Серия «Математика»
2022. Т. 41. С. 107—120

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского государственного университета

Research article

УДК 518.517
MSC 34A08, 34B10, 34B15
DOI https://doi.org/10.26516/1997-7670.2022.41.107

Existence and Stability of Solutions for a Class of Stochastic Fractional Partial Differential Equation with a Noise

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Abstract. In this work, we will introduce a fractional Duhamel principle and use it to establish the well-boundedness and stability of a mild solution to an original fractional stochastic equation with initial data.

Keywords: stochastic fractional partial differental equation, fractional derivative, mild solution, stability

For citation: Bouteraa N. Existence and Stability of Solutions for a Class of Stochastic Fractional Partial Differential Equation with a Noise. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 41, pp. 107–120.
https://doi.org/10.26516/1997-7670.2022.41.107

Научная статья

Существование и устойчивость решений одного класса стохастических дифференциальных уравнений в частных производных дробного порядка с шумом

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Аннотация. В работе вводится дробный принцип Дюамеля и используется для установления ограниченности и устойчивости слабого решения исходного стохастического уравнения с дробными производными с начальными данными.

Ключевые слова: стохастическое дробное дифференциальное уравнение в частных производных, дробная производная, слабое решение, устойчивость

Ссылка для цитирования: Bouteraa N. Existence and Stability of Solutions for a Class of Stochastic Fractional Partial Differential Equation with a Noise // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 41. С. 107–120.
https://doi.org/10.26516/1997-7670.2022.41.107

1. Introduction

In this paper, we are interested in the existence of solutions for an original nonlinear fractional difference equation

$$ {}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) = g(x,t)\dot{W}(t), \quad x \in D,\ t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \varphi(x),\ x \in D, \tag{1.2}$$

and the Dirichlet boundary condition

$$u(x,t) = 0,\ x \in \partial D, \tag{1.3}$$

where $D \subset \mathbb{R}$, $u(x,t)$ represents the velocity field of the fluid, the state $u(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle \cdot,\cdot \rangle$, $g(x,t)$ continuously differentiable function, $g(x,0) = 0$, the term $g(x,t)\dot{W}(t) = g(x,t)\frac{d}{dt}W(t)$ describes a state dependent random noise, where $W(t)_{t\in[0,T]}$ is a F_t-adapted Wiener process defined in completed probability space (Ω, F, P) with expectation E and associate with the normal filtration

$$F_t = \sigma\{W(s) : 0 \le s \le t\}.$$

The operator Δ is the Laplacian. Here, ${}_0^cD_t^\alpha$ denotes the Caputo type derivative of order α $(0 < \alpha < 1)$ for the function $u(x,t)$ with respect time t which is defined by

$$\begin{cases} {}_0^cD_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha},\ 0<\alpha<1, \\ \frac{\partial u(x,t)}{\partial t}, \qquad \alpha = 1, \end{cases}$$

where $\Gamma(\cdot)$ is the Euler gamma function and Caputo fractional derivative of order α also defined as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = I_t^{n-\alpha}\frac{\partial^n}{\partial t^n}u(x,t). \tag{1.4}$$

Fractional stochastic equaions have been considered in the literature, first by introducing a stochastic forcing term which cames from a Brownian motion see [1; 22]. The addition of the white noise driven term to the basic governing equations is natural for both practical and theoretical applications to take into account for numerical and empirical uncertainties, and have been proposed as a model for turbulence. Later on other kinds of noises have been studied. Stochastic differential equations have attracted great interest because of their applications in characterizing many problems in physics, mecanics, electrical engineering, biology, ecology and so on. On this matter, we refer the reader to [6; 14] and references therein. We quote also stochastic partial differential equations, for more details see [4; 10; 15–17]. To identify a few models, there has been a widespread interest during the last decade in constructing a stochastic integration theory with respect to fractional Brownian motion (FBM) and solving stochastic differential equations driven by FBM. In fact, stochastic perturbation factors, such as precipitation, absolute humidity, and temperature, have a significant impact on the infection force of all types of virus diseases to humans. Taking this into consideration enables us to present randomness into deterministic biological models to expose the environmental variability effect, whether it is environmental fluctuations in parameters or random noise in the differential systems, for more details, see [18]. The reason that we use Caputo type fractional-order time derivative is not only to make the equation original but also to employ the advantages of fractional-order operators in an equation having a noise term. The most-well known advantage of Caputo derivatives with respect to the classical derivatives is its capability of taking into account the previous historical effects of model at each time step. This feature of fractional-order operators makes them more accurate and appropriate in modeling of the systems.

The stability theory for functional equations developed and it got popularity so quikly. Subsequently, a large number of mathematicians took differents types of stabilities. In particular, the stability theory of stochastic differential equations has been popularly applied in variety fields of science and technology. Several authors have established the stability results of mild solutions for these equations by using various techniques. Govindan [8] considered the existence and stability for mild solution of stochastic partial differential equations by applying the comparison theorem. Caraballo and Liu [12] proved the exponential stability for mild solution to stochastic partial differential equations with delays by utilizing the well-known Gronwall inequality. The exponential stability of the mild solutions for semilinear stochastic delay evolution equations have been discussed by using Lyapunov functionals in [13]. The author in [12] considered the exponential stability for stochastic partial functional differential equations by means of the Razuminkhin-type theorem. Further, Sakthivel et al. [20] established the asymptotic stability and exponential stability of second-order stochastic evolution equations in Hilbert spaces. The study of this area has grown to be one of the central and most essential subjects in the mathematical analysis area. For more details on the recent advances on the study of stability solution one can see the research papers [3; 21; 25].

The classical Duhamel principle, established nearly two centuries ago by Jean-Marie-Constant Duhamel, reduces the Cauchy problem for an inhomogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. In [23; 24], Umarov generalized the classical Duhamel principle for the Cauchy problem to general inhomogeneous fractional distributed differential-operator equations. In this paper we formulate and prove fractional analogue of this famous principle for the stochastic time-fractional partial differential equation. We first establish a fractional Duhamel principle for the nonhomogeneous stochastic time-fractional partial differential equation. Then based on it and the superposition principle, the solution of the above initial value problem is represented. Finally, we obtain the stability and boundedness of the solution. To the best of the authors knowledge, literature on study of stability by fractional Duhamel principle for stochastic fractional problems is rather limited and even remain open. For this raison and motivated by the above articles, the aim of this paper is to derive stability and boudedness of solutions for initial value problems of an original stochastic fractional equations $(1.1)-(1.3)$. Furthermore, The goal of this paper is to enrich this academic area. Hence, this paper will contribute a slightly general way. The current paper is organized as follows. In Section 2, we shall present basic definitions, lemmas and preliminary results that are needed in the sequel. In Section 3, we develop a fractional version of Duhamel's principle and conditions for existence, boundedness and stability of mild solution are establish for the problem $(1.1)-(1.3)$.

2. Preliminaries

In this section, we give some notions and certain important preliminaries, which will be used in the subsequent discussions.
Let $\left(\Omega, F, P, \{F\}_{t\geq 0}\right)$ be a filtered probability space with a normal filtration, where P is a probability measure on (Ω, F) and F is the Borel $\sigma-$algebra. Let $\{F\}_{t\geq 0}$ satisfying that F_0 contains all P-null sets.
The operator A is the infinitesimal generator of a strongly continuous semigroup on a separable real Hilbert space $H = L^2(D)$.
Denote the basic functional space $L^p(D)$, $1 \leq p < \infty$ and $H^s(D)$ by the usual Lebesgue and Sobolev space, respectively. We assume that

$$A = -\Delta,\ D(A) = H_0^1(D) \cap H^2(D),$$

since the operator A is self-adjoint, i.e. there exist the eigenvectors e_k corresponding to eigenvalues λ_k such that

$$Ae_k = \lambda_k e_k,\ e_k = \sqrt{2}\sin(k\pi),\ \lambda_k = \pi^2 k^2,\ k \in \mathbb{N}^+.$$

For any $\sigma > 0$, let H^σ be the domain of the fractional power $A^{\frac{\sigma}{2}} = (-\Delta)^{\frac{\sigma}{2}}$, which can be defined by

$$\sigma > 0, \quad A^{\frac{\sigma}{2}} e_k = \gamma_k^{\frac{\sigma}{2}} e_k,\ k = 1, 2, \ldots$$

and

$$H^\sigma = D\left(A^{\frac{\sigma}{2}}\right) = \left\{ v \in L^2(D),\ s.t.\ \|v\|_{H^\sigma}^2 = \sum_{k=1}^{\infty} \gamma_k^{\frac{\sigma}{2}} v_k^2 < \infty \right\},$$

where $v_k = \langle v, e_k \rangle$ with the inner product $\langle \cdot, \cdot \rangle$ in $L^2(D)$ and the norm $\|H^\sigma v\| = \left\| A^{\frac{\sigma}{2}} v \right\|$. Then we can rewrite the equation $(1.1)-(1.3)$ as follows in the abstract form

$$\begin{cases} {}_0^cD_t^\alpha u(x,t) + Au(x,t) = g(x,t)\frac{d}{dt}W(t),\ t > 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{2.1}$$

where $\{W(t),\ t \geq 0\}$ is a Q-Wiener process with linear bounded covariance operator Q such that a trace class operator Q denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots$, then the Wiener process is given by

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k,$$

where $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of real-valued standard Brownian motions.
Let $L_0^2 = L^2\left(Q^{\frac{1}{2}}(H), H\right)$ be a Hilbert-Schmidt space of operators from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\phi\|_{L_0^2} = \left\| \phi Q^{\frac{1}{2}} \right\|_{H^\sigma} = \left(\sum_{n=1}^{\infty} \phi Q^{\frac{1}{2}} e_n \right)^{\frac{1}{2}},$$

i.e.,

$$L_0^2 = \left\{ \phi \in L(H) : \sum_{n=1}^{\infty} \left\| \lambda_n^{\frac{1}{2}} \phi Q^{\frac{1}{2}} e_n \right\|^2 < \infty \right\},$$

where $L(H)$ is the space of bounded linear operators from H to H.
For an arbitrary Banach space Y, we denote

$$\|v\|_{L^p(\Omega,Y)} = \left(E\|v\|_Y^p\right)^{\frac{1}{p}},\ \forall v \in L^p(\Omega, F, P, Y),\ for\ any\ p \geq 2.$$

We shall also need the following result with respect to the operator A (see [26]).

Lemma 1. *For any $\nu > 0$, an analytic semigroup $T(t) = e^{-tA}$, $t \geq 0$ generated by the operator A on L^p, there exists a constant C_ν dependent on ν such that*

$$\|AT(t)\|_{L(L^p)} \leq C_\nu t^{-\nu},\ t > 0,$$

in which $L(Y)$ denotes the Banach space of all bounded operators from Y to itself.

Next, we will introduce the following lemmas and definitions which will be used in the sequel.

Definition 1. *[19]. The Laplace transform of the Caputo fractional derivative ${}_0^cD_t^\alpha f(t)$ is*

$$L\left[{}_0^cD_t^\alpha f(t); s\right] = \int_0^\infty e^{-st}\left({}_0^CD_t^\alpha f(t)\right) dt$$

$$= s^\alpha \widehat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where $n-1 < \alpha \leq n$, $\widehat{f}(s)$ is the Laplace transform of $f(t)$ and s is a parameter. In particular, for $0 < \alpha \leq 1$,

$$L\left[{}_0^cD_t^\alpha f(t); s\right] = s^\alpha \widehat{f}(s) - s^{\alpha-1} f(0).$$

Definition 2. *[19] The Fourier transform of a continuous function $h(t)$ absolutely integrable in $\mathbb{R}$ is defined by*

$$\widetilde{h}(\xi) = F[h(x); \xi] = \int_{\mathbb{R}} e^{i\xi x} h(x)\, dx,\ \xi \in \mathbb{R}.$$

ξ is a parameter. The inverse Fourier transform is defined by

$$h(x) = F^{-1}\left[\widetilde{h}(\xi); x\right] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} h(\xi)\, d\xi,\ x \in \mathbb{R}.$$

Lemma 2. *[24] Let X be a reflexive Banach space and suppose $v(t,\tau)$ is an $X-$ valued function defined for all $t \geq \tau \geq 0$, the derivatives $\frac{\partial^i v(t,\tau)}{\partial t^j}$, $0 \leq j \leq k-1$, are jointly continuous in the $X-$ norm and $\frac{\partial^k v(t,\tau)}{\partial t^k} \in L^1(0,t;X)$ for all $t > 0$. Let $u(t) = \int_0^t v(t,\tau)\, d\tau$. Then*

$$\frac{d^k}{dt^k} u(t) = \sum_{j=0}^{k-1}\left[\frac{\partial^{k-1-j}}{\partial t^{k-1-j}} v(t,\tau)\mid_{\tau=t}\right] + \int_0^t \frac{\partial^k}{\partial t^k} v(t,\tau)\, d\tau.$$

Lemma 3. *[11] Let $0 < \alpha < 1$. Then*

$$ {}_0I_t^\alpha \left[{}_0^cD_t^\alpha f(t)\right] = f(t) - f(0). $$

Lemma 4. *[9]. The Fourier transform of the Dirac delta function $\delta(\cdot)$ has the following property*

$$ F[\delta(x), \xi] = \int_{\mathbb{R}} e^{i\xi x} \delta(x)\, dx = 1, $$

and the inverse Fourier transform of the Dirac delta function $\delta(\cdot)$ is

$$ \delta(x) = F^{-1}[1] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \delta(x)\, dx = 1. $$

Lemma 5. *[9] The Dirac delta function $\delta(\cdot)$ has the following property*

$$ \int_{\mathbb{R}} \delta(x)\, dx = 1. $$

Lemma 6. *([9] Hausdorff-Young inequality) If $g \in L^p\,(p \geq 1)$, $f \in L^1$, then $h = f * g \in L^p$ and*

$$ \|h\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p}, $$

where $f * g = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$ denotes the convolution between f and g.

3. Main results

Inspired by the definition of the mild solution to the time-fractional differential equations (see [21]), we give the following definition of mild solution for our stochastic time-fractional partial differential equation. The mild solution (3.1) obtained by using the Definitions 1, 2 and Lemma 4.

Definition 3. *An F_t-adapted stochastic process $(u(t),\, t \in [0,T])$ is called a mild solution to homogeneous equation in (2.1) if the following integral equation is satisfied*

$$ u(x,t) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty e^{-i\xi(x-y)} E_\alpha(t)\, d\xi \varphi(y)\, dy, \tag{3.1} $$

where the generalized Mittag-Leffler operator $E_\alpha(t)$ is defined by

$$ E_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta)\, S(t^\alpha \theta)\, d\theta, $$

where $S(t) = e^{-tA}$, $t \geq 0$ *is an analytic semi group generated by the operator* $-A$ *and the Mainardi's Wright-type function with* $\alpha \in (0,1)$ *is given by*

$$\zeta_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^k}{k!\Gamma(1-\alpha(1+k))}.$$

Note that the classical Duhamel principle reduces the Cauchy problem for an inhomogeneous partial differential equation to Cauchy problem for corresponding homogeneous equation.

3.1. Fractional Duhamel principle

We consider the equation (1.1) subject to the initial data $u(x,0) = \varphi(x) = 0$, that is we study the nonhomogenuous problem

$$ {}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) = g(x,t)\dot{W}(t), \quad x \in D,\ t > 0, \tag{3.2}$$

$$u(x,0) = 0, \quad x \in D. \tag{3.3}$$

A fractional Duhamel principle is firstly given, which can reduce the nonhomogeneous problem $(3.2) - (3.3)$ to the corresponding homogeneous prolem that is the problem $(3.2) - (3.3)$ without the right hand part of (3.2).

Theorem 1. *(Fractional Duhamel principle) Let* τ *is a parameter. If* $w(x,t;\tau)$ *is the solution of the homogeneuos equation.*

$$ {}_0^cD_t^\alpha w(x,t) - \Delta w(x,t) = 0, \quad x \in D,\ t > 0, \tag{3.4}$$

satisfying when $t = \tau$,

$$w(x,\tau) = {}_0^cD^{1-\alpha} g(x,\tau)\dot{W}(\tau), \tag{3.5}$$

where $g(x,\tau)$ *is continuously differentiable function and* $\{W(\tau),\ \tau \geq 0\}$ *Brownian motion starting at* 0 *i.e.,* $W(0) = 0$. *Then, the solution of problem* $(3.2) - (3.3)$ *is given by*

$$u(x,t) = \int_0^t w(x,t;\tau)\,d\tau.$$

Proof. Suppose that $w(x,t,\tau)$ is the solution of $(3.4) - (3.5)$ and we will prove that

$$u(x,t) = \int_0^t w(x,t;\tau)\,d\tau.$$

Let $k = 1$ in Lemma 2, then we have

$$\frac{d}{dt}u(x,t) = w(x,t;\tau)\mid_{t=\tau} + \int_0^t \frac{\partial}{\partial t}w(x,t;\tau)\,d\tau.$$

Thus, by (1.4), (3.7) and $g(x,0)\dot{W}(0) = 0$, we have

$$\begin{aligned}
{}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) &= {}_0I_t^{1-\alpha}\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = \\
&= {}_0I_t^{1-\alpha}\frac{\partial}{\partial t}\int_0^t w(x,t;\tau)\,d\tau - \Delta\int_0^t w(x,t;\tau)\,d\tau = \\
&= {}_0I_t^{1-\alpha}\left[w(x,t;\tau)\mid_{t=\tau} + \int_0^t \frac{\partial}{\partial t}w(x,t;\tau)\,d\tau\right] - \Delta\int_0^t w(x,t;\tau)\,d\tau = \\
&= {}_0I_t^{1-\alpha}\left({}_0^cD_t^{1-\alpha}g(x,t)\,W(t)\right) + \int_0^t {}_0I_t^{1-\alpha}\frac{\partial}{\partial t}w(x,t;\tau)\,d\tau - \int_0^t \Delta w(x,t;\tau)\,d\tau = \\
&= g(x,t)\dot{W}(t) - g(x,0)\dot{W}(0) + \int_0^t \left[{}_0^cD_t^\alpha w(x,t;\tau) - \Delta w(x,t;\tau)\right]d\tau = \\
&= g(x,t)\dot{W}(t).
\end{aligned}$$

In addition, $u(x,0) = 0$. Thus $u(x,t) = \int_0^t w(x,t;\tau)\,d\tau$ is the solution of (3.2) – (3.3). The proof is complete. □

Corollary 1. *The solution of* (3.4) – (3.5) *can be expressed as*

$$\begin{aligned}
u(x,t) &= \int_0^t w(x,t-\tau;\tau)\,d\tau = \\
&= \frac{1}{2\pi}\int_0^t\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t-\tau)\,{}_0^cD_t^{1-\alpha}g(y,\tau)\,\dot{W}(\tau)\,d\xi dy d\tau.
\end{aligned}$$

Proof. In fact, let $t' = t - \tau > 0$ in (3.4) – (3.5), then (3.4) – (3.5) turned in the form,

$${}_0^cD_t^\alpha w\left(x,t';\tau\right) - \Delta w\left(x,t',\tau\right) = 0, \quad ,0 < \alpha < 1,\ x \in D, \tag{3.6}$$

$$t' = 0,\ w(x,0;\tau) = {}_0^cD_t^{1-\alpha}\varphi(x),\ x \in D. \tag{3.7}$$

From Definition 3, the solution of (3.6) – (3.7), expressed by

$$w(x,t-\tau;\tau)=\frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t-\tau)\,{}_0^cD_t^{1-\alpha}g(y,\tau)\dot{W}(\tau)\,d\xi dy.$$

Further, in view of Theorem 1, the solution of (3.2) – (3.3) has the form

$$u(x,t)=\int_0^t w(x,t-\tau;\tau)\,d\tau=$$
$$=\frac{1}{2\pi}\int_0^t\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t-\tau)\,{}_0^cD_t^{1-\alpha}g(y,\tau)\dot{W}(\tau)\,d\xi dyd\tau.$$

□

Combining Definition 3 with Corollary 1, we have the following result.

Theorem 2. *The solution of the nonhomogeneous* (1.1) – (1.3) *has the form*

$$u(x,t)=u_1(x,t)+u_2(x,t), \tag{3.8}$$

where $u_1(x,t)$, $u_2(x,t)$ *are solution of homogeneous problem and nonhomogeneous problem* (3.2) – (3.3) *respectively. That is*

$$u(x,t)=\frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t)\varphi(y)\,d\xi dy+$$
$$+\frac{1}{2\pi}\int_0^t\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t-\tau)\,{}_0^cD_t^{1-\alpha}g(y,\tau)\dot{W}(\tau)\,d\xi dyd\tau. \tag{3.9}$$

Theorem 3. *When* $t\to 0$, *the solution* (3.9) *of the problem* (1.1) – (1.3) *is bounded.*

Proof. From (3.9) and the definition of convolution between two functions, we have

$$\lim_{t\to 0}\|u(x,t)\|_{L^p(\mathbb{R})}=\left\|\frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}u(y,0)\,d\xi dy\right\|_{L^p(\mathbb{R})}=$$
$$=\left\|\int_0^\infty\delta(x-y)u(y,0)\,dy\right\|_{L^p(\mathbb{R})}=$$
$$=\|\delta(x)*u(y,0)\|_{L^p(\mathbb{R})},\ p\geq 1. \tag{3.10}$$

Applying Haousdorff-Yong inequality (Lemma 6) to (3.10) and the property of Dirac delta function $\delta(x)$, we have

$$\lim_{t\to 0}\|u(x,t)\|_{L^p(\mathbb{R})} = \|\delta(x) * u(y,0)\|_{L^p(\mathbb{R})} \leq$$
$$\leq \|\delta(x)\|_{L^1(\mathbb{R})} \cdot \|u(x,0)\|_{L^p(\mathbb{R})} \leq \|u(x,0)\|_{L^p(\mathbb{R})},$$

for $p \geq 1$. The proof is complete. □

3.2. Stability of solution

We will concerns with stability of the solution of problem (1.1) – (1.3).

Definition 4. *Suppose that H is a linear normed space with the norm $\|\cdot\|_H$, $u_1(x,t)$, $u_2(x,t)$ are solution of problem* (1.1) – (1.3) *corresponding to initial datum $\varphi_1(x)$, $\varphi_2(x)$ respectively. For any $\varepsilon > 0$, if there exists a constant $\delta > 0$ such that $\|\varphi_1(x) - \varphi_2(x)\| < \delta$ implies*

$$\|u_1(x,t) - u_2(x,t)\| < \varepsilon,$$

then we say that the solution of problem (1.1) – (1.3) *is stable.*

Now, we are in position to give our main result of nonhomogeneous problem (1.1) – (1.3).

Theorem 4. *(Stability) Assume $\varphi(x) \in L^p(\mathbb{R})$, $p \geq 1$. Then the solution $u(x,t)$ of the nonhomogeneous* (1.1) – (1.3) *is stable*

Proof. Suppose that $u_1(x,t)$ is the solution of the nonhomogeneous

$${}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) = g(x,t)\dot{W}(t), \quad x \in D,\ t > 0,$$
$$u(x,0) = \varphi_1(x), \quad x \in D.$$

and that $u_2(x,t)$ is the solution of the nonhomogeneous

$${}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) = g(x,t)\dot{W}(t), \quad x \in D,\ t > 0,$$
$$u(x,0) = \varphi_2(x), \quad x \in D.$$

Then, the superposition principle implies that $u_1(x,t) - u_2(x,t)$ is the solution of the following homogenuous problem

$${}_0^cD_t^\alpha u(x,t) - \Delta u(x,t) = 0, \quad x \in D,\ t > 0,$$
$$u(x,0) = \varphi_1(x) - \varphi_2(x), \quad x \in D.$$

By Definition 3, we have

$$u_1(x,t) - u_2(x,t) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-i\xi(x-y)}E_\alpha(t)\left[\varphi_1(y) - \varphi_2(y)\right]d\xi dy. \quad (3.11)$$

Taking the L^p-norm $p \geq 1$ on both sides of equation (3.11) and using Lemma 6, we have

$$\|u_1(x,t) - u_2(x,t)\|_{L^p(\mathbb{R})} =$$
$$= \left\| \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^\infty e^{-i\xi(x-y)} E_\alpha(t) \left[\varphi_1(y) - \varphi_2(y)\right] d\xi dy \right\|_{L^p(\mathbb{R})} \leq$$
$$\leq \left\| \frac{1}{2\pi} \int\limits_0^\infty e^{-i\xi(x-y)} E_\alpha(t)\, d\xi \right\|_{L^1((\mathbb{R}))} * \left\| \left[\varphi_1(y) - \varphi_2(y)\right] \right\|_{L^p(\mathbb{R})},$$

for $t > 0$.
For any $\varepsilon > 0$, choose $\delta < \frac{\varepsilon}{C}$. Then

$$\left\| \left[\varphi_1(y) - \varphi_2(y)\right] \right\|_{L^p(\mathbb{R})} < \delta,$$

this implies

$$\|u_1(x,t) - u_2(x,t)\|_{L^p(\mathbb{R})} < \varepsilon.$$

Thus, from the Definition 4, we deduce that the solution $u(x,t)$ of the nonhomogeneous (1.1) – (1.3) is stable.

□

4. Conclusion

In this paper, we transform the study of the stochastic fractional partial differential equation into the study of abstract stochastic fractional difference equation with initial data. The concept of superposition principle and fractional Duhamel principle and it's help to obtain the solution of our problem is introduced. Finally, we obtain the stability and boundedness of solutions of the proplem.

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Поступила в редакцию / Received 26.01.2022
Поступила после рецензирования / Revised 21.05.2022
Принята к публикации / Accepted 28.05.2022