformation $\alpha \in I(G)$ is an anti-endomorphism if and only if for any $x, y \in G$ equality

$$(\alpha \cdot l_x)(y) = (r_{\alpha(x)} \cdot \alpha)(y) \tag{2.3}$$

is true.

The conditions (2.3) are equivalent to the fact that for any $x \in G$ the equality of transformations

$$\alpha \cdot l_x = r_{\alpha(x)} \cdot \alpha \tag{2.4}$$

is true. Indeed, since the equality (2.3) holds for every $y \in G$, we obtain a pointwise equality of the transformations on the left and right sides of the equality (2.4).

2. Let us show that the equality of the sets is true:

$$\mathrm{Aend}(G) = \bigcap_{s \in G} (U^{(1)}(s) \cup U^{(2)}(s)). \tag{2.5}$$

Let α be the transformation from the right side of the equality (2.5). Let us show that $\alpha \in \mathrm{Aend}(G)$. Indeed, if α is included in the set from the right side of the equality (2.5), then for any $s \in G$ the condition is performed:

$$\alpha \in W(s) := (U^{(1)}(s) \cup U^{(2)}(s)).$$

Therefore, at least one of the condition is satisfied: $\alpha \in U^{(1)}(s)$ or α in $U^{(2)}(s)$. If the condition $\alpha \in U^{(1)}(s)$ is true, then we have the equality $l_s = r_{\alpha(s)}$, α is commute with l_s and the following implications are valid:

$$\alpha \cdot l_s = \alpha \cdot l_s \Rightarrow \alpha \cdot l_s = l_s \cdot \alpha \Rightarrow \alpha \cdot l_s = r_{\alpha(s)} \cdot \alpha.$$

In this case, the equality (2.4) turns into an identity for the element $x = s$. If $\alpha \in U^{(2)}(s)$, then equality $\alpha \cdot l_s = r_{\alpha(s)} \cdot \alpha$ is true. In the latter case, the equality (2.4) holds for the element $x = s$.

Since the condition $\alpha \in W(s)$ is valid for any $s \in G$, then α satisfies the equality (2.4) for any $x \in G$. So we get the entry $\alpha \in \mathrm{Aend}(G)$.

We have proved that the set from the right side of the equality (2.5) is a subset from the left side of this equality. Next, we show the reverse inclusion.

Now let α be an arbitrary anti-endomorphism from $\mathrm{Aend}(G)$. Then the equality (2.4) hold for each $x \in G$. We fix an arbitrary element $x = s$ from G. It is easy to see that exactly one of the cases below is true.

Case 1. The following relations are fulfilled: $l_s = r_{\alpha(s)}$, $\alpha \cdot l_s = r_{\alpha(s)} \cdot \alpha$. In this case, we have the condition $\alpha \in U^{(1)}(s)$ is true.

Case 2. The following relations are fulfilled: $l_x \neq r_{\alpha(s)}$, $\alpha \cdot l_s = r_{\alpha(s)} \cdot \alpha$. We get the condition $\alpha \in U^{(2)}(s)$ is true. Case 1 and case 2 are obviously not compatible (it can be seen from their definition).

From what has been said above, we obtain that the anti-endomorphism α belongs to the set $W(s)$. The transformation α belongs to $\mathrm{Aend}(G)$ if and only if the equality (2.4) holds for any $x \in G$. Therefore, we get that if $\alpha \in \mathrm{Aend}(G)$, then it is necessary that the condition $\alpha \in W(s)$ be valid for any $s \in G$. Therefore, the anti-endomorphism α belongs to the set from the right-hand side of the equality (2.5). Thus, we have shown that the equality (2.5) holds.

3. Let us show that the equality (2.1) holds. We assume that the transformation α is anti-endomorphism. By virtue of the equality (2.5),

the condition $\alpha \in W(s)$ is valid for any $s \in G$. Hence, for any $s \in G$, the transformation α belongs to the set $U^{(1)}(s)$ or belongs to the set $U^{(2)}(s)$ (recall that the intersection of these sets are empty). Thus, for any element $s \in G$, one can define the number $d_\alpha(s) \in \{1, 2\}$, which is determined by the equivalences:

$$d_\alpha(s) = 1 \Leftrightarrow \alpha \in U^{(1)}(s), \quad d_\alpha(s) = 2 \Leftrightarrow \alpha \in U^{(2)}(s).$$

For each fixed α the numbers $d_\alpha(s)$ (for different $s \in G$) define a mapping from $\mathrm{Bte}(G)$. By definition of the function $d_\alpha : G \to \{1, 2\}$ we get that the anti-endomorphism α belongs to all sets of the family of sets $\{U^{(d_\alpha(s))}(s)\}_{s\in G}$, hence the condition $\alpha \in T(d_\alpha)$ is true. Thus, we have shown the inclusion of a set within a set:

$$\mathrm{Aend}(G) \subseteq \bigcup_{\gamma\in \mathrm{Bte}(G)} T(\gamma).$$

Let us prove that the last inclusion also holds in the opposite direction. Let the condition be true:

$$\alpha \in \bigcup_{\gamma\in \mathrm{Bte}(G)} T(\gamma).$$

Consequently, the condition $\alpha \in T(\gamma_0)$ is true for some fixed type $\gamma_0 \in \mathrm{Bte}(G)$. Hence, by the definition of the sets $T(\gamma)$, the transformation α belongs to the set $U^{(\gamma_0(s))}(s)$ for any $s \in G$; therefore, the transformation α belongs to the set $W(s)$ for any $s \in G$. Hence, due to the equality (2.5), we obtain the validity of the condition $\alpha \in \mathrm{Aend}(G)$. The inclusion in the opposite direction is proved. Thus, we have shown the equality (2.1).

4. Let us show that the intersection of the sets $T(\xi)$ and $T(\omega)$ is empty if the types ξ and ω are distinct. We assume that for a fixed $s \in G$ the conditions $\xi(s) = 1$ and $\omega(s) = 2$ are satisfied. Then if the anti-endomorphism α belongs to the set $T(\xi)$, then α belongs to $U^{(1)}(s)$. But the sets $U^{(1)}(s)$ and $U^{(2)}(s)$ have empty intersection, hence $\alpha \notin U^{(2)}(s)$. Therefore, the condition $\alpha \notin T(\omega)$ is satisfied. The theorem has been proven. □

As usual, $S(X)$ is the symmetric group of all permutations of the set X.

Definition 6. *The set $T_a(\gamma) := S(G) \cap T(\gamma)$ will be called the base set of type anti-automorphisms γ of the groupoid G.*

Since an arbitrary anti-endomorphism of a groupoid G is an antiautomorphism of this groupoid if and only if it is invertible, we obtain the following

Corollary 1. *The set of all anti-automorphisms of any groupoid is the union of all possible base sets of anti-automorphisms of the given groupoid. The base sets of anti-automorphisms of a groupoid of various types have an empty intersection.*

Definition 7. *An anti-endomorphism α of a groupoid G will be called a anti-endomorphism of the first (second) type if α belongs to the basic set of anti-endomorphisms of the first (second) type. If an anti-endomorphism α belongs to the base set of anti-endomorphisms of mixed type, then we say that α is a mixed-type anti-endomorphism. We will say in more detail that an anti-endomorphism α has type γ from* $\mathrm{Bte}(G)$ *if the condition $\gamma \in T(\gamma)$ holds.*

3. Semiheaps and monotypic subsemiheaps of anti-endomorphisms

An algebraic system $S = (S, f)$ is called a 3-*groupoid* if f is a ternary algebraic operation on the set S. If in a 3-groupoid S for any x_1, x_2, x_3, x_4, x_5 from G the relations

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = \tag{3.1}$$
$$= f(x_1, x_2, f(x_3, x_4, x_5))$$

are true, then such groupoids are called *semiheaps*. This relation is an analogue of associativity in a 3-groupoid. According to [16], the operation f is called *skew-associative*.

Let $G = (G, *)$ be a groupoid. Then the set $\mathrm{End}(G)$ is a semigroup under the composition of transformations. The set of all anti-endomorphisms is not closed under the composition of transformations, but the product of any three anti-endomorphisms is again an anti-endomorphism. Indeed, for any $x, y \in G$ and any $\alpha, \beta, \tau \in \mathrm{Aend}(G)$ the equalities are true

$$(\alpha \cdot \beta \cdot \tau)(x * y) = \alpha \cdot \beta(\tau(y) * \tau(x)) = \alpha(\beta(\tau(x)) * \beta(\tau(y))) =$$
$$= [\alpha(\beta(\tau(y)))] * [\alpha(\beta(\tau(x)))] = (\alpha \cdot \beta \cdot \tau)(y) * (\alpha \cdot \beta \cdot \tau)(x).$$

Thus, on the set $\mathrm{Aend}(G)$, one can introduce a ternary algebraic operation q defined by the identity:

$$q(\alpha, \beta, \tau) := \alpha \cdot \beta \cdot \tau \quad (\alpha, \beta, \tau \in \mathrm{Aend}(G)).$$

The associativity of the composition of transformations leads to the fact that the relations (3.1) will hold for the operation q.

Definition 8. *The algebraic system* $\mathrm{Aend}(G) = (\mathrm{Aend}(G), q)$ *will be called the semiheap of all anti-endomorphisms.*

A subsystem in $\mathrm{Aend}(G)$ is any subset of the set of all anti-endomorphisms that is closed under the operation q. Thus, H is a subsystem in $\mathrm{Aend}(G)$ if and only if for any $\alpha, \beta, \tau \in H$ the condition holds: $q(\alpha, \beta, \tau) \in H$. It is clear that for each subsystem of the semiheap $\mathrm{Aend}(G)$ the condition (3.1) will be satisfied. Therefore, the subsystem of the semiheap of all anti-endomorphisms will be called the *subsemiheap*.

Next, we construct a subsystem in the semiheap $\mathrm{Aend}(G)$ consisting of anti-endomorphisms of the first type.

As before, G is a groupoid. For each element $g \in G$ we define sets:

$$M_g := \{s \in G \mid r_s = l_g\},\ O_g := \begin{cases} \{\alpha \in I(G) \mid \alpha(M_g) \subseteq M_g\}, & \text{if } M_g \neq \varnothing, \\ \varnothing, & \text{if } M_g = \varnothing. \end{cases}$$

In the semigroup $I(G)$, we define a set of transformations:

$$\mathrm{Mantiend}(A, G) := \bigcap_{g \in G} (O_g \cap C(l_g)).$$

Theorem 2. *If the set* $\mathrm{Mantiend}(A, G)$ *is not empty for the groupoid G, then it is a subsemiheap in the semiheap* $\mathrm{Aend}(G)$*. The set* $\mathrm{Mantiend}(A, G)$ *consists of anti-endomorphisms of the first type.*

Proof. We assume that the set $\mathrm{Mantiend}(A, G)$ is not empty for the groupoid G. Therefore, for any $g \in G$ the set O_g is not empty.

1. Let us show that the set $\mathrm{Mantiend}(A, G)$ is closed under the operation g applied to transformations from $I(G)$. Let the transformations α, β, τ belongs to the set $\mathrm{Mantiend}(A, G)$; therefore, for any element g from G the transformations α, β, τ belongs to the sets: $C(l_g)$ and O_g. Let us show that the condition $q(\alpha, \beta, \tau) \in O_g \cap C(l_g)$ holds. Indeed, since there are inclusions

$$\alpha(M_g) \subseteq M_g, \quad \beta(M_g) \subseteq M_g \quad \tau(M_g) \subseteq M_g,$$

then the following relations hold:

$$\tau(M_g) \subseteq M_g \quad \Rightarrow \quad \beta(\tau(M_g)) \subseteq M_g \quad \Rightarrow \quad \alpha(\beta(\tau(M_g))) \subseteq M_g.$$

Hence the transformation $g(\alpha, \beta, \tau)$ belongs to the set O_g.

It is well known that the centralizer $C(l_g)$ is a submonoid in the symmetric semigroup $I(G)$. Since the conditions $\alpha, \beta, \tau \in C(l_s)$ are satisfied and $q(\alpha, \beta, \tau)$ is a composition of three transformations, then the condition $q(\alpha, \beta, \tau) \in C(l_g)$ is true. Therefore, for each $g \in G$ the transformation $q(\alpha, \beta, \tau)$ belongs to the set $O_g \cap C(l_s)$. Hence, the $q(\alpha, \beta, \tau)$ belongs to the set $\mathrm{Mantiend}(A, G)$.

2. Let us show that the subsystem $\mathrm{Mantiend}(A, G)$ consists of anti-endomorphisms of the first type. By the definition of the set $U^{(1)}(s)$, for all

$g \in G$ the inclusion $(O_g \cap C(l_s)) \subseteq U^{(1)}(g)$ is true. Therefore, the relations are true

$$\mathrm{Mantiend}(A, G) = \bigcap_{g \in G} (O_g \cap C(l_g)) \subseteq \bigcap_{g \in G} U_g = T(A).$$

Thus, by virtue of Theorem 1, the semiheap $\mathrm{Mantiend}(A, G)$ consists of endomorphisms of the first type. The theorem has been proven. □

Let $S = \{1, ..., n\}$ be a finite set of n elements and α be a transformation from $I(S)$. Then we will use the notation: $\alpha = (\alpha(1), \alpha(2), ..., \alpha(n))$.

Example 1. Let G be a groupoid defined by its left translation:

$$l_1 = (1,1,2,2), \quad l_2 = (1,1,2,2), \quad l_3 = (2,2,1,1), \quad l_4 = (2,2,1,1).$$

Let us show that the set $\mathrm{Mantiend}(A, G)$ is not empty. The following equalities are fulfilled:

$$M_i = \{1,2\}, \quad O_i = \{\alpha \in I(G) \mid \alpha(\{1,2\}) \subseteq \{1,2\}\}, \quad i = 1,2;$$

$$M_j = \{3,4\}, \quad O_j = \{\alpha \in I(G) \mid \alpha(\{3,4\}) \subseteq \{3,4\}\}, \quad j = 3,4.$$

Now it is not difficult to see that the set $\mathrm{Mantiend}(A, G)$ contains the identity transformation ε, and hence is not empty. The latter is possible because the groupoid G is commutative.

Next, we construct a subsystem in the semiheap $\mathrm{Aend}(G)$ consisting of anti-endomorphisms of the second type.

For each element $g \in G$ we define sets:

$$\overline{M}_g := \{s \in G \mid r_s \neq l_g\}, O'_g := \begin{cases} \{\alpha \in I(G) \mid \alpha(\overline{M}_g) \subseteq \overline{M}_g\}, & \text{if } \overline{M}_g \neq \varnothing, \\ \varnothing, & \text{if } \overline{M}_g = \varnothing. \end{cases}$$

In the semigroup $I(G)$, we define a set of transformations:

$$\mathrm{Mantiend}(\Omega, G) := \bigcap_{g \in G} (O'_g \cap U^{(2)}(g)).$$

Theorem 3. *If the set* $\mathrm{Mantiend}(\Omega, G)$ *is not empty for the groupoid* G*, then it is a subsemiheap in the semiheap* $\mathrm{Aend}(G)$*. The set* $\mathrm{Mantiend}(\Omega, G)$ *consists of anti-endomorphisms of the second type.*

Proof. We assume that the set $\mathrm{Mantiend}(\Omega, G)$ is not empty for the groupoid G. Therefore, for any $g \in G$ the set O'_g is not empty.

1. Let us show that the set $\mathrm{Mantiend}(\Omega, G)$ is closed under q. Let transformations α, β, τ belongs to the set $\mathrm{Mantiend}(\Omega, G)$. Then the transformations α, β, τ belongs to each of the sets O'_g and $U^{(2)}(g)$ for any $g \in G$.

The proof that $q(\alpha,\beta,\tau) \in O'_g$ for all $g \in G$ is similar to the proof that $q(\alpha,\beta,\tau) \in O_g$ (from the theorem 2).

Let us show that $q(\alpha,\beta,\tau) \in U^{(2)}(g)$. To do this, we will use the following property: for all $x, y \in G$, the equality $l_x(y) = r_y(x)$ holds. Since the transformations α, β, τ belongs to the intersection $U^{(2)}(g) \cap O'_g$ for each $g \in G$, then for any transformation $\omega \in \{\alpha, \beta, \tau\}$ and for any element $x \in G$ the following relations hold:

$$\omega \cdot l_x = r_{\omega(x)} \cdot \omega, \quad \omega(x) \in \overline{M}_x, \quad \omega(\overline{M}_x) \subseteq \overline{M}_x. \tag{3.2}$$

For any $x, z \in G$, the chain of equalities holds:

$$((\alpha \cdot \beta \cdot \tau) \cdot l_x)(z) = ((\alpha \cdot \beta \cdot r_{\tau(x)}) \cdot \tau)(z) = (\alpha \cdot \beta)(r_{\tau(x)}(\tau(z))) =$$

$$= ((\alpha \cdot \beta) \cdot l_{\tau(z)})(\tau(x)) = (\alpha \cdot (r_{\beta(\tau(z))} \cdot \beta))(\tau(x)) = (\alpha \cdot r_{\beta(\tau(z))})(\beta(\tau(x))) =$$

$$= (\alpha \cdot l_{\beta(\tau(x))})(\beta(\tau(z))) = (r_{\alpha(\beta(\tau(x)))} \cdot \alpha)(\beta(\tau(z))) = (r_{\alpha(\beta(\tau(x)))} \cdot (\alpha \cdot \beta \cdot \tau))(z).$$

Since the last chain of equalities holds for any $z \in G$, then the equality of transformations holds:

$$q(\alpha,\beta,\tau) \cdot l_x = r_{q(\alpha,\beta,\tau)(x)} \cdot q(\alpha,\beta,\tau).$$

Let us show that the condition $l_x \neq r_{q(\alpha,\beta,\tau)(x)}$ is satisfied. Due to the inclusions from (3.2), we have the relations are true

$$\tau(\overline{M}_x) \subseteq \overline{M}_x \quad \Rightarrow \quad \beta(\tau(\overline{M}_x)) \subseteq \overline{M}_x \quad \Rightarrow \quad \alpha(\beta(\tau(\overline{M}_x))) \subseteq \overline{M}_x.$$

Therefore, $l_x \neq r_{q(\alpha,\beta,\tau)(x)}$ and the transformation $q(\alpha,\beta,\tau)$ belongs to the set $U^{(2)}(g)$ for all $g \in G$. Hence the set $\mathrm{Mantiend}(\Omega, G)$ is closed under the operation q.

2. Let us show that the set $\mathrm{Mantiend}(\Omega, G)$ consists of anti-endomorphisms of the second type. Since the inclusion $O'_g \cap U^{(2)}(g) \subseteq U^{(2)}(g)$ holds for any $g \in G$, the relations are true

$$\mathrm{Mantiend}(\Omega, G) = \bigcap_{g \in G} (O'_g \cap U^{(2)}(g)) \subseteq \bigcap_{g \in G} U^{(2)}(g) = T(\Omega).$$

Thus, by virtue of Theorem 1, the semiheap $\mathrm{Mantiend}(\Omega, G)$ consists of endomorphisms of the second type. The theorem has been proven. □

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Example 2. Let G be a groupoid defined by its left translation: $l_1 = (1,1)$, $l_2 = (2,2)$. Let us show that the set $\mathrm{Mantiend}(\Omega, G)$ is not empty. Equalities are fulfilled:

$$\overline{M}_i = \{1,2\}, \quad O'_i = \{\alpha \in I(G) \mid \alpha(\{1,2\}) \subseteq \{1,2\}\} = I(G), \quad i = 1,2.$$

Let $\alpha(z) = 1$ for any $z \in G$ (α is a constant). Since l_1, l_2 do not belong to the set $\{r_1, r_2\}$ and for any $z \in G$ the equalities

$$(\alpha \cdot l_1)(z) = \alpha(1) = 1, \quad (r_1 \cdot \alpha)(z) = r_1(1) = 1, \quad (\alpha \cdot l_1)(z) = \alpha(2) = 1$$

are true, then α belongs to the sets $U^{(2)}(1)$ and $U^{(2)}(2)$. Hence, the transformation α belongs to the set $\mathrm{Mantiend}(\Omega, G)$. Thus, the set $\mathrm{Mantiend}(\Omega, G)$ is not empty.

Let γ be a mixed type from $\mathrm{Bte}(G)$. Then we introduce the notation

$$Q_1(\gamma) := \{g \in G \mid \gamma(g) = 1\}, \quad Q_2(\gamma) := \{g \in G \mid \gamma(g) = 2\}.$$

Let us define the set of transformations in $I(G)$:

$$\mathrm{Mantiend}(\gamma, G) := \left(\bigcap_{g \in Q_1(\gamma)} (O_g \cap C(l_g)) \right) \cap \left(\bigcap_{g \in Q_2(\gamma)} (O'_g \cap U^{(2)}(g)) \right).$$

Conjecture 1. If the set $\mathrm{Mantiend}(\gamma, G)$ is not empty for the groupoid G, then it is a subsemiheap of anti-endomorphisms of type γ in the semiheap of all anti-endomorphisms.

Conjecture 2. If the set $\mathrm{Mantiend}(A, G)$ is not empty for the groupoid G, then it consists of endomorphisms.

4. Conclusion

Theorem 1 can be used to describe element by element the set of all anti-endomorphisms of some particular groupoid. In this case, the calculation of the base sets of anti-endomorphisms in the general case is a non-trivial problem. Theorems 2 and 3 can be useful for studying the structure of the semiheap of all anti-endomorphisms of a particular groupoid.

The bipolar classification of anti-endomorphisms of some groupoid (expressed by Definition 7 on the basis of Theorem 1) can be used for theoretical studies in which anti-endomorphisms are used as a technical means of reasoning and formulating objects and statements.

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