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Optimal Behavior of Agents in a Piecewise Linear Taxation Environment

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Abstract. We study analytical properties of the optimal income taxation model. In this model we consider the maximization of *utility* of an agent of the given type. The real meaning of the utility is the net profit of the legal entity. The mathematical consideration of the taxation optimization uses methods of probability theory, functional analysis and optimal control. The totality of all agents in the economy is represented by the probability space of their types. Optimal income taxation differs from commodity taxation, another branch of the optimal tax theory. Actual taxes are commonly linear or segmented, which naturally suggests us to consider such cases in this research. To be more precise, we describe the general piecewise linear taxation model with increasing linear coefficients. The latter is necessary for the tax function to be convex. An explicit description of optimal functioning of agents depending on their types is obtained. In particular, we consider optimal labour effort and optimal utility.

Keywords: optimal income taxation, taxation theory, piecewise linear optimization, mathematical economics

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Научная статья

Оптимальное функционирование субъектов при кусочно-линейном налогообложении

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Аннотация. Рассматриваются аналитические свойства модели оптимального налогообложения прибыли и максимизация *полезности* субъекта налогообложения. При этом используются методы теории вероятностей, оптимального управления и функциональный анализ. Указывается, что совокупность всех субъектов в экономике представлена вероятностным пространством их типов, модель оптимального дохода отличается от налогообложения товаров. Исследуется общая кусочно-линейная модель налогообложения с возрастающими линейными коэффициентами. Последнее условие необходимо, чтобы налоговая функция была выпуклой. В результате получено явное описание оптимального функционирования субъектов в зависимости от их типов. В частности, рассмотрены оптимальные трудозатраты и оптимальная полезность.

Ключевые слова: оптимальное налогообложение, оптимальный подоходный налог, кусочно-линейная оптимизация, математическая экономика

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1. Introduction

The impact of taxation on the economy has been considered in practice only in a rather intuitive way. However, a lot of cases in finance, physics and technology show that intuitive approaches to optimize a system could provide you a result far from the real optimum. Specifically, situations occur where adding an additional resource to the system leads to deterioration of the overall performance (famous Braess's paradox [5] or other situations of that kind [6]).

This has motivated mathematical research on the optimization of taxation models. We will focus on the optimal income taxation theory, described mainly by the Nobel prize winner J. A. Mirrlees ([7], [8]). Another branch of the optimal tax theory is the commodity taxation (see [2]). The consideration of this problem was motivated by our discussions with A. Tsyvinski, in particular on the papers [10] and [9]. In the paper [3] some results were obtained for a general case with several smooth conditions.

Actual taxes are commonly linear or segmented, which naturally suggests us to consider such cases. To be more precise, we describe the general piecewise linear taxation model with increasing linear coefficients. The latter is necessary for the tax function to be convex. Note that this research is different from the one given in [1]. In that article the model is a bit different, and a 2-step case is considered thoroughly, including concave tax functions. However, for the general piecewise linear case only numerical results are obtained in a discrete model. In this paper we obtain some general analytical results about the behavior of the system in a generalized model.

2. Problem statement

We denote by $\theta \in \Theta \subset \mathbb{R}^n_+$ a productivity type of a given agent, by $l \in \mathbb{R}^n_+$ its labour effort. There is the probability measure H on Θ, which defines the distribution of agents among the types. As a result of the scalar product we have the income $y = (\theta, l)$ and the utility $U(\theta, l) = y - T(y) - v(l)$, where $T(y)$ is the tax liability of the income y, defined by the government, and $v(l)$ is a twice continuously differentiable, increasing, strictly convex dis-utility function of labour effort. Common sense induces that $T(y)$ and $y - T(y)$ are increasing non-negative continuous functions. Usually we also assume T to be convex, which makes sense as it is supposed to be growing faster for greater incomes. For a given type θ and a fixed tax function T we solve the optimization problem

$$\max_{l>0} U(\theta, l) = \max_{l\in\mathbb{R}^n_+} ((\theta, l) - T(\theta, l) - v(l)). \tag{2.1}$$

Having found the points of maxima l_{max} for each θ and, henceforth, $y_{max} = (\theta, l_{max})$, we define the government revenue as

$$R(T) = \int_\Theta T(y_{max}(\theta))dH. \tag{2.2}$$

By analogy, the overall utility in the economy can be defined as

$$\int_\Theta U(\theta, l_{max}(\theta))dH.$$

In this paper we consider $v(l) = \frac{l^2}{2}$, which is equivalent to $v(l) = c||l||^2$ for each $c > 0$, as it will be shown later. The main aim of this study is to describe $l_{max}(\theta)$, $y_{max}(\theta)$ and the resulting utility of the “optimal” behavior.

2.1. Dimensionality reduction

Suppose that vectors θ and l are non-collinear. Take the projection l' of l onto θ. Then $(\theta, l') = (\theta, l)$, while $v(l') > v(l)$. This implies that we should always consider only collinear vectors l and θ, which gives us $(l, \theta) = ||l||\,||\theta||$. Hence, we could consider positive numbers θ and l instead of vectors and the usual multiplication of numbers instead of the scalar product.

2.2. Piecewise linear model

We consider a tax function represented by N linear segments. The segments are described by partition points $m_1 \leq \ldots \leq m_{N-1}$ and coefficients $k_1 \leq \ldots \leq k_N$, so that

$$T(y) = \begin{cases} k_1 y \ \text{ if } y \leq m_1 \\ k_1 m_1 + k_2(y - m_1) \ \text{ if } m_1 \leq y \leq m_2 \\ \vdots \\ k_1 m_1 + k_2(m_2 - m_1) + \ldots + k_N(y - m_{N-1}) \ \text{ if } m_{N-1} \leq y. \end{cases} \tag{2.3}$$

3. Results

Recall that for each θ the solution to the optimization problem 2.1 is the optimal labour effort $l_{max}(\theta)$, which optimizes the utility. Then we have the optimal income $y_{max} = \theta l_{max}(\theta)$ and the optimal utility $U_{max}(\theta) = U(\theta, l_{max}(\theta))$.

Theorem 1. *The optimal income $y_{max}(\theta)$ is increasing with respect to θ.*

The proof is given in Section 4.2.

Theorem 2. *In the piecewise linear case (2.3), the optimal income $y_{max}(\theta)$ is piecewise constant. To be more precise, $[0, +\infty)$ is divided into consecutive intervals $I_1, \ldots, I_{2N+1}$ and y_{max} is constant on the intervals with odd numbers and strictly increasing on the ones with even numbers.*

Corollary 1. *The conclusions of Theorem 2 are also valid for the resulting tax $T(y_{max}(\theta))$.*

Remark 1. Note that the rate of increase of the optimal income is diminishing from segment to segment. The same is in general not true for $T(y_{max}(\theta))$.

Theorem 2 is proved in section 5.2.

4. General case

First of all, let us note that we speak about the individual income y without giving any special constraints. However, there are some, especially if we take $v(l) = \frac{l^2}{2}$:

$$U(\theta, l) = \frac{l(2\theta - l)}{2} - T(\theta l), \text{ which implies that } l < 2\theta, \text{because } U > 0.$$

Hence $y < 2\theta^2$. Moreover, it can be easily shown that $l_{max}(\theta) \leq \theta$:

$$\begin{aligned} U(\theta, \theta + \epsilon) &= \theta^2 + \theta\epsilon - T(\theta^2 + \theta\epsilon) - \frac{\theta^2}{2} - \theta\epsilon - \frac{\epsilon^2}{2} \\ &= \frac{\theta^2}{2} - \frac{\epsilon^2}{2} - T(\theta^2 + \theta\epsilon) < \frac{\theta^2}{2} - T(\theta^2) = U(\theta, \theta). \end{aligned}$$

Therefore, we have $y_{max}(\theta) \leq \theta^2$ for each θ.

4.1. Currency conversion

Now we will convert our currency into another one with the coefficient c:

$$\begin{aligned} y_{new} &= cy \\ T_{new}(y_{new}) &= cT(y) \\ v_{new}(l_{new}) &= cv(l). \end{aligned}$$

To be more precise, $c = c_\theta c_l$, so that $l_{new} = c_l l$ and $\theta_{new} = c_\theta \theta$. We can see that

$$U_{new}(\theta_{new}, l_{new}) = l_{new}\left(\theta_{new} - l_{new}\frac{c}{2c_l^2}\right) - T_{new},$$

and the first term on the right is positive if and only if $l < 2\theta$. As a result, it is now proved that if we have some solutions for some v and Θ, then we can transform the entire system into another one, where $v(l) = \frac{l^2}{2}$.

4.2. Optimal income properties

Consider $\theta_1 = \theta_2 - \delta$, where $\delta > 0$ and $l_i = l_{max}(\theta_i), y_i = l_i\theta_i$. Then

$$\begin{aligned} U(\theta_2, l_1) &= \theta_2 l_1 - T(\theta_2 l_1) - \frac{l_1^2}{2} = l_1(\theta_1 + \delta) - T(l_1(\theta_1 + \delta)) - \frac{l_1^2}{2} \\ &= U(\theta_1, l_1) + l_1\delta - \Delta T > U(\theta_1, l_1), \end{aligned}$$

because we know that $y - T(y)$ is an increasing function. Thus,

$$U(\theta_2, l_2) \geq U(\theta_2, l_1) > U(\theta_1, l_1) \tag{4.1}$$

Moreover, by the definition of y_1

$$y_1 - T(y_1) - \frac{y_1^2}{2\theta_1^2} > y - T(y) - \frac{y^2}{2\theta_1^2} \quad \forall y < y_1. \tag{4.2}$$

If we look at

$$d(y) = \frac{y^2}{2\theta_1^2} - \frac{y^2}{2\theta_2^2},$$

we notice that this function is increasing, so we can add $d(y_1)$ to the left part of (4.2) and $d(y)$ to the right part. Therefore,

$$y_1 - T(y_1) - \frac{y_1^2}{2\theta_2^2} > y - T(y) - \frac{y^2}{2\theta_2^2} \quad \forall y < y_1. \tag{4.3}$$

Hence, given (4.1), we have $y_1 \le y_2$. Theorem 1 is proved.

5. Piecewise linear case

5.1. Simple observations

We will first assume that

$$T(y) = \begin{cases} k_1 y & \text{if } y \le m \\ k_1 m + k_2(y-m) & \text{otherwise,} \end{cases}$$

where $0 \le k_1 \le k_2 < 1$, and $m > 0$ is some given point on the incomes scale.

(A) $l \le m/\theta$:
Here $U(\theta, l) = U_1(l) = -\frac{1}{2}l^2 + l(\theta - k_1\theta)$. Argmax is, obviously, $l_1 = \theta(1-k_1)$. But we need to check first whether l_1 is inside boundaries for this case. That is equivalent to the following condition:

$$\theta^2 \le \frac{m}{1-k_1} = \theta_1^2, \tag{5.1}$$

where the maximum value is

$$M_1 = U_1(l_1) = \frac{\theta^2(1-k_1)^2}{2}$$

which we will note and later return to.

(B) $l \ge m/\theta$:
In this case $U(\theta, l) = U_2(l) = -\frac{1}{2}l^2 + l(\theta - k_2\theta) + m(k_2 - k_1)$. It is maximized at $l_2 = \theta(1-k_2)$, and, by analogy with the previous case

$$\theta^2 \ge \frac{m}{1-k_2} = \theta_2^2, \tag{5.2}$$

where the maximum value is

$$M_2 = U_2(l_2) = \frac{\theta^2(1-k_2)^2}{2} + m(k_2 - k_1).$$

Remark 2. Let us compare these two obtained values by analyzing their difference $M_1 - M_2$:

$$\frac{\theta^2}{2}\Big((1-k_1)^2 - (1-k_2)^2\Big) + m(k_1 - k_2)$$
$$= \frac{\theta^2}{2}(k_2 - k_1)(2 - k_1 - k_2) + m(k_1 - k_2) > 0$$
$$\iff \frac{\theta^2}{2}(2 - k_1 - k_2) > m \iff \theta^2 > \frac{2m}{2 - k_1 - k_2} = \frac{m}{1 - k^\star} = \theta^\star, \quad (5.3)$$

where $k^\star$ is the arithmetic mean of the coefficients.

It should be noted, however, that there seems to be a contradiction between conditions (5.3) and (5.1) + (5.2). Indeed, $\theta_1^2 \le \theta^\star \le \theta_2^2$, and for $\theta \le \theta_1$ we have the first mode of tax liability (5.1), so $U_{max} = M_1$. But at the same time $M_1 < M_2$, because $\theta^2 < \theta^\star$. An explanation lies in the behavior of M_i as a function of θ: M_1 grows faster than M_2 and its value becomes greater once the condition (5.3) is met. Also, it is very important to note that $l_1 > l_2$.

In other words, every θ generates two parabolic curves $U_1(l)$ and $U_2(l)$ the maximum points of which we have just compared. Their crossing is $l = m/\theta$, so the final $U(l)$ is $U_1|_{l \le m/\theta} \bigcup U_2|_{l > m/\theta}$.

5.2. The behavior analysis

Now assume that the tax function has N pieces described by coefficients $k_1 \le \ldots \le k_N$ and partition points $m_1 \le \ldots \le m_{N-1}$, as it was defined in 2.3. Each point m_i produces two boundary values, as was shown in (5.1). We denote them by

$$\theta_{i,1} = \sqrt{\frac{m_i}{1 - k_i}}, \ \theta_{i,2} = \sqrt{\frac{m_i}{1 - k_{i+1}}}. \quad (5.4)$$

For a given θ, the linear segment corresponding to the coefficient k_i, generates a curve

$$U_i(l) = -\frac{1}{2}l^2 + l\theta(1 - k_i) + \sum_{j<i} m_j(k_{j+1} - k_j), \quad (5.5)$$

with the argmax $l_i = \theta(1 - k_i)$, and we switch from U_i to U_{i+1} once we approach $l = m_i/\theta$.

Given all that it is now possible to describe the behavior of $l_{max}(\theta)$ and $U_{max}(\theta)$.

Lemma 1. *In the given environment the dependence of the optimal solution on θ is as follows:*

1) $\theta < \theta_{1,1}$.
This means that $l_2 \le l_1 \le m_1/\theta$. *In this case* U_1 *and* U_2 *look as shown on Fig.1(a). Obviously,* $l_{max} = l_1 = \theta(1-k_1)$. *A generalization of this case is to be found further.*

2) $\theta_{i,1} \le \theta < \theta_{i,2}$.
Here $l_{i+1} \le m_i/\theta < l_i$ *and* U *is maximized at* $l = m_i/\theta$ *(Figures 1(b), 1(d)) . Therefore,* $y_{max} = \theta l = m_i$ *is constant on this interval. So,*

$$\begin{aligned} l_{max}(\theta) &= m_i/\theta \\ y_{max}(\theta) &= m_i \\ T_{max}(\theta) &= k_1 m_1 + \ldots + k_i(m_i - m_{i-1}) \\ U_{max}(\theta) &= -\frac{m_i^2}{2\theta^2} + m_i - T_{max} \end{aligned} \tag{5.6}$$

3) $\theta_{i,2} \le \theta < \theta_{i+1,1}$ *(Figure 1(c)). Which is* $m_i/\theta < l_{i+1} \le l_i$. *Thus,*

$$\begin{aligned} l_{max}(\theta) &= l_{i+1} = \theta(1-k_{i+1}), \\ y_{max}(\theta) &= \theta^2(1-k_{i+1}), \\ T_{max}(\theta) &= k_{i+1}(1-k_{i+1})\theta^2 + \\ &\quad + k_1 m_1 + \ldots + k_i(m_i - m_{i-1}) - k_{i+1} m_i, \\ U_{max}(\theta) &= \theta^2 \frac{(1-k_{i+1})^2}{2} - \\ &\quad - \left(k_1 m_1 + \ldots + k_i(m_i - m_{i-1}) - k_{i+1} m_i\right). \end{aligned} \tag{5.7}$$

This covers the first case, once we take $i = 0$, $\theta_{0,2} = 0$. *Also, when* $i = N-1$, *we have a particular case described below.*

4) $\theta_{N-1,2} < \theta$. *This is a particular situation of the case above, with* $\theta_{N,1} = \infty$. $l_{max} = l_N = \theta(1-k_N)$. *(Figure 1(e)).*

Note that all boundaries on θ *are actually not strict, as the functions remain continuous.*

Theorem 2 follows immediately from Lemma 1.

5.3. Pictures

Pictures on Fig. 1 detail the case of three tax coefficients. Fig. 1(f) illustrates how the optimal labour l depends on the productivity type θ.

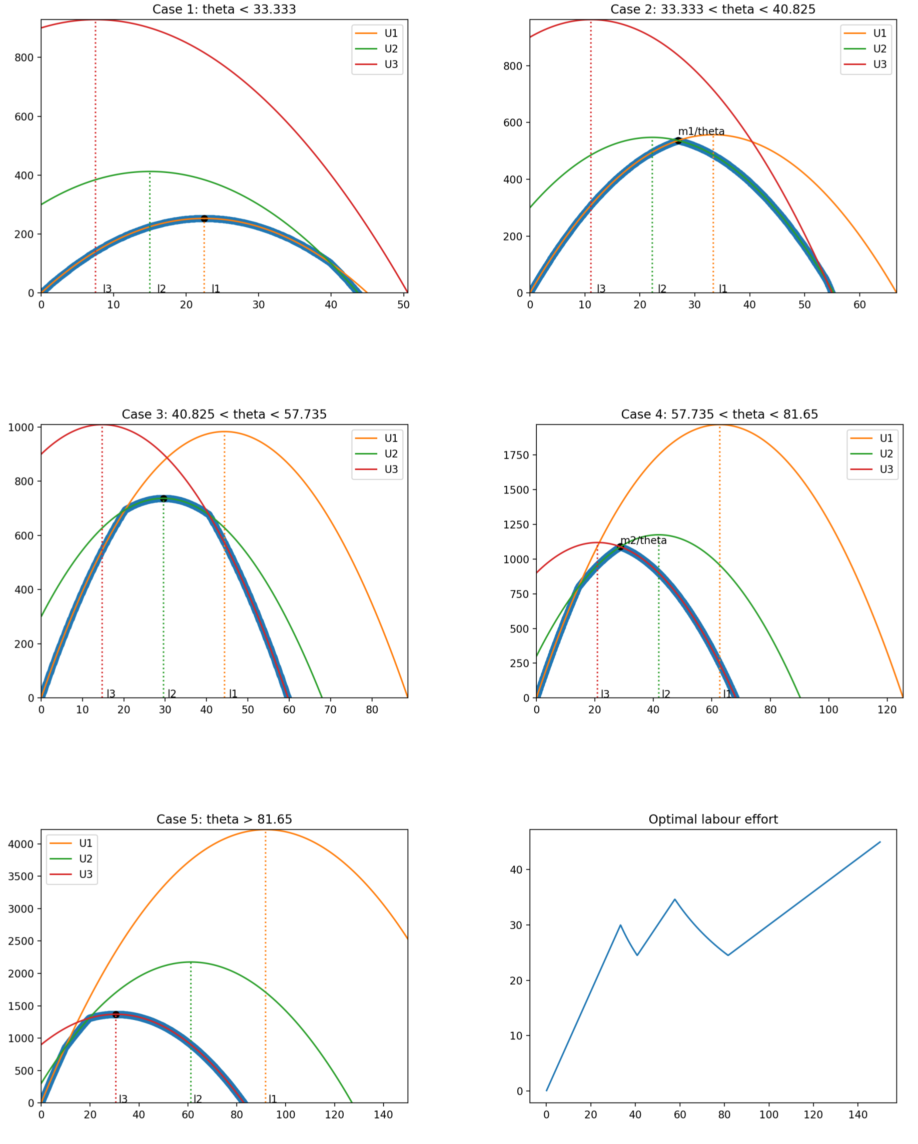

Figure 1. 3 tax coefficients

6. Conclusion

As a result, explicit formula is obtained for the optimal income for each type of agent and, thus, also explicit formulas for the government revenue (2.2) and the overall utility in the economy. Maximization of those is an open problem usually being of a particular interest for application purposes.

It would be interesting to investigate possible connections of the considered problems with nonlinear Kantorovich problems of optimal transportation (see, e.g., [4] and the references therein).

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