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Elliptic Equations with Arbitrarily Directed Translations in Half-Spaces

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Abstract. In this paper, we investigate the half-space Dirichlet problem for elliptic differential-difference equations with superpositions of differential operators and translation operators acting in arbitrary directions parallel to the boundary hyperplane. The summability assumption is imposed on the boundary-value function of the problem. The specified equations, substantially generalizing classical elliptic partial differential equations, arise in various models of mathematical physics with nonlocal and (or) heterogeneous properties or the process or medium: multi-layer plates and envelopes theory, theory of diffusion processes, biomathematical applications, models of nonlinear optics, etc. The theoretical interest to such equations is caused by their nonlocal nature: they connect values of the desired function (and its derivatives) at different points (instead of the same one), which makes many classical methods unapplicable.

For the considered problem, we establish the solvability in the sense of generalized functions, construct Poisson-like integral representations of solutions, and prove the infinite smoothness of the solution outside the boundary hyperplane and its uniform convergence to zero (together with all its derivatives) as the timelike variable tends to infinity. We find a power estimate of the velocity of the specified extinction of the solution and each its derivative.

Keywords: differential-difference equations, elliptic equations, half-space Dirichlet problems, summable boundary-value functions

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Научная статья

Эллиптические уравнения со сдвигами произвольных направлений в полупространстве

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Аннотация. Исследуется задача Дирихле в полупространстве для эллиптических дифференциально-разностных уравнений с операторами, представляющими собой суперпозиции дифференциальных операторов и операторов сдвига в произвольных направлениях, параллельных краевой гиперплоскости. На краевую функцию задачи накладывается условие суммируемости. Указанные уравнения, существенно обобщающие классические эллиптические уравнения в частных производных, возникают в различных моделях математической физики, для которых имеют место нелокальные и (или) неоднородные свойства процесса или среды: теория многослойных пластин и оболочек, теория диффузионных процессов, биоматематические приложения, модели нелинейной оптики и др. В теоретическом плане интерес к таким уравнениям обусловлен их нелокальной природой — они связывают между собой значения неизвестной функции (и ее производных) не в одной точке, а в разных, что делает неприменимыми многие классические методы.
Для рассматриваемой задачи устанавливается разрешимость в смысле обобщенных функций, строится интегральное представление решения формулой пуассоновского типа, доказывается его бесконечная гладкость вне краевой гиперплоскости и его равномерное стремление к нулю (вместе со всеми его производными) при стремлении времениподобной независимой переменной к бесконечности. Доказывается степенная оценка скорости указанного равномерного затухания решения и каждой его производной.

Ключевые слова: дифференциально-разностные уравнения, эллиптические уравнения, задача Дирихле в полупространстве, суммируемые краевые функции

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1. Introduction

We consider the Dirichlet problem (with summable boundary-value functions) for functional-differential equations containing superpositions of differential operators and arbitrarily directed translation operations. Namely, in the half-space $\Big\{(x,y)\Big|x\in\mathbb{R}^n, y>0\Big\}$, we consider the problem

$$\sum_{j=1}^{n} u_{x_jx_j}(x,y)+u_{yy}(x,y)+\sum_{j=1}^{n} a_j u_{x_jx_j}(x+h_j,y)=0, \qquad (1.1)$$

$$u\Big|_{y=0}=u_0(x),\ x\in\mathbb{R}^n, \qquad (1.2)$$

where

$$a_0:=\max_{j=\overline{1,n}}|a_j|<1, \qquad (1.3)$$

$h_j:=(h_{j1},\dots,h_{jn})$, $j=\overline{1,n}$, are arbitrary vectors from $\mathbb{R}^n$, and $u_0\in L_1(\mathbb{R}^n)$.
This substantially generalizes the problem investigated in [9], where translation operators act only with respect to coordinate directions. In the present paper, this restriction is taken off, which means that translations of the independent (vector) variable x are not supposed to be either collinear or orthogonal to each other; instead, each angle between the directions of those translations is admitted.

The current interest to *differential-difference* equations, i. e., to equations where translation operators act on the desired function apart from differential operators, is caused both by their numerous applications not covered by classical models of mathematical physics (see, e. g., [13–15; 21] and references therein) and by purely theoretical reasonings: the *nonlocal* nature of such equations generates qualitatively new phenomena not observed in the classical case of differential equations, while various research methods proved their efficiency in the theory of differential equations (e. g., such as the maximum principle) turn to be inapplicable. A comprehensive and profound explanation of the theory of bounded-region problems for elliptic differential-difference equations (as well as for the very close theory of nonlocal problems for *differential* elliptic equations) can be found in [3; 13; 16–18] (see references therein as well). Problems in unbounded regions are investigated not so profoundly. Moreover, the case of *bounded* boundary-value functions is mainly investigated (see [8] and references therein). In the present paper, the case of *summable* boundary-value functions is studied. This decomposition of half-space problems into two classes (problems with *bounded* boundary-value functions and problems with *summable* boundary-value functions) takes place for the classical case of differential equations (both elliptic and parabolic ones) as well. Such a separation is reasonable because the specified difference in the problem settings generates solutions with qualitatively different sets of properties: for example, constant solutions are possible only for problems from the first of the above classes, while only finite-energy solutions are possible for problems from the second class. In other words, the Repnikov–Eidel'man stabilization criterion (see [10; 11]), according to which solutions with and without limits (in general, different from zero) are possible, is valid only for problems with bounded boundary-value (initial-value in the parabolic case) functions. For problems with summable boundary-value (initial-value) functions, the solution always has a limit and it is always equal to zero, while the main research interest in such problems is concentrated on the estimate of the decay rate for the solution.

2. Operational scheme

We use the classical Gel'fand–Shilov operational scheme (see, e. g., [1, Sec. 10]): apply (formally) the Fourier transformation with respect to the (vector) variable x to problem (1.1)-(1.2). This leads to the following initial-value problem to an ordinary differential equation:

$$\frac{d^2\widehat{u}}{dy^2} = \left(|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 e^{ih_j\cdot\xi}\right)\widehat{u}, \ y \in (0, +\infty), \tag{2.1}$$

$$\widehat{u}(0;\xi) = \widehat{u}_0(\xi). \tag{2.2}$$

Note that this is not a Cauchy problem because the order of the equation is equal to two, while the initial-value condition is unique.
Thus, (2.1) is a second-order linear differential equation (depending on the n-dimensional parameter ξ) with constant coefficients, such that its characteristic equation has two roots

$$\pm\sqrt{|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 e^{ih_j\cdot\xi}} = \pm\sqrt{|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi + i\sum_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}$$
$$=: \pm\rho(\cos\theta + i\sin\theta),$$

where

$$\rho(\xi) = \left[\left(|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2 + \left(\sum_{j,k=1}^{n} a_{jk}\xi_j^2 \sin h_j\cdot\xi\right)^2\right]^{\frac{1}{4}},$$

$$\theta(\xi) = \frac{1}{2}\operatorname{arctg}\frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}{|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi}.$$

Arguing as in [6, Sec. 1], we solve problem (2.1)-(2.2), suitably select the "free" arbitrary constant (it arises because the amount of initial-value conditions is less than the order of the equation), and (formally) apply the inverse Fourier transformation to the obtained solution. This yields the convolution of the boundary-value function with the integral (with respect to ξ) of the function $e^{-y\rho(\xi)\cos\theta(\xi)}\cos[x\cdot\xi - y\rho(\xi)\sin\theta(\xi)]$:

$$u(x,y) = \int\limits_{\mathbb{R}^n} \mathcal{E}(x-\xi,y)u_0(\xi)d\xi, \tag{2.3}$$

where

$$\mathcal{E}(x,y) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-yG_1(\xi)}\cos\left[x\cdot\xi - yG_2(\xi)\right]d\xi, \tag{2.4}$$

$$G_1(\xi) = \rho(\xi)\cos\theta(\xi), \quad \text{and} \quad G_2(\xi) = \rho(\xi)\sin\theta(\xi). \tag{2.5}$$

Denote $\sum\limits_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi$ by $a(\xi)$. Denote $\sum\limits_{j=1}^{n} a_j\xi_j^2\sin h_j\cdot\xi$ by $b(\xi)$. Then $\rho(\xi) = \left(\left[|\xi|^2 + a(\xi)\right]^2 + b^2(\xi)\right)^{\frac{1}{4}}$ and $\theta(\xi) = \frac{1}{2}\operatorname{arctg}\frac{b(\xi)}{|\xi|^2 + a(\xi)}$. This implies that $|\theta(\xi)| \le \frac{\pi}{4}$, i. e., $\cos\theta(\xi) \ge \frac{\sqrt{2}}{2}$ and $\cos 2\theta(\xi) \ge 0$. Hence, $\cos\theta(\xi)$ can be represented in the form $\sqrt{\frac{1+\cos 2\theta(\xi)}{2}}$. Now, apply the relation

$$\cos^2 2\theta(\xi) = \frac{1}{1+\operatorname{tg}^2 2\theta(\xi)}$$

and take into account the nonnegativity of $\cos 2\theta(\xi)$ again. We obtain that

$$\cos 2\theta(\xi) = \frac{1}{\sqrt{1+\operatorname{tg}^2 2\theta(\xi)}}.$$

Further, since

$$2\theta(\xi) = \operatorname{arctg}\frac{b(\xi)}{|\xi|^2 + a(\xi)},$$

it follows that

$$\operatorname{tg} 2\theta(\xi) = \frac{b(\xi)}{|\xi|^2 + a(\xi)}$$

and, therefore,

$$\cos 2\theta(\xi) = \left(1 + \frac{b^2(\xi)}{\left[|\xi|^2 + a(\xi)\right]^2}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{\left[|\xi|^2 + a(\xi)\right]^2}{\left[|\xi|^2 + a(\xi)\right]^2 + b^2(\xi)}} = \frac{|\xi|^2 + a(\xi)}{\sqrt{\left[|\xi|^2 + a(\xi)\right]^2 + b^2(\xi)}}$$

because Condition (1.3) ensures the nonnegativity of the function $|\xi|^2+a(\xi)$ for each $\xi \in \mathbb{R}^n$ and its positivity everywhere apart from the origin:

$$|\xi|^2 + \sum_{j=1}^{n} a_j \xi_j^2 \cos h_j \cdot \xi \geq |\xi|^2 - \sum_{j=1}^{n} |a_j| \xi_j^2$$
$$\geq |\xi|^2 - \max_{j-\overline{1,n}} |a_j| \sum_{j=1}^{n} \xi_j^2 = (1-a_0)|\xi|^2 > 0$$

provided that $\xi \neq 0$. Then

$$\cos\theta(\xi) = \frac{1}{\sqrt{2}} \left[1 + \frac{|\xi|^2 + a(\xi)}{\sqrt{[|\xi|^2 + a(\xi)]^2 + b^2(\xi)}} \right]^{\frac{1}{2}}.$$

In $\mathbb{R}^n$, introduce the function

$$\varphi(\xi) := \sqrt{[|\xi|^2 + a(\xi)]^2 + b^2(\xi)} = \sqrt{|\xi|^4 + a^2(\xi) + b^2(\xi) + 2a(\xi)|\xi|^2}.$$

Then

$$G_1(\xi) = \sqrt{\varphi(\xi)} \frac{1}{\sqrt{2}} \left[1 + \frac{|\xi|^2 + a(\xi)}{\sqrt{[|\xi|^2 + a(\xi)]^2 + b^2(\xi)}} \right]^{\frac{1}{2}}$$
$$= \sqrt{\varphi(\xi)} \frac{1}{\sqrt{2}} \sqrt{\frac{\varphi(\xi) + |\xi|^2 + a(\xi)}{\varphi(\xi)}} = \frac{1}{\sqrt{2}} \sqrt{\varphi(\xi) + |\xi|^2 + a(\xi)}$$
$$\geq \frac{1}{\sqrt{2}} \sqrt{|\xi|^2 + a(\xi)}$$

(by virtue of the nonnegativity of the function φ). Therefore, the function $G_1(\xi)$ is bounded from below by the expression

$$\frac{1}{\sqrt{2}} \sqrt{|\xi|^2 - \sum_{j=1}^{n} |a_j| \xi_j^2} = \frac{1}{\sqrt{2}} \sqrt{|\xi|^2 - \sum_{j=1}^{n} \xi_j^2 \sum_{j=1}^{n} |a_j|} \geq \frac{1}{\sqrt{2}} \sqrt{|\xi|^2 - a_0 \sum_{j=1}^{n} \xi_j^2}$$
$$= \sqrt{\frac{1-a_0}{2}}\, |\xi|,$$

which guarantees that function (2.4) is well defined in $\mathbb{R}^n \times (0, +\infty)$.

Note that, applying the direct and inverse Fourier transformations in the present section, we do not care about the justification of the convergence of the integrals and the legibility of the change of the order of integrating. This entirely corresponds to the general scheme of [1, Sec. 10] because solutions in the sense of generalized functions are meant. Lemma 1 (see the next section) deals with smooth functions, but it is proved independently.

3. Constructing of Poisson kernels

The following assertion is valid.

Lemma 1. *The function* $\mathcal{E}(x,y)$ *defined by relation* (2.4) *is well defined in the half-space* $\mathbb{R}^n \times (0,+\infty)$ *and satisfies* (*in the classical sense*) *Eq.* (1.1).

Proof. It is shown in the previous section that the function $G_1(\xi)$ is estimated from below by the function $\sqrt{\frac{1-a_0}{2}}\,|\xi|$. Then the function $\mathcal{E}(x,y)$ is well defined in $\mathbb{R}^n \times (0,+\infty)$. Really, the modulus of the integrand function in (2.4) is majorized by the integrable function *const* $\exp\left(-\sqrt{1-|a_0|}\,y|\xi|\right)$. Now, compute the Laplacian of function (2.4). We have the relations

$$\mathcal{E}_{x_jx_j}(x,y) = -\int\limits_{\mathbb{R}^n} \xi_j^2 e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi,\ j=\overline{1,n},$$

$$\mathcal{E}_y(x,y) = -\int\limits_{\mathbb{R}^n} G_1(\xi) e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$+\int\limits_{\mathbb{R}^n} G_2(\xi) e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi,$$

and

$$\mathcal{E}_{yy}(x,y) = \int\limits_{\mathbb{R}^n} G_1^2(\xi) e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$-\int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi) e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$-\int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi) e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$-\int\limits_{\mathbb{R}^n} G_2^2(\xi) e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi =$$
$$= \int\limits_{\mathbb{R}^n} \left[G_1^2(\xi) - G_2^2(\xi)\right] e^{-yG_1(\xi)} \cos\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$
$$-2\int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi) e^{-yG_1(\xi)} \sin\left[x\cdot\xi - yG_2(\xi)\right] d\xi$$

(note that the differentiating inside the integral is legible in all these cases because the arising integrand factors have no singularities and their growth with respect to ξ is at most polynomial).

Further, it follows from (2.5) that

$$2G_1(\xi)G_2(\xi) = 2\rho(\xi)\cos\theta(\xi)\rho(\xi)\sin\theta(\xi) = \rho^2(\xi)\sin 2\theta(\xi).$$

It is proved above that $\cos 2\theta(\xi) \geq 0$. Therefore,

$$\sin 2\theta(\xi) = \frac{\operatorname{tg} 2\theta(\xi)}{\sqrt{1+\operatorname{tg}^2 2\theta(\xi)}}$$

$$= \frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}{|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi}\left[1 + \frac{\left(\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi\right)^2}{\left(|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2}\right]^{-\frac{1}{2}}.$$

Hence, the following relation is valid:

$$\sin 2\theta(\xi) = \frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}{|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi}$$

$$\times \sqrt{\frac{\left(|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2}{\left(|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2 + \left(\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi\right)^2}}.$$

Since the sum $|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi$ is nonnegative by virtue of Condition (1.3), we eventually obtain the relation

$$\sin 2\theta(\xi) = \frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}{\sqrt{\left(|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j\cdot\xi\right)^2 + \left(\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi\right)^2}}.$$

Therefore, the following relation holds:

$$2G_1(\xi)G_2(\xi) =$$

$$= \rho^2(\xi)\frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j\cdot\xi}{\sqrt{\left(|\xi|^2+\sum\limits_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi\right)^2+\left(\sum\limits_{j=1}^{n} a_j\xi_j^2\sin h_j\cdot\xi\right)^2}} =$$

$$= \sum_{j=1}^{n} a_j\xi_j^2\sin h_j\cdot\xi = b(\xi).$$

Now, using (2.5), we obtain that

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\Big[\cos^2\theta(\xi)-\sin^2\theta(\xi)\Big]\rho^2(\xi)\cos 2\theta(\xi) =$$

$$= \rho^2(\xi)\sqrt{1-\frac{\operatorname{tg}^2 2\theta(\xi)}{1+\operatorname{tg}^2 2\theta(\xi)}} = \rho^2(\xi)\frac{1}{\sqrt{1+\operatorname{tg}^2 2\theta(\xi)}},$$

which is equal to

$$\rho^2(\xi)\sqrt{\frac{\left(|\xi|^2+\sum\limits_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi\right)^2}{\left(|\xi|^2+\sum\limits_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi\right)^2+\left(\sum\limits_{j=1}^{n} a_j\xi_j^2\sin h_j\cdot\xi\right)^2}} =$$

$$= |\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2\cos h_j\cdot\xi = |\xi|^2 + a(\xi).$$

This means that

$$\mathcal{E}_{yy}(x,y) = \int\limits_{\mathbb{R}^n}\left[|\xi|^2+a(\xi)\right]e^{-yG_1(\xi)}\cos\left[x\cdot\xi - yG_2(\xi)\right]d\xi -$$

$$- \int\limits_{\mathbb{R}^n} b(\xi)e^{-yG_1(\xi)}\sin\left[x\cdot\xi - yG_2(\xi)\right]d\xi.$$

Now, substitute the function $\mathcal{E}(x,y)$ in Eq. (1.1):

$$\sum_{j=1}^{n}\mathcal{E}_{x_jx_j}(x,y)+\mathcal{E}_{yy}(x,y)+\sum_{j=1}^{n}a_j\mathcal{E}_{x_jx_j}(x+h_j,y)=$$

$$=\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\Big(a(\xi)\cos\left[x\cdot\xi-yG_2(\xi)\right]-b(\xi)\sin\left[x\cdot\xi-yG_2(\xi)\right]\Big)d\xi-$$

$$-\sum_{j=1}^{n}a_j\int\limits_{\mathbb{R}^n}\xi_j^2e^{-yG_1(\xi)}\cos\left[(x+h_j)\cdot\xi-yG_2(\xi)\right]d\xi.$$

This is equal to

$$\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\Big(\cos\left[x\cdot\xi-yG_2(\xi)\right]\sum_{j=1}^{n}a_j\xi_j^2\cos h_j\cdot\xi-$$

$$-\sin\left[x\cdot\xi-yG_2(\xi)\right]\sum_{j=1}^{n}a_j\xi_j^2\sin h_j\cdot\xi-$$

$$-\sum_{j=1}^{n}a_j\xi_j^2\cos\left[(x+h_j)\cdot\xi-yG_2(\xi)\right]\Big)d\xi=$$

$$=\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\sum_{j=1}^{n}a_j\xi_j^2\Big(\cos h_j\cdot\xi\cos\left[x\cdot\xi-yG_2(\xi)\right]-$$

$$-\sin h_j\cdot\xi\sin\left[x\cdot\xi-yG_2(\xi)\right]-\cos\left[(x+h_j)\cdot\xi-yG_2(\xi)\right]\Big)d\xi,$$

which is equal to

$$\int\limits_{\mathbb{R}^n}e^{-yG_1(\xi)}\sum_{j=1}^{n}a_j\xi_j^2\Big(\cos\left[(x+h_j)\cdot\xi-yG_2(\xi)\right]-$$

$$-\cos\left[(x+h_j)\cdot\xi-yG_2(\xi)\right]\Big)d\xi.$$

This is equal to zero, i.e., the function $\mathcal{E}(x,y)$ satisfies (in the classical sense) Eq. (1.1) in the half-space $\mathbb{R}^n\times(0,+\infty)$. □

4. Convolutions with summable functions

Now, majorize the function $\mathcal{E}(x,y)$ itself as well as its derivatives of arbitrary orders:

$$\int\limits_{\mathbb{R}^n} |\xi|^m e^{-y|\xi|\sqrt{\frac{1-|a_0|}{2}}} d\xi = \frac{1}{(1-|a_0|)^{\frac{m+n}{2}} y^{m+n}} \int\limits_{\mathbb{R}^n} |\eta|^m e^{-|\eta|} d\eta =$$

$$= \frac{const}{y^{m+n}} \int\limits_0^\infty \rho^{m+n-1} e^{-\rho} d\rho = \frac{const}{y^{m+n}}.$$

Taking into account Lemma 1, we obtain the following assertion.

Theorem 1. *If $u_0 \in L_1(\mathbb{R}^n)$, then the function $u(x,y)$ defined by relation* (2.3) *is an infinitely differentiable solution of Eq.* (1.1) *in* $\mathbb{R}^n \times (0,+\infty)$. *This function and each its partial derivative tend to zero as $y \to +\infty$ uniformly with respect to $x \in \mathbb{R}^n$.*

Note that, similarly to less general cases investigated earlier (see [9] and references therein), a greater smoothness is established for the solution of the equation (not of the boundary-value problem). At the moment, the solvability of problem (1.1)-(1.2) is proved only in the Gel'fand–Shilov sense (see [1, Sec. 10] and cf. [7, Remark 2]): (2.3) is a *generalized function* of the n-dimensional variable x, depending on the real parameter y and differentiable with respect to this parameter on the positive semiaxis (see, e. g., [12, §9, Sec. 5]), Eq. (1.1) is treated as the equality of generalized functions of variable x, satisfied for each positive value of the parameter y, and Condition (1.2) is treated as a limit relation in the topology of generalized functions of variable x as the real parameter y tends to zero from the right (see, e. g., [12, §9, Sec. 4]).

However, it is established by Theorem 1 that this solution possesses a greater smoothness (more exactly, the infinite smoothness) *outside the boundary-value hyperplane*, i. e., in the *open* half-space $\mathbb{R}^n \times (0,+\infty)$.

Remark 1. Though the problem is set in an unbounded region, it is not a Cauchy problem (unlike the parabolic case, see, e. g., [4; 5]). This phenomenon (an elliptic equation considered in an anisotropic region acquires qualitative properties typical for *nonstationary* equations) is well known for the classical case of *differential* equations (see, e. g., [19; 20]). In the *nonlocal* case, this phenomenon is still actively investigated.

Remark 2. The solvability result obtained above complements the results of [2] where the solvability of the investigated problem is proved for various classes of boundary-value functions (in particular, it is known for classes of functions belonging to $L_p(\mathbb{R}^n), 1 < p < \infty$, and such that their Fourier transforms are compactly supported in a domain of $\mathbb{R}^n$). As far as the authors are aware, the case where $p = 1$ was not considered earlier.

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