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Nonlinear Kantorovich Problems with a Parameter

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Abstract. We consider nonlinear Kantorovich problems with marginal distributions and cost functions depending measurably on a parameter and prove that there exist optimal transportation plans that are also measurable with respect to the parameter. Unlike the classical linear Kantorovich problem of minimization of the integrals of a given cost function with respect to transportation plans, we deal with nonlinear cost functionals in which integrands depend on transportation plans. Dependence of cost functions on conditional measures of transportation plans is also allowed.

Keywords: Kantorovich problem, optimal plan, measurability with respect to a parameter

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Научная статья

Нелинейные задачи Канторовича с параметром

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Аннотация. Рассматриваются нелинейные задачи Канторовича с маргинальными распределениями и функциями стоимости, измеримо зависящими от параметра. Доказывается существование оптимальных планов транспортировки, также измеримо зависящих от параметра. В отличие от классической линейной задачи Канторовича минимизации интегралов заданной функции стоимости по транспортным планам мы имеем дело с нелинейными функционалами стоимости, в которых подынтегральные функции зависят от транспортных планов. Допускается также зависимость от условных мер транспортных планов.

Ключевые слова: задача Канторовича, оптимальный план, измеримость по параметру

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1. Introduction

In this paper we study nonlinear Kantorovich problems with marginal distributions and cost functions depending measurably on a parameter. We consider measures on completely regular Luzin spaces, i.e., images of complete separable metric spaces under continuous injective mappings. This important class of spaces contains all Borel subspaces of complete separable metric spaces (see [5, Corollary 6.8.5]), but also many nonmetrizable spaces encountered in applications. The parameter takes values in a Souslin space (Souslin spaces are images of complete separable metric spaces under continuous mappings). For the classical linear Kantorovich problem the existence of optimal plans measurable with respect to the parameter has recently been proved in [9] under broad conditions. In particular, if X, Y are complete separable metric spaces or, more generally, completely regular Luzin spaces, a parameter set T is a Souslin space, a cost function $h\colon T \times X \times Y \to [0, +\infty)$ is Borel measurable and for every fixed $t \in T$ the function $(x, y) \mapsto h(t, x, y)$ is lower semicontinuous (which means that the sets $\{(x, y)\colon h(t, x, y) \le c\}$ are closed), then, for any given Borel mappings $t \mapsto \mu_t$ and $t \mapsto \nu_t$ with values in the spaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ of Borel probability measures on X and Y, where the spaces of measures are equipped with their weak topologies, one can find optimal Kantorovich plans σ_t that are Borel measurable in t.

Recall (see [2], [8], [17], [18], and [19] for more details) that for a single Borel measurable cost function $h\colon X \times Y \to [0, +\infty)$ and two given measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ an optimal Kantorovich plan for the triple (μ, ν, h) is a Borel probability measure on $X \times Y$ belonging to the set $\Pi(\mu, \nu)$ of measures with projections μ and ν on the factors and minimizing the integral of h with respect to measures from $\Pi(\mu, \nu)$. Such an optimal plan exists if h is lower semicontinuous. In other words, the value

$$K_h(\mu, \nu) := \inf_{\sigma \in \Pi(\mu, \nu)} \int_{X \times Y} h \, d\sigma$$

is attained.

A nonlinear Kantorovich problem, investigated recently by several authors (see [1], [3], [4], and [14]), deals with minimization of more general integrals of the form

$$J_h(\sigma) = \int_{X \times Y} h(x, y, \sigma) \, \sigma(dx \, dy), \tag{1.1}$$

where the cost function $h\colon X \times Y \times \mathcal{P}(X \times Y) \to [0, +\infty)$ can now depend on the measure σ with respect to which it is integrated. The term "weak transport cost" used in some of these works does not look appropriate. So we call problems of this new type "nonlinear" to emphasize that the cost functional is not linear with respect to the plan.

In the parametric version the cost function also depends on a parameter t from a Souslin space T. Our main result states that this more general problem has solutions measurably dependent on the parameter. There is an interesting special case of dependence of the cost function on the plan (actually, the cost functions considered in [1], [3], [4], and [14] are of this type):

$$h(x, y, \sigma) = H(x, \sigma^x), \tag{1.2}$$

where H is defined on $X \times \mathcal{P}(Y)$ and σ^x are conditional measures for σ with respect to its projection σ_X on X, that is, $x \mapsto \sigma^x$ is a Borel mapping from X to $\mathcal{P}(Y)$ such that

$$\int_{X\times Y} f(x,y)\,\sigma(dx\,dy) = \int_X \int_Y f(x,y)\,\sigma^x(dy)\,\sigma_X(dx)$$

for every bounded Borel function f on $X \times Y$. Such cost functions have worse continuity properties, because conditional measures depend on σ measurably, but not always continuously. It is also possible to define conditional measures σ^x as measures on $X \times Y$ concentrated on the sets $\{x\} \times Y$, in which case the function H is defined on $X \times \mathcal{P}(X \times Y)$ and can be identified with h. Our second main result states that for cost functions of type (1.2) with a parameter there are also solutions depending measurably on this parameter, but here the convexity of h in σ is additionally required as in the cited papers. For the classical Kantorovich problem measurable dependence on parameters has been studied in [7], [9], [10], [16], and [19], including the case of problems with pointwise constraints in the spirit of [15] (see also [12]).

2. Measurable plans for general nonlinear costs

Before giving exact formulations, we recall that the weak topology on the set $\mathcal{P}(X)$ of Borel probability measures on a completely regular topological space X is induced by the weak topology on the whole space $\mathcal{M}(X)$ of signed Borel measures on X generated by all seminorms of the form

$$\mu \mapsto \int_X f\,d\mu,$$

where f belongs to the space of bounded continuous functions on X. If X is a complete metric space, then $\mathcal{P}(X)$ with the weak topology is metrizable with a complete metric, for example, one can use the Kantorovich–Rubinshtein norm

$$\|\mu\|_{KR} = \sup\biggl\{\int_X f\,d\mu\colon f \in \mathrm{Lip}_1(X),\ |f| \le 1\biggr\},$$

where $\mathrm{Lip}_1(X)$ is the space of 1-Lipschitz functions. Recall that on a Souslin space every measure $\mu \in \mathcal{P}(X)$ is Radon: for all Borel sets B one has

$$\mu(B) = \sup\{\mu(K)\colon K \subset B,\ K \text{ is compact}\}.$$

If X is a Luzin completely regular space, then $\mathcal{P}(X)$ is also Luzin in the weak topology and its compact subsets are metrizable (see [5, Theorem 8.9.6] or [6, Theorem 5.1.8]). In particular, every uniformly tight set is metrizable, that is, a set $M \subset \mathcal{P}(X)$ such that for every $\varepsilon > 0$ there is a compact set $K \subset X$ with $\mu(X \backslash K) < \varepsilon$ for all $\mu \in M$. Uniformly tight sets have compact closures in the weak topology by the Prohorov theorem. For more details on this background material see [5] and [6].

Given a nonnegative Borel function h on $X \times Y \times \mathcal{P}(X \times Y)$, we set

$$J_h(\sigma) = \int_{X\times Y} h(x,y,\sigma)\,\sigma(dx\,dy), \quad \sigma \in \mathcal{P}(X \times Y).$$

We need a simple observation: the function J_h with values in $[0,+\infty]$ is lower semicontinuous on uniformly tight subsets of $\mathcal{P}(X \times Y)$ provided that h is lower semicontinuous on every set of the form $S \times M$, where S is compact in $X \times Y$ and $M \subset \mathcal{P}(X \times Y)$ is compact and uniformly tight. Indeed, since the values $J_{\max(h,n)}(\sigma)$ increase to $J_h(\sigma)$, the assertion reduces to the case $h \le 1$. Next, J_h can be uniformly approximated on M by functions J_g with g lower semicontinuous on $X \times Y \times M$, because for each $\varepsilon \in (0,1)$ there is a compact set $K \subset X \times Y$ with $\sigma(K) > 1 - \varepsilon$ for all $\sigma \in M$. On the set $K \times M$ the function h can be represented as $\sup_n h_n$ for a sequence of continuous functions $h_n \ge 0$ (see [13, 1.7.15(c)]), but the same expression defines a lower semicontinuous function g on all of $X \times Y \times M$ and $|J_h(\sigma) - J_g(\sigma)| \le 2\varepsilon$ for all $\sigma \in M$. Finally, dealing with continuous h, we obtain yet another uniform approximation of J_h by J_f with f of the form $f(x,y,\sigma) = \sum_{i=1}^m \varphi_i(x,y)\psi_i(\sigma)$, where φ_i is a bounded continuous function on $X \times Y$ and ψ_i is a bounded continuous function on M. For such f the function J_f is continuous. In order to construct f we apply the Stone–Weierstrass theorem and find f of the indicated form such that $|h(x,y,\sigma) - f(x,y,\sigma)| \le \varepsilon$ for all $(x,y) \in K$, $\sigma \in M$. Finally, the functions φ_i can be redefined in such a way that $\varphi_i = 0$ outside a suitable neighborhood U of K, the values on K remain the same and f takes values in $[0,2]$ (the original function f takes values in $[0,2]$ only on $K \times M$).

This implies that the sets $\{\sigma \in M_1 \colon J_h(\mu) \le c\}$ are closed. In addition, the function J_h attains a minimum on every closed uniformly tight set on which it is finite. If X and Y are complete separable metric spaces and the function h is lower semicontinuous, then the functional J_h is lower semicontinuous on all of $X \times Y \times \mathcal{P}(X \times Y)$, which is verified shorter (see [3]).

Theorem 1. *Suppose that $t \mapsto \mu_t$, $T \to \mathcal{P}(X)$ and $t \mapsto \nu_t$, $T \to \mathcal{P}(Y)$ are Borel mappings and*

$$h\colon T \times X \times Y \times \mathcal{P}(X \times Y) \to [0,+\infty)$$

is a Borel function such that for every $t \in T$ the function

$$h_t\colon (x,y,\sigma) \mapsto h(t,x,y,\sigma)$$

is lower semicontinuous on sets of the form $K \times \Pi(\mu_t,\nu_t)$ with compact $K \subset X \times Y$ and $K_{h_t}(\mu_t,\nu_t)$ is finite for every $t \in T$. Then the function

$t \mapsto K_{h_t}(\mu_t, \nu_t)$ is Borel measurable and there is a Borel mapping $t \mapsto \sigma_t$ from T to $\mathcal{P}(X \times Y)$ such that σ_t is optimal for (μ_t, ν_t, h_t) for each t. Moreover, there exists a sequence of Borel mappings ξ_n from T to $\mathcal{P}(X \times Y)$ such that for each t the sequence of measures $\xi_n(t)$ is everywhere dense in the set of optimal plans for the triple (μ_t, ν_t, h_t).

Proof. It is worth noting that we can assume from the very beginning that the spaces X and Y are complete separable metric. Indeed, a Luzin space is obtained from such a space by weakening the topology, but the Borel σ-algebra remains the same. In addition, Borel measures on the original space remain Radon measures with respect to the metric (because on complete separable metric spaces all Borel measures are Radon) and every set $\Pi(\mu, \nu)$ is uniformly tight and compact in the weak topology on the space $\mathcal{P}(X \times Y)$ corresponding to the metrics on X and Y, hence on such sets we have coincidence of the weak topologies on $\mathcal{P}(X \times Y)$ generated by the original Luzin topology and the stronger metrizable topology. On the whole space $\mathcal{P}(X \times Y)$ these two topologies are different if $X \times Y$ is not a Polish space in the original topology. Finally, the lower continuity is also preserved in the stronger metrizable topology.

The set

$$S = \{(t, \sigma) \in T \times \mathcal{P}(X \times Y) \colon \sigma \in \Pi(\mu_t, \nu_t)\}$$

is Borel, because the inclusion $\sigma \in \Pi(\mu_t, \nu_t)$ can be written as a countable family of equalities

$$\int_{X \times Y} f_j(x)\, \sigma(dx\, dy) = \int_X f_j(x)\, \mu_t(dx),$$

$$\int_{X \times Y} g_j(y)\, \sigma(dx\, dy) = \int_Y g_j(y)\, \nu_t(dy),$$

where $\{f_j\}$ and $\{g_j\}$ are some sequences of bounded continuous functions on X and Y separating measures (such sequences exist on all completely regular Souslin spaces, see [5, Corollary 6.7.5, Theorem 6.7.7, Lemma 8.10.38]). The sections

$$S_t = \{\sigma \colon (t, \sigma) \in S\} = \Pi(\mu_t, \nu_t)$$

are compact.

Let us recall the following classical result ("Measurable Choice Theorem"), see [11, p. 224, 225]. Let T be a Souslin space, E a Luzin space, and let $B \subset T \times E$ be a Borel set such that for all $t \in T$ the sections B_x are σ-compact (countable unions of compact sets). Then B admits a Borel uniformization, which means that the projection $\pi_T(B)$ of B on T is a Borel set and there is a Borel mapping $f \colon \pi_T(B) \to E$ whose graph is contained in B.

By this theorem applied to $E=\mathcal{P}(X\times Y)$, in which case the projection of M on T is T, there exists a sequence of Borel mappings $\zeta_n\colon T\to\mathcal{P}(X\times Y)$ such that for each t the sequence $\{\zeta_n(t)\}$ is everywhere dense in $\Pi(\mu_t,\nu_t)$.

The function $(t,\sigma)\mapsto J_{h_t}(\sigma)$ with values in $[0,+\infty]$ is Borel measurable on $T\times\mathcal{P}(X\times Y)$. Indeed, this function is the result of substitution of σ for ζ in the function

$$(t,\sigma,\zeta)\mapsto\int_{X\times Y}h(t,x,y,\zeta)\,\sigma(dx\,dy),$$

which is Borel measurable. This is shown in [9, Lemma 3.1] for bounded h, but for unbounded h the result follows by considering $\max(h,N)$ and letting $N\to\infty$.

Next, if for every fixed t the function $\sigma\mapsto J_{h_t}(\sigma)$ is continuous on $\Pi(\mu_t,\nu_t)$, then

$$K_{h_t}(\mu_t,\nu_t)=\inf_n J_{h_t}(\zeta_n(t)).$$

Each function $J_{h_t}(\zeta_n(t))$ is Borel in t, because $\zeta_n(t)$ is Borel in t and $(t,\sigma)\mapsto J_{h_t}(\sigma)$ is jointly Borel measurable. Hence the left-hand side is Borel measurable in t. But the functions J_{h_t} can be only lower semicontinuous, when the previous argument does not work. In this case we apply the measurable selection theorem to verify that $K_{h_t}(\mu_t,\nu_t)$ is Borel measurable. With the aid of this theorem we shall construct a sequence of Borel functions J_n on $T\times\mathcal{P}(X\times Y)$ such that $J_n(t,\sigma)$ increases to $J_t(\sigma)$ for each $\sigma\in\Pi(\mu_t,\nu_t)$ and is continuous in the second argument. It suffices to find a countable family of functions J_n of this kind such that $J=\sup_n J_n$ and pass to the functions $\max(J_1,\dots,J_n)$. The required family is constructed as follows. For each rational number r the set

$$Z_r=\{(t,\sigma)\colon \sigma\in\Pi(\mu_t,\nu_t),\ J_{h_t}(\sigma)\le r\}$$

is Borel in $T\times\mathcal{P}(X\times Y)$ and its sections $S_{r,t}=\{\sigma\colon (t,\sigma)\in Z_r\}$ are compact by the compactness of $\Pi(\mu_t,\nu_t)$ and the lower semicontinuity of J_{h_t}. Hence the projection A_r of this set on T is Borel and there is a sequence of Borel mappings $\tau_{r,j}\colon A_r\to\mathcal{P}(X\times Y)$ such that $\{\tau_{r,j}(t)\}$ is dense in $S_{r,t}$ for each $t\in A_r$. The function $\delta_r(t,\sigma)=\inf_j\|\sigma-\tau_{r,j}(t)\|_{KR}$, i.e., the KR-distance from σ to $S_{r,t}$, is Borel measurable on $T\times\mathcal{P}(X\times Y)$. For each n, let $\theta_{r,n}$ be the function on $\mathbb{R}$ defined by $\theta_{r,n}(s)=0$ if $s\le 1/n$, $\theta_{r,n}(s)=r$ if $s\ge 2/n$, and $\theta_{r,n}(s)=nr(s-1/n)$ if $1/n<s<2/n$. Set $J_{r,n}(t,\sigma)=\theta_{r,n}(\delta_r(t,\sigma))$ if $t\in A_r$ and $J_{r,n}(t,\sigma)=r$ otherwise. It is readily seen that $J_{r,n}$ is Borel measurable and continuous in σ. In addition, $J_{r,n}(t,\sigma)\le J_{h_t}(\sigma)$ whenever $\sigma\in\Pi(\mu_t,\nu_t)$, since $J_{r,n}\le r$ and $J_{h_t}(\sigma)>r$ if $\sigma\notin S_{r,t}$, while $J_{r,n}(t,\sigma)=0$ if $\sigma\in S_{r,t}$. Finally,

$$J_{h_t}(\sigma)=\sup_{r,n}J_{r,n}(t,\sigma)\quad\text{for all }\sigma\in\Pi(\mu_t,\nu_t).$$

Indeed, otherwise there is a rational number r with

$$J_{h_t}(\sigma) > r > \sup_{r,n} J_{r,n}(t,\sigma).$$

By the lower semicontinuity of J_{h_t} there is a number n with

$$\|\sigma - \tau_{r,j}(t)\|_{KR} \geq 2/n$$

for all j. Then $J_{r,n}(t,\sigma) = r$, a contradiction. It remains to use the following simple fact: if continuous functions f_n on a compact space K increase pointwise to a bounded function f, then their minima increase to the minimum of f, which exists since f is lower semicontinuous. Indeed, let m be the minimum of f. The functions $f_n \wedge m$ increase to m and by Dini's theorem convergence is uniform. Hence their minima increase to m, which yields our claim. In our situation, having Borel functions $J_n(t,\sigma)$ increasing to $J_{h_t}(\sigma)$ and continuous in σ, we obtain that their minima on $\Pi(\mu_t,\nu_t)$ are Borel measurable in t and increase to $K_{h_t}(\mu_t,\nu_t)$.

Our next step is to repeat the same reasoning for the set

$$M = \{(t,\sigma) \in S \colon J_{h_t}(\sigma) = K_{h_t}(\mu_t,\nu_t)\},$$

which is also Borel in the space $T \times \mathcal{P}(X \times Y)$. The sections M_t are compact by the lower continuity of the function J_h and the fact that the set of points of minimum of a lower continuous function on a compact set is compact. Now we get a sequence of mappings ξ_n such that $\{\xi_n(t)\}$ is dense in M_t. $\square$

Remark 1. It is clear from the proof that the same assertion is true if h takes values in $[0,+\infty]$. Moreover, if we agree that the minimum is attained in the case where $J_{h_t}(\sigma) = +\infty$ for all $\sigma \in \Pi(\mu_t,\nu_t)$, then the assumption that $K_{h_t}(\mu_t,\nu_t) < \infty$ can be removed. It follows from the proof that the sets $\{t \in T \colon K_{h_t}(\mu_t,\nu_t) \leq C\}$ are Borel, so we can apply the theorem to such parametric sets.

3. Nonlinear costs with conditional measures

We now turn to cost functions of the form (1.2). In this case, an additional restriction is imposed on cost functions.

Theorem 2. *Suppose that in Theorem* 1 *the cost function h is of the form* (1.2), *where H is defined on $T \times X \times \mathcal{P}(Y)$, the functions*

$$H_t \colon (x,p) \mapsto H(t,x,p)$$

are lower semicontinuous and, in addition, the functions $p \mapsto H(t,x,p)$ are convex for all t,x. Then the conclusion of Theorem 1 *is true.*

Proof. In order to obtain a Borel function when substituting σ^x for p in $H(t,x,p)$, we use the fact established in [9] that conditional measures σ^x (which are unique only up to a redefinition on a measure zero set) can be selected in such a way that the functions $(x,\sigma) \mapsto \sigma^x(B)$ will be Borel measurable for each Borel set $B \subset Y$. It follows from [3] that the function $\sigma \mapsto J_{h_t}(\sigma)$ is lower semicontinuous on $\Pi(\mu_t,\nu_t)$. To be more precise, this is shown in [3] in the case of metric spaces, but as explained at the beginning of the proof of Theorem 1 this is sufficient for our purposes. Actually, the reasoning in [3] directly applies in the case of Luzin spaces (however, it is important to consider J_{h_t} on uniformly tight sets). Therefore, the sets

$$M_t = \left\{(t,\sigma)\colon \sigma \in \Pi(\mu_t,\nu_t),\ \int_X H(t,x,\sigma^x)\,\mu_t(dx) = K_{h_t}(\mu_t,\nu_t)\right\}$$

are compact. Hence the same reasoning as in Theorem 1 applies. □

What we have said in Remark 1 is also valid in the present situation.

It would be interesting to consider more general cost functions in (1.2) of the form $H(t,x,y,\sigma^x)$ or $H(t,x,y,\sigma^x,\sigma^y)$.

Note that in our paper [9] the case of Souslin spaces X and Y was also considered with appropriate concepts of measurability. This leads to some complications in the proof, because the measurable choice theorem applied above is not valid for such spaces. However, it is likely that the constructions from [9] extend to Souslin spaces in the present more general setting, which will be the subject of another paper.

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