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Method of Limiting Differential Inclusions and Asymptotic Behavior of Systems with Relay Controls

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Abstract. In this paper, problems of asymptotic behavior of non-autonomous controlled systems with a matrix of derivatives and the feedbacks of relay type are considered. The research is based on the method of limiting equations in combination with the direct method of Lyapunov functions with semidefinite derivatives. The method of the limiting equations has arisen in works G.R. Sell (1967) and Z. Artstein (1977, 1978) on topological dynamics of nonautonomous systems. Now this method is advanced for a wide class of systems, including the systems with delay. Nevertheless the method of the limiting equations till now has not received development with reference to nonautonomous differential inclusions and discontinuous systems for which it has fragmentary character. The main results are bound up with development of this method for discontinuous systems represented in the form of differential inclusions. In this case, specific methods of multivalued analysis and development of new methods for constructing limiting differential inclusions were required. The structure of the systems under scrutiny makes it possible, in particular, to study mechanical systems controlled on the decomposition principle of E.S. Pyatnitsky, and systems with dry friction submitted by equations Lagrange of 2-nd kind.

Keywords: limiting differential inclusion, Lyapunov function with semidefinite derivative, controlled mechanical systems, relay control, dry friction

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Научная статья

Метод предельных дифференциальных включений и асимптотическое поведение систем с релейными управлениями

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Аннотация. Рассматриваются вопросы стабилизации неавтономных управляемых систем с матрицей при производных и обратными связями релейного типа. Основой исследований служит метод предельных уравнений в сочетании с прямым методом функций Ляпунова со знакопостоянными производными. Метод предельных уравнений восходит к работам G. R. Sell (1967) и Z. Artstein (1977, 1978) по топологической динамике неавтономных систем и в настоящее время развит для широкого класса систем (в том числе для систем с запаздывающим аргументом), но не получил развития применительно к неавтономным дифференциальным включениям и разрывным системам, для которых он носит фрагментарный характер. Основные результаты связаны с развитием этого метода для разрывных систем, представленных в форме дифференциальных включений. Здесь потребовались специфические методы многозначного анализа и разработка новых способов построения предельных дифференциальных включений. Структура уравнений систем позволяет, в частности, изучать механические системы, управляемые на принципе декомпозиции Е. С. Пятницкого, и системы с сухим трением, представленные уравнениями Лагранжа 2-го рода.

Ключевые слова: предельное дифференциальное включение, функция Ляпунова со знакопостоянной производной, релейное управление, управляемые механические системы, сухое трение

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1. Introduction

The main ideas of the method of the limiting differential equations for the study of the asymptotic behavior of non-autonomous systems, were incorporated in the papers [1; 16]. At present, this method is well developed [13] and extended to many classes of differential equations, such as the delay equation [2; 3; 11]. For differential inclusions the first this method was introduced in the paper [4], and later extended to non-autonomous functional-differential inclusions [6] and differential equations with discontinuous right-hand part. [5]. Solutions of these equations are understood in the sense of Filippov, as solutions of differential inclusions [8].

The method of limiting equations is an effective method for studying a group of problems that can be combined under one name, as the problem of the asymptotic behavior of non-autonomous systems. These include problems of attraction, localization of right limiting sets, asymptotic stability of solutions and a number of others. In the framework of the method of limiting differential equations, all these problems are related to generalizations of the Krasovskii-LaSalle invariance principle (for example, see [10]).

In this paper we consider a system of the form

$$P(x)\dot{x} = R(t,x) + u, \tag{1.1}$$

where t be a scalar variable and $x = (x_1, \ldots, x_n)$ be a vector in space R^n with the Euclidean norm $\|\cdot\|$, $P(x)$ — a continuous, symmetric, positive definite $n \times n$ matrix, $R(t,x) = (R_1, \ldots, R_n)$ — continuous vector function, $u = (u_1, \ldots, u_n)$, $u_i(t,x) = -H_i(t,x)\,\mathrm{sign}\,\phi_i(t,x)$ under the condition $\phi_i(t,x) \neq 0$, $H_i(t,x) \geq 0$, $i = 1, \ldots, n$. While using well-known methods of the theory of differential equations with a discontinuous right-hand side, assume that $u_i = [-H_i, H_i]$ under the condition $\phi_i = 0$ for each $i = 1, \ldots, n$. Then equation (1.1) may be represented in the form of a differential inclusion

$$P(x)\dot{x} \in R(t,x) + U(t,x), \tag{1.2}$$

where $U(t,x) = u_1 \times \cdots \times u_n$. Denote

$$F(t,x) = P^{-1}(x)(R(t,x) + U(t,x)) \tag{1.3}$$

and write the differential inclusion (1.2) in the standard form

$$\dot{x} \in F(t,x). \tag{1.4}$$

The main problem for study differential inclusion is to construct the limiting differential relations. For this purpose, specific methods of multivalued analysis were needed. New limiting differential relations were also obtained in the form of differential inclusions [4] and were developed in [5] for the general form for differential equations with a discontinuous right-hand part. The result of this paper is the method of limiting differential inclusions for equation (1.1).

The structure of systems allows, in particular, to study mechanical controlled systems of E.S. Pyatnitsky [14] and systems with dry friction wich described by the Lagrange equations of the 2-nd kind [12].

2. General theorems

Let us introduce in consideration the following multivalued mapping

$$F^*(x) = \bigcap_{b\geq 0} \overline{\text{co}} \bigcup_{a\geq b} F(t+a,x).$$

Note, the multivalued mapping F^* does not depend on the variable t.

Definition 1. *The differential inclusion*

$$\dot{x} \in F^*(x) \tag{2.1}$$

is called the limiting.

General properties and some equivalent ways of describing limiting differential inclusions may be found in [4]. In particular, the set $F^*(x)$ at each fixed point x is the convex closed hull of all limiting values of the functions $y(t) \in F(t,x)$ under condition $t \to +\infty$. This fact is further used without reservations.

Let us denote by

$$\dot{V}^*(x) = \sup\{\langle \nabla V(x), y\rangle : y \in F^*(x)\}$$

the upper derivative of a continuously differentiable function $V(x)$ by virtue of the limiting differential inclusion (2.1).

Definition 2. *We will say that a function $Z(t,x)$ is continuous in x uniformly with respect t, if for any x_0 and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that*

$$\|Z(t,x) - Z(t,x_0)\| < \epsilon$$

for all x satisfying the inequality $\|x - x_0\| < \delta$ and for all $t \in R^1$.

Definition 3. *Set D is called semi-ivariant if for any point $y_0 \in D$ there exists a solution $y(t)$ of inclusion (2.1) with the initial condition $y(0) = y_0$ such that $y(t) \in D$ for all $t \geq 0$.*

Theorem 1. *Let the functions $R(t,x)$, $H_i(t,x)$ and the matrix $P(t,x)$ be bounded, continuous with respect to the variables (t,x) and continuous in x uniformly with respect to variable t, $H_i(t,x) \geq 0$, the functions $\phi_i(t,x)$ be continuously differentiable and $\nabla\phi_i(t,x) \neq 0$ for each $i \in I(t,x) \triangleq \{s = 1,\dots,n : \phi_s(t,x) = 0\}$. Let's assume there exists a continuously* *differentiable function $V(x)$ such that at the points of continuity of function $u(t,x)$ holds*

$$\dot{V}(t,x) \triangleq \langle \nabla V(x), F(t,x) \rangle \leq 0.$$

Then the ω-limit set $\Lambda^+(x)$ of any bounded solution of inclusion (1.4) belongs to the largest semi-invariant subset of set

$$E^* \triangleq \{x : \dot{V}^*(x) = 0\}.$$

Proof. Under the assumptions made for the multivalued mapping from the right-hand part of (1.4) all conditions of Lemmas 2, 3 and Theorem 1 from the paper [5] are hold. Then the set $F(tx)$ is upper semicontinuous and bounded set-valued mapping with convex and compact values. The Lyapunov function $V(x)$ satisfies the conditions of Theorem 5 from [5], from which follows, that the set $\Lambda^+(x)$ is semi-invariant and $\Lambda^+(x) \subset E^*$. □

We denote $\operatorname{sign}^+ \phi_i(t,x)$ (respectively, $\operatorname{sign}^- \phi_i(t,x)$) the function $\operatorname{sign} \phi_i(t,x)$, extended with the value 1 (respectively, the value –1) for $\phi(t,x) = 0$, $i = 1, \ldots, m$.

For each x, $i = 1, \ldots, n$ let us put

$$\begin{cases} a_i(x) = \underline{\lim}_{t \to +\infty}(-H_i(t,x)\operatorname{sign}^+ \phi_i(t,x)), \\ b_i(x) = \overline{\lim}_{t \to +\infty}(-H_i(t,x)\operatorname{sign}^- \phi_i(t,x)). \end{cases} \tag{2.2}$$

Let

$$R^*(x) = \bigcap_{b \geq 0} \overline{\operatorname{co}} \bigcup_{a \geq b} R(t+a, x)$$

and $U^{**}(x) = U_1^{**}(x) \times \ldots \times U_n^{**}(x)$, where

$$U_i^{**}(x) = [a_i(x), b_i(x)].$$

for each $i = 1, \ldots, n$.

We will consider also the limiting differential inclusion of the form

$$\dot{x} \in F^{**}(t,x), \tag{2.3}$$

where

$$F^{**}(x) = \dot{x} \in P(x)^{-1}(R^*(x) + U^{**}(x))$$

or, equivalent, differential inclusion of the form

$$P(x)\dot{x} \in (R^*(x) + U^{**}(x)).$$

Remark 1. Due to Lemma 1 from [5], the limiting multivalued mapping $F^*(x)$ for each fixed x is a convex closed hull of all limit points of the mapping $t \to F(t,x)$. Then $F^*(x) \subset F^{**}(x)$ and

$$\dot{V}^*(x) \leq \dot{V}^{**}(x), \tag{2.4}$$

where
$$\dot{V}^{**}(x) = \sup\{\langle \nabla V(x), y\rangle : P(x)y \in R^*(x) + U^{**}(x)\}$$
is the upper derivative of function $V(x)$ by vertue of the differential inclusion (2.3).

We denote
$$E^{**} = \{x : \dot{V}^{**}(x) = 0\}.$$

Theorem 2. *Let all the conditions of Theorem 1 be satisfied and*
$$\dot{V}^{**}(x) \leq 0. \tag{2.5}$$
Then the ω-limit set $\Lambda^+(x)$ of any bounded solution of inclusion (1.4) belongs to the largest semi-invariant subset of the set E^{**}.

Condition (2.5) follows that $E^* \subset E^{**}$. Then proof follows from the Theorem 1.

Remark 2. The Theorem 2 is a corollary of Theorem 1. They are equivalent only in the one-dimensional case. But the functions (2.2) can be easily computed and structure of the map $F^{**}(x)$ is simpler. In particular, Theorem 2 is more constructive, if function $R(t, x)$ does not depend on t or has a limit under condition $t \to +\infty$ at each fixed x. We also note that the structure of the original equations and the Lyapunov functions play an important role. This will be seen below for the equations of mechanical systems with friction

3. Mechanical systems

Let us consider the problems arising in dynamics of controlled mechanical systems based on the E. S. Pyatnitsky decomposition principle [14] and mechanical systems with dry friction represented in the form of Lagrange 2-nd kind equations [12].

3.1. Controlled Mechanical Systems

Lagrange equations for the system under consideration write in the expanded form as follows
$$A(q)\ddot{q} = g(t, q, \dot{q}) + Q^A(t, q, \dot{q}) + u, \tag{3.1}$$
where $A(q)$ — positive definite, symmetric $k \times k$ matrix of coefficients of the quadratic form of generalized velocities; $g(t, q, \dot{q}) = (g_1, \ldots, g_k)$, $Q^A(t, q, \dot{q}) = (Q^A_1, \ldots, Q^A_k)$ — continuous vector functions describing various generalized forces acting upon the system; $u = (u_1, \ldots, u_k)$ — vector

of generalized controls satisfying the following constrants $|u_i| \leq H_i(t, q, \dot{q})$, where $H_i = H_i(t, q, \dot{q}) > 0$ are continuous functions.

The structure of the controls is determined by the problem synthesis of control systems for mechanical systems based on the decomposition principle [14]: find such controls u_i, which (under some additional assumptions) would keeps that motions of the system (3.1) on the target set

$$M = \{(t, q, \dot{q}) : \dot{q}_i = f_i(t, q),\, i = 1, \dots, k\}.$$

The controls are defined in the form $u_i = -H_i \operatorname{sign}(\dot{q}_i - f_i(t, q))$.

With the use of appropriate replacements of the variables, equation (3.1) may be rewritten in the form of the differential inclusion (1.2) and to investigate the problems of attraction and stabilization.

3.2. Systems with friction

We consider the equations of motion of a mechanical system with Coulomb's sliding friction in the expanded vector form

$$A(q)\ddot{q} = g(q, \dot{q}) + Q^A(q, \dot{q}) + Q^T(t, q, \dot{q}). \tag{3.2}$$

The friction forces under the condition $\dot{q}^i \neq 0$ have the form

$$Q_i^T(t, q, \dot{q}) = -f_i(t, q, \dot{q})|N_i(q, \dot{q})|\operatorname{sign} \dot{q}^i,$$

where $|\, N_i(q, \dot{q}) \,|$ are the moduli of normal reactions at the points of contact between the rubbing bodies. We suppose that the friction coefficients $f_i(t, q, \dot{q}) \geq 0$ for all $i = 1, \dots, k_*$, $k_* \leq k$. For all $i = k^* + 1, \dots, k$ we assume $f_i(t, q, \dot{q}) \equiv 0$.

Friction is a very complex physical process, which has not been completely studied so far. The dependence of friction coefficients on the variable t (time) may arise in virtue of various causes, such as variations of the temperature of rubbing bodies.

We assume that the system's kinetic energy is represented by the quadratic form

$$T(q, \dot{q}) = \frac{1}{2} \sum_{i,j=1}^{k} a_{i,j}(q)\dot{q}^i \dot{q}^j, \tag{3.3}$$

where $a_{i,j}(q)$ — coefficients of inertia (elements of matrix $A(q)$).

Extending the definition of the friction forces at the points of discontinuity, we may obtain the general expression for the friction forces in form

$$Q_i^T(t, q, \dot{q}) = \begin{cases} -f_i|N_i|\operatorname{sign} \dot{q}, \text{ if } \dot{q}^i \neq 0, \\ [-f_i|N_i|, f_i|N_i|], \text{ if } \dot{q}^i = 0 \end{cases} \tag{3.4}$$

for each $i = 1, \dots, k^*$.

Equations (3.2) with friction forces (3.4) describe the systems with single-degree kinematic pairs with friction (for example, the mechanisms consisting of connecting rods and sliders, pendulum systems with friction in hinges and supports).

Let us put

$$a_i(q,\dot q) = \underline{\lim}_{t\to+\infty} f_i(t,q,\dot q),\ b_i(q,\dot q) = \overline{\lim}_{t\to+\infty} f_i(t,q,\dot q)$$

for each $i = 1,\dots,k^*$ and define the multivalued function with the values $Q^{**}(q,\dot q) = Q_1^{**}(q,\dot q) \times \dots \times Q_k^{**}(q,\dot q)$, where

$$Q_i^{**}(q,\dot q) = \begin{cases} [a_i|N_i|, b_i|N_i|], \text{if } \dot q^i < 0, \\ [-b_i|N_i|, -a_i|N_i|], \text{if } \dot q^i > 0, \\ [-b_i|N_i|, b_i|N_i|], \text{if } \dot q^i = 0. \end{cases} \tag{3.5}$$

Let us consider the limiting differential inclusion the following form

$$A(q)\ddot q \in g(q,\dot q) + Q^A(q,\dot q) + Q^{**}(q,\dot q). \tag{3.6}$$

Let $Q_i^A = D_i(q,\dot q) + K_i(q)$, where $K_i(q) = -\partial\Pi/\partial q^i$, $\Pi(q)$ is system's potential energy, $D_i(q,\dot q)$ — dissipative forces with the conditions $D_i(q,0) = 0$, $\sum_{i=1}^k D_i(q,\dot q)\dot q_i \le 0$, which represent viscous friction or environmental resistance forces.

The function $g(q,\dot q) = (g_1(q,\dot q),\dots,g_k(q,\dot q))$ in the Lagrange's equations of the second kind of motion of mechanical systems is determined by the equalities

$$g_i(q,\dot q) = \frac{1}{2}\sum_{\nu=1}^k\sum_{j=1}^k \frac{\partial a_{\nu j}}{\partial q^i}\dot q^\nu \dot q^j - \sum_{j=1}^k\sum_{\nu=1}^k \frac{\partial a_{ij}}{\partial q^\nu}\dot q^\nu \dot q^j. \tag{3.7}$$

In the capacity of the Lyapunov function we assume the complete system's energy: $V = T + \Pi$. Direct calculations, definitions of derivatives $\dot V$, $\dot V^*$ and formulas (3.2)–(3.7) lead to equalities

$$\dot V = -\sum_{i=1}^{k_*} f_i(t,q,\dot q)|N_i(q,\dot q)||\dot q^i| + \sum_{i=1}^k D_i(q,\dot q)\dot q^i,$$

$$\dot V^{**} = -\sum_{i=1}^{k^*} a_i(q)|N_i(q,\dot q)||\dot q^i| + \sum_{i=1}^k D_i(q,\dot q)\dot q^i.$$

Under above assumptions the condition $\dot V \le 0$ and $\dot V^{**} \le 0$ holds, and due to Theorem 2, the problem of the asymptotic behavior of the system (3.2) reduces to investigation of the set

$$M = \{(q,\dot q) : \sum_{i=1}^k D_i(q,\dot q)\dot q^i - \sum_{i=1}^{k^*} a_i(q)|N_i(q,\dot q)||\dot q^i| = 0\},$$

which, in particular, may coincide with the set of equilibrium positions of inclusion (3.6).

Example 1. A body of mass m, considered as a material point, moves along a horizontal line Ox under the action of an elastic force $F^e(x) = -kx$ $(k > 0)$ with a point unstressed state $x = 0$, $P = mg$ — weight of body, $F^{fr}(t, v) = -f(t)P \operatorname{sgn} \dot{x}$ — Coulomb's dry friction force, which is discontinuous under condition $\dot{x} = 0$. The function $f(t)$ is continuous and bounded. The equations of motion of the system have the form:

$$m\ddot{x} = -kx - f(t)P \operatorname{sgn} \dot{x}. \tag{3.8}$$

Our goal is to most accurately describe the set of points to which the bounded solutions of the equation (3.8) are aimed.

In accordance with the above methods for studying systems with friction, we pass to the differential inclusion

$$m\ddot{x} \in -kx + F^{fr}(t, \dot{x}),$$

where $F^{fr}(t, \dot{x}) = -f(t)P \operatorname{sgn} \dot{x}$ under the condition $\dot{x} \neq 0$ and $F^{fr}(t, 0) = [-f(t)P, f(t)P]$.

Denote

$$a = \underline{\lim}_{t\to+\infty} f(t),\ b = \overline{\lim}_{t\to+\infty} f(t).$$

Let $\Lambda = \Lambda^+(x, \dot{x})$ — ω-limit set of solution $(x(t), \dot{x}(t))$ of system (3.8).

We will consider the limiting differential inclusion in the form

$$\ddot{x} \in -kx + Q^{**},\ Q^{**} = \begin{cases} [-b, -a]P, \text{ if } \dot{x} > 0, \\ [a, b]P, \text{ if } \dot{x} < 0, \\ [-b, b]P, \text{ if } \dot{x} = 0. \end{cases} \tag{3.9}$$

under condition $a > 0$.

We take the Lyapunov function in the form

$$V(x, \dot{x}) = \frac{1}{2}(m\dot{x}^2 + kx^2).$$

Then $\dot{V}(t, x, \dot{x}) = -f(t)P|\dot{x}| \leq 0$ and $\dot{V}^{**} = -a|\dot{x}|$. Thus, we get

$$E^{**} = \{(x, \dot{x}) : \dot{x} = 0\}.$$

.

The conclusions that can be drawn from this are the following.

1. $\Lambda^+(x, \dot{x})$ consists of equilibria of (1).

2. Level lines of the Lyapunov function $V(x, \dot{x}) = \frac{1}{2}(m\dot{x}^2 + kx^2)$ under the condition $\dot{x} = 0$ are two-point sets $\{(x_0, 0), (-x_0, 0)\}$. Due to the connectivity property of the set $\Lambda^+(x, \dot{x})$, it can consist of only one point

and therefore always represents one equilibrium position. Obviously, this could be the point $(0, 0)$.

Further analysis, not related to Theorems 1 and 2, shows that any bounded solution $(x(t), \dot{x}(t))$ of the equation (3.8) tends to a point $(x_0, 0)$ or to a point $(-x_0, 0)$ such that $|x_0| \leq Pa/k$, if $t \to +\infty$.

4. Conclusion

The conclusion, which Theorems 1, 2 allow us to draw is that the ω-limit sets of system (1.1) belong to the set of zeros of the derivative of the Lyapunov function due to the limiting differential inclusions. This does not completely solve the problem of attraction, asymptotic stability or stabilization of the system. Investigation of sets E^* and E^{**} in the general form is not always possible and hardly ever expedient. Therefore, further investigations shall take into account the structure of these sets, which depends on the equation (1.1), the choice of the Lyapunov function (which is ambiguous) and also relies on the properties of ω-limit sets. This can be seen from the example above.

Theorems 1 and 2 are new. We also note that the asymptotic behavior of the systems with variable friction coefficients in such a degree of generality as the Lagrange equation of the 2-nd kind (3.2) with friction forces (3.4) not previously studied and received on the basis of the method of limiting differential inclusions from the works [4], [5].

The total mechanical energy, as a Lyapunov function, has a semidefinite derivative, and its use in the problems of stability and attraction causes difficulties even in the autonomous case (see [12]).

Further investigations may be bound up with the study of controlled mechanical systems on the decomposition principle, which are simultaneously under the action of Coulomb friction forces.

The method of limiting differential inclusions may be considered within the frames of the universal method of implications of properties of related mathematical models [17], when the initial systems and the limiting systems are considered as structurally similar. In such a case the issue of choosing an auxiliary system remains open. But the choice of the limiting differential relation is also ambiguous.

In the present paper, we considered bounded solutions of differential equations and inclusions. If solution is unbounded (but not infinite) then it is possible to state only that their trajectories are weakly tending to the ω-limit set $\Lambda(x)$. It means there exists a sequence of points $t_n \to +\infty$ such that $d(x(t_n), E^*) \to 0$. This property, in combination with the property of stability gives the asymptotic stability and is close to the property of "invariance with the visit" from paper [17, Example 2].

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