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Optimal Location Problem for Composite Bodies with Separate and Joined Rigid Inclusions

Nyurgun P. Lazarev[1✉], **Galina M. Semenova**[1]

[1] North-Eastern Federal University, Yakutsk, Russian Federation
✉ nyurgun.ngs.ru

Abstract. Nonlinear mathematical models describing an equilibrium state of composite bodies which may come into contact with a fixed non-deformable obstacle are investigated. We suppose that the composite bodies consist of an elastic matrix and one or two built-in volume (bulk) rigid inclusions. These inclusions have a rectangular shape and one of them can vary its location along a straight line. Considering a location parameter as a control parameter, we formulate an optimal control problem with a cost functional specified by an arbitrary continuous functional on the solution space. Assuming that the location parameter varies in a given closed interval, the solvability of the optimal control problem is established. Furthermore, it is shown that the equilibrium problem for the composite body with joined two inclusions can be considered as a limiting problem for the family of equilibrium problems for bodies with two separate inclusions.

Keywords: optimal control problem, composite body, Signorini conditions, rigid inclusion, location

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Научная статья

Задача об оптимальном расположении включений для композитных тел с отдельными и соединенными жесткими включениями

Н. П. Лазарев[1✉], **Г. М. Семенова**[1]

[1] Северо-Восточный федеральный университет, Якутск, Российская Федерация
✉ nyurgun@ngs.ru

Аннотация. Исследуются нелинейные математические модели, описывающие состояние равновесия композитных тел, которые могут контактировать с неподвижным недеформируемым препятствием. Предполагается, что композитные тела состоят из упругой матрицы и одного или двух встроенных объемных жестких включений, эти включения имеют прямоугольную форму, при этом одно из них может изменять свое расположение вдоль прямой линии. Рассматривая параметр расположения как параметр управления, сформулирована задача оптимального управления с функционалом качества, заданным произвольным непрерывным функционалом на пространстве решений. В предположении, что параметр расположения изменяется на заданном замкнутом интервале, доказывается разрешимость задачи оптимального управления. Кроме того, установлено, что задачу о равновесии композитного тела с двумя соединенными включениями можно рассматривать как предельную задачу для семейства задач о равновесии тел с двумя отдельными включениями.

Ключевые слова: задача оптимального управления, композитное тело, условия Синьорини, жесткое включение, расположение

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1. Introduction

Clear advantages of using of composite parts in industry have increased the need for high-precision mathematical models in order to design and optimize in an efficient way composite structures. Along with tasks of improving the physicochemical properties of the elements of composite bodies, one of the important issues related to the creation of reinforced composites is investigation of the best location and geometric shape of built-in components. The direction of research related to nonlinear problems describing deformation of elastic bodies with rigid or elastic inclusions is an actual area of applied mathematics, see, for example, [8–14; 24–27]. Nonlinear model approach using well-known Signorini type boundary conditions can be applied for contact problems [1; 15; 18; 21]. This approach leads to variational problems with an unknown contact zone. Optimal control of volume or Neumann forces in the framework of Signorini type problems was studied, for example, in [2;23]. A classification of the different optimality systems of strong stationarity for the case of optimal control for obstacle problems can be found in [5; 28]. The researches on the shape and topological sensitivity analysis of variational inequalities have been actively elaborating [4; 6; 20; 22]. A shape-topological control problem for nonlinear crack - defect interaction was investigated in [16].

We study an optimal control problem for nonlinear mathematical models describing an equilibrium state of composite bodies contacting with a fixed non-deformable obstacle. We suppose that the composite bodies consist of an elastic matrix and two built-in volume (bulk) rigid inclusions. These inclusions have a rectangular shape and one of them can vary its location along a straight line. For the optimal control problem under consideration, a cost functional is specified by an arbitrary continuous functional defined on the solution's space, while the location parameter of one rigid inclusion serves as a control. In [19] the solvability of optimal location problem for a family of contact problems with finite number of inclusions was established. Despite of the arbitrariness of the number of rigid inclusions, the solvability of a relevant optimal control problem was established under the restriction of a given nonzero distance between inclusions. In contrast to that result, the current study deals with the case of arbitrarily close two inclusions. Moreover, it should be noted that in the limit case, when the distance between inclusions is equal to zero, we have one united rigid inclusion that geometrically corresponds to the union of relevant sets. Assuming that the location parameter varies in a closed interval, the solvability of the optimal control problem is established. Furthermore, it is shown that the equilibrium problem for the composite body with joined two inclusions can be considered as a limiting problem for the family of equilibrium problems for bodies with two separate inclusions.

2. Formulation of variational problems

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma \in C^{0,1}$, $\Gamma = \Gamma_0 \cup \Gamma_c$, $\mathrm{meas}(\Gamma_0) > 0$. We consider two square subdomains $\omega, \omega_s \subset \Omega$, $s \in [2, S]$, $S > 2$, which are defined by the following relations:

$$\omega = (-1,1) \times (-1,1),$$

$$\omega_s = \{(x_1, x_2) \,:\, x_1 = y_1 + s, x_2 = y_2, \quad (y_1, y_2) \in \omega\}.$$

We suppose that both domains lie strictly inside in the domain Ω, i.e.

$$dist(\overline{\omega}, \partial\Omega) > 0,$$

$$dist(\overline{\omega}_s, \partial\Omega) > 0 \quad \text{for each} \quad s \in [2, S].$$

Remark 1. This assumption allows us to apply trace theorems and well-known results concerning characterization of Sobolev spaces in Lipschitz domains $\Omega \setminus \overline{\omega}$, $\Omega \setminus \overline{\omega}_s$.

Denote by $W = (w_1, w_2)$ the displacement vector. Introduce the tensors describing the deformation of an elastic part of the inhomogeneous body

$$\varepsilon_{11}(W) = \frac{\partial w_1}{\partial x_1}, \quad \varepsilon_{12}(W) = \varepsilon_{21}(W) = \frac{1}{2}\left(\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\right), \varepsilon_{22}(W) = \frac{\partial w_2}{\partial x_2}.$$

$$\sigma_{ij}(W) = c_{ijkl}\varepsilon_{kl}(W), \ i, j = 1, 2,$$

where c_{ijkl} is the given elasticity tensor, assumed to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, \quad c_{ijkl} = const,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const, \quad c_0 > 0.$$

By the assumption concerning the domain Ω and the Korn's inequality [7], the following inequality holds

$$\int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(W) d\Omega \geq c\|W\|^2_{H(\Omega)}, \ \ \forall W \in H(\Omega), \tag{2.1}$$

with a constant $c > 0$ independent of W.

Remark 2. The inequality 2.1 yields the equivalence of the standard norm in $H(\Omega)$ and the semi-norm determined by the left-hand side of 2.1.

To formulate mathematical models for a composite body with volume (bulk) rigid inclusions, we will use the notion of a rigid inclusion which in general can occupy an arbitrary subdomain $\mathcal{O} \subset \Omega$. In this case the displacements on the domain $\mathcal{O}$ should have a special structure $W|_{\mathcal{O}} = \rho$, where $\rho \in R(\mathcal{O})$ and $R(\mathcal{O})$ is the space of infinitesimal rigid displacements on $\mathcal{O}$

$$R(\mathcal{O}) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x_1, x_2) = b(x_2, -x_1) + (c_1, c_2);$$
$$b, c_1, c_2 \in \mathbb{R}, \ (x_1, x_2) \in \mathcal{O}\},$$

see, [13]. In the sequel we deal with two type of problems, the first describes an equilibrium of a composite body with a single rigid inclusion prescribed with the set $\omega \cup \omega_2$, and the second one corresponds to a composite body with two separate rigid inclusion prescribed with the sets ω, ω_s, $s \in (2, S]$.

For both types of problems, we have common conditions on the external boundary Γ. We suppose that the body is fixed on the part Γ_0 of the boundary, i.e.

$$W = (0, 0) \quad \text{on} \quad \Gamma_0. \tag{2.2}$$

According to the last condition, we deal with the following Sobolev spaces

$$H^{1,0}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_0\}, \quad H(\Omega) = H^{1,0}(\Omega)^2.$$

The Signorini condition of contact interaction is written as

$$W\nu \leq 0 \quad \text{on} \quad \Gamma_c,$$

where $\nu = (\nu_1, \nu_2)$ is an outward normal to Γ. We introduce the energy functional

$$\Pi(W) = \frac{1}{2} \int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(W) d\Omega - \int\limits_{\Omega} FW d\Omega, \tag{2.3}$$

where $F = (f_1, f_2) \in L^2(\Omega)^2$ is a given vector of exterior forces.

Now we formulate an equilibrium problem describing a contact of a composite body with a single united rectangular inclusion which corresponds to the set $\omega_+ = \text{int}(\overline{\omega \cup \omega_2})$. Furthermore the remaining part of the domain $\Omega \backslash \omega_+$ corresponds to the elastic matrix. It is required to

$$\text{Find } U_2 \in K(2),$$
$$\text{such that} \quad \Pi(U_2) = \inf_{W \in K(2)} \Pi(W), \tag{2.4}$$

where the set of admissible displacements is defined as follows

$$K(2) = \{W \in H(\Omega) \mid W\nu \leq 0 \quad \text{on} \quad \Gamma_c,$$
$$W|_{\omega_+} = \rho, \text{ where } \rho \in R(\omega_+)\}.$$

It should be noted that, without loss of generality, due to properties of functions $W \in H(\Omega)$, we can require the relation $W \in R(\omega \cup \omega_2)$ instead of $W \in R(\omega_+)$. The problem 2.4 has a unique solution $U_2 \in K(2)$, and can be represented in the equivalent form of the variational inequality [3]

$$\int\limits_{\Omega} \sigma_{ij}(U_2)\varepsilon_{ij}(W - U_2) d\Omega \geq \int\limits_{\Omega} F(W - U_2) d\Omega, \tag{2.5}$$

for all $W \in K(2)$.

Consider a family of equilibrium problems, where sets ω, ω_s of rigid inclusions are located at some distance from each other. Next, we fix the coordinate parameter $s \in (2, S]$, which defines a location of the inclusion domain ω_s, while the set

$$\Omega \backslash (\omega \cup \omega_s),$$

corresponds to the elastic part of the body. An equilibrium problem of a composite body with two separate rigid inclusions can be formulated as the following minimization problem

$$\text{Find } U_s \in K(s),$$

$$\text{such that} \quad \Pi(U_s) = \inf_{W \in K(s)} \Pi(W), \tag{2.6}$$

where the set of admissible displacements is defined as follows

$$K(s) = \{W \in H(\Omega) \mid W\nu \le 0 \quad \text{on} \quad \Gamma_c,$$

$$W|_{\omega} = \rho, \quad W|_{\omega_s} = \rho_s, \text{ where } \rho \in R(\omega),\ \rho_s \in R(\omega_s)\}.$$

The problem 2.6 is known to have a unique solution $U_s \in K(s)$, which satisfies the variational inequality [3]

$$\int_{\Omega} \sigma_{ij}(U_s)\varepsilon_{ij}(W - U_s) d\Omega \ge \int_{\Omega} F(W - U_s) d\Omega, \tag{2.7}$$

for all $W \in K(s)$.

3. Optimal control problem

Let's define a cost functional $J : [2, S] \to \mathbb{R}$ of an optimal control problem with the use of the equality $J_G(s) = G(U_s)$, where U_2 is the solution of the problem 2.4 for $s = 2$ and U_s represents the solution of the problem 2.6 for $s \in (2, S]$, a functional $G : H(\Omega) \to \mathbb{R}$ satisfies continuity property in $H(\Omega)$.

As examples of such functionals having physical sense, we can provide the functional $G_1(W) = \|W - W_0\|_{H(\Omega)}$ characterizing the deviation of the displacement vector from a given function W_0. Consider the optimal control problem:

$$\text{Find} \quad s^* \in [2, S] \quad \text{such that} \quad J_G(s^*) = \sup_{s \in [2,S]} J_G(s). \tag{3.1}$$

This means that we want to find the best location of one of the separate two rigid inclusions or to reveal that the optimal configuration fits one united single rigid inclusion which provides the maximal value for the cost functional. The following is the main result of the paper.

Theorem 1. *There exists a solution of the optimal control problem 3.1.*

Proof. Let $\{s_n\} \subset [2, S]$ be a maximizing sequence. By the compactness of the set $[2, S] \subset \mathbb{R}$, we can extract a convergent number subsequence of real numbers $\{s_{n_k}\} \subset \{s_n\}$ such that

$$s_{n_k} \to s^* \quad \text{as} \quad k \to \infty, \quad s^* \in [2, S].$$

Let us consider two possible different cases. The first case corresponds to the inequality $s^* > 2$, and the second one to the equality $s^* = 2$. For the first case we can see that $s_{n_k} > 2$ and $dist(\omega, \omega_{s_{n_k}}) > \hat{\delta}$ for some $\hat{\delta} > 0$ and for sufficiently large k. In this case of nonzero minimal distance between rigid inclusions we can apply the results of the paper [19], where the solvability of the problem 3.1 was established.

Now we consider the second case when $s_{n_k} \to 2$ as $k \to \infty$. This case models the passage to the limit when inclusions tend to each other in order to get as a limit the single joined inclusion. Taking into account Lemma 2 proved below, we have a convergence $U_{s_{n_k}} \to U_2$ strongly in $H(\Omega)$ as $k \to \infty$. This allows us to obtain the convergence

$$J_G(s_{n_k}) \to J_G(2),$$

indicating that

$$J_G(2) = \sup_{s \in [2,S]} J_G(s).$$

The theorem is proved. □

4. Auxiliary lemmas

Now we have to justify some auxiliary lemmas which had to be used within the proof of the above theorem. In establishing the proof, we needed Lemma 2; however before proceeding further we need first prove the following lemma.

Lemma 1. *Let $\{s_n\} \subset [2, S]$ be a sequence of real numbers converging to 2 in $\mathbb{R}$ as $n \to \infty$. Then for an arbitrary function $W \in K(2)$ there exist a subsequence $\{s_k\} = \{s_{n_k}\} \subset \{s_n\}$ and a sequence of functions $\{W_k\}$ such that $W_k \in K(s_k)$, $k \in \mathbb{N}$ and $W_k \to W$ strongly in $H(\Omega)$ as $k \to \infty$.*

Proof. We construct new subdomains $\hat{\omega}_s = (-1, 1+s) \times (-1, 1)$, $s \in (2, S]$. One can note that ω and ω_s are subsets of $\hat{\omega}_s$. As the next step, we can consider auxiliary problems related to $\hat{\omega}_s$

$$\text{Find } \hat{U}_s \in \hat{K}(s),$$
$$\text{such that} \quad \Pi(\hat{U}_s) = \inf_{W \in \hat{K}(s)} \Pi(W),$$

where the set of admissible displacements is defined as follows

$$\hat{K}(s) = \{W \in H(\Omega) \mid W\nu \le 0 \quad \text{on} \quad \Gamma_c,$$
$$W|_{\hat{\omega}_s} = \rho, \text{ where } \rho \in R(\hat{\omega}_s)\}.$$

For this kind of problems, in [17] was proved that there exists a sequence of functions $W_k \in \hat{K}(s_k)$ such that $W|_{\hat{\omega}_{s_k}} \in R(\hat{\omega}_{s_k})$ and $W_k \to W$ strongly in $H(\Omega)$ as $k \to \infty$. Since $\hat{K}(s_k) \subset K(s_k)$, we obtain the assertion of the lemma. □

Now, we are in a position to prove an auxiliary statement which was used in the proof of the theorem.

Lemma 2. *Let $\{s_n\} \subset [2, S]$ be a sequence of real numbers converging to 2 in $\mathbb{R}$ as $n \to \infty$. Then $U_{s_n} \to U_2$ strongly in $H(\Omega)$ as $n \to \infty$, where U_{s_n}, are the solutions of 2.6 corresponding to parameters s_n, and U_2 is the solution of 2.4.*

Proof. We proceed by contradiction. Let us assume that there exist a number $\epsilon_0 > 0$ and a sequence $\{s_n\} \subset [2, S]$ such that $s_n \to 2$, $\|U_{s_n} - U_2\| \geq \epsilon_0$.

Because of $W^0 \equiv (0, 0) \in K(s_n)$ for all $n \in \mathbb{N}$, we can insert $W = W^0$ in 2.5 for fixed $n \in \mathbb{N}$. This provides

$$\int\limits_{\Omega} \sigma_{ij}(U_{s_n})\varepsilon_{ij}(U_{s_n})d\Omega \leq \int\limits_{\Omega} FU_{s_n}d\Omega, \quad \forall\, n \in \mathbb{N}.$$

From here, we conclude that for all $n \in \mathbb{N}$ the following uniform estimate holds

$$\|U_{s_n}\|_{H(\Omega)} \leq c$$

with some constant $c > 0$ independent of $n \in \mathbb{N}$. Consequently, replacing U_{s_n} by its subsequence if necessary, we can assume that U_{s_n} converges to some function $\tilde{U}$ weakly in $H(\Omega)$.

Now we show that $\tilde{U} \in K(2)$. Indeed, we have

$$U_{s_n}|_{\omega_{s_{n_k}}} = \rho_n \in R(\omega_{s_n}),$$

for all $n \in \mathbb{N}$. Due to the Sobolev embedding theorem [11], we conclude that

$$U_{s_n}|_{\omega_2} \to \tilde{U}|_{\omega_2} \quad \text{strongly in } L_2(\omega_2)^2 \text{ as } n \to \infty, \tag{4.1}$$

$$U_{s_n}|_{\Gamma} \to \tilde{U}|_{\Gamma} \quad \text{strongly in } L_2(\Gamma)^2 \text{ as } n \to \infty. \tag{4.2}$$

Choosing a subsequence, if necessary, we assume that $U_{s_n} \to \tilde{U}$ a.e. in ω_2 as $n \to \infty$.

In the next step we fix an arbitrary strictly inner subdomain $D \subset \omega_2$. For the sufficiently large numbers n we have $D \subset \omega \cap \omega_{s_n}$ and, as a consequence, the sequence $\{\rho_n\}$ converges to $\tilde{U}$ a.e. on D as n tends to infinity. This allows us to conclude that each of the numerical sequences $\{b^n\}$, $\{c_1^n\}$, $\{c_p^n\}$, defining the structure of functions ρ_n, $n = 1, 2, ...$ on D is bounded in $\mathbb{R}$. Thus, we can extract subsequences (retain notation) such that

$$b^n \to b, \quad c_i^n \to c_i, \quad i = 1, 2, \quad \text{as} \quad n \to \infty.$$

Therefore, we can choose a subsequence $\{s_{n_k}\}$ such that

$$U_{s_{n_k}} \to (bx_2 + c_1, -bx_1 + c_2) \quad \text{a.e.} \quad \text{in} \quad D \quad \text{as} \quad k \to \infty. \tag{4.3}$$

Consequently, we obtain that

$$\tilde{U} = (bx_2 + c_1, -bx_1 + c_2) \quad \text{a.e.} \quad \text{in} \quad D.$$

Due to arbitrariness of the domain $D \subset \omega_2$, we infer that

$$\tilde{U} = (bx_2 + c_1, -bx_1 + c_2) \quad \text{a.e.} \quad \text{in} \quad \omega_2.$$

On the other hand, we have for the fixed domain ω that

$$\tilde{U} = (\hat{b}x_2 + \hat{c}_1, -\hat{b}x_1 + \hat{c}_2) \quad \text{in} \quad \omega.$$

Since $\tilde{U} \in H(\Omega)$, then the jump of function $\tilde{U}$ on the intersection curve (the common side of two closed squares) $\overline{\omega} \cap \overline{\omega}_2$ is equal to zero. This means that $\hat{b} = b$, $\hat{c}_1 = c_1$, $\hat{c}_2 = c_2$, and, therefore we have

$$\tilde{U} = (bx_2 + c_1, -bx_1 + c_2) \quad \text{a.e.} \quad \text{in} \quad \omega \cup \omega_2,$$

i.e. $\tilde{U} \in R(\omega \cup \omega_2)$ holds.

We now show that $\tilde{U}$ satisfies the inequality $\tilde{U}\nu \leq 0$ on Γ_1. Taking into account the convergence 4.2, if necessary, we can once again extract a subsequence satisfying $U_{s_n}|_\Gamma \to \tilde{U}|_\Gamma$ a.e. on Γ. Therefore, we can perform the passage to the limit in the following inequality

$$U_{s_n}\nu \leq 0 \quad \text{on} \quad \Gamma_s.$$

This leads to $\tilde{U}\nu \leq 0$ on Γ_s. Thus, we reveal the inclusion $\tilde{U} \in K(2)$.

Our next goals are to prove the following equality $\tilde{U} = U_2$ and to establish the existence of a sequence U_{s_n}, $n = 1, 2...$ of solutions strongly converging in $H(\Omega)$ to U_2. Now, let us prove that $\tilde{U} = U_2$. For this purpose we will analyze the variational inequality 2.5 and its limiting case. From Lemma 1, for any $W \in K(2)$ there exist a subsequence $\{s_{n_k}\} \subset \{s_n\}$ and a sequence of functions $\{W_k\}$ such that $W_k \in K(s_{n_k})$ and $W_k \to W$ strongly in $H(\Omega)$ as $k \to \infty$.

The properties established above for the convergent sequences $\{W_k\}$ and $\{U_n\}$ allow us to pass to the limit as $k \to \infty$ in following inequalities derived from 2.5 for $\{s_{n_k}\}$ and with the test functions $W_k \in K(s_{n_k})$

$$\int\limits_\Omega \sigma_{ij}(U_{s_{n_k}})\varepsilon_{ij}(W_{s_{n_k}} - U_{s_{n_k}})d\Omega \geq \int\limits_\Omega F(W_{s_{n_k}} - U_{s_{n_k}})d\Omega. \tag{4.4}$$

As a result, we have

$$\int\limits_\Omega \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})d\Omega \geq \int\limits_\Omega F(W - \tilde{U})d\Omega \quad \forall\ W \in K(2).$$

The unique solvability of this variational inequality ensures that $\tilde{U} = U_2$.

To complete the proof, it is sufficient to establish the strong convergence $U_{s_n} \to U_2$. By substituting $W = 2U_{s_n}$ and $W = (0,0)$ into the variational inequalities 2.5 for $n \in \mathbb{N}$, we get

$$\int_\Omega \sigma_{ij}(U_{s_n})\varepsilon_{ij}(U_{s_n})d\Omega = \int_\Omega FU_{s_n}d\Omega \quad \forall n \in \mathbb{N}. \tag{4.5}$$

The equalities 4.5 together with the weak convergence $U_{s_n} \to U_2$ in $H(\Omega)$ as $n \to \infty$ imply

$$\lim_{n\to\infty} \int_\Omega \sigma_{ij}(U_{s_n})\varepsilon_{ij}(U_{s_n})d\Omega = \lim_{n\to\infty} \int_\Omega FU_{s_n}d\Omega =$$

$$\int_\Omega FU_2 d\Omega = \int_\Omega \sigma_{ij}(U_2)\varepsilon_{ij}(U_2)d\Omega.$$

Since we have the equivalence of norms (see Remark 2), one can see that $U_{s_n} \to U_2$ strongly in $H(\Omega)$ as $n \to \infty$. But this contradicts to the initial assumption. The Lemma is proved. □

5. Conclusion

Equilibrium problems for composite bodies which may come into contact with a fixed non-deformable obstacle were investigated. The solvability of the optimal control problem 3.1 is established. Also, it is shown that the equilibrium problem for the composite body with joined two inclusions can be considered as a limiting problem for the family of equilibrium problems for bodies with two separate inclusions. Namely, the strong convergence of the solutions U_s of the family of problems 2.4 to the solution U_2 of the limiting problem 2.6 in the Sobolev space $H(\Omega)$ was established. As can be seen from the proofs of the present paper, the main result remains true in 3D case for rigid cubic inclusions, as well as for equilibrium problems related to the two-dimensional solids with classical linear conditions.

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Об авторах

Лазарев Нюргун Петрович, д-р физ.-мат. наук, Северо-Восточный федеральный университет, Российская Федерация, 677000, г. Якутск, nyurgun@ngs.ru, https://orcid.org/0000-0002-7726-6742

Семенова Галина Михайловна, канд. пед. наук, доц., Северо-Восточный федеральный университет, Российская Федерация, 677000, г. Якутск, sgm.08@yandex.ru, https://orcid.org/0000-0003-1923-2904

About the authors

Nyurgun P. Lazarev, Dr. Sci. (Phys.–Math.), North-Eastern Federal University, Yakutsk, 677000, Russian Federation, nyurgun@ngs.ru, https://orcid.org/0000-0002-7726-6742

Galina M. Semenova, Cand. Sci. (Ped.), Assoc. Prof., North-Eastern Federal University, Yakutsk, 677000, Russian Federation, sgm.08@yandex.ru, https://orcid.org/0000-0003-1923-2904

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