Серия «Математика»
2022. Т. 39. С. 111—126

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

Research article

УДК 512.577+519.68:007.5
MSC 08A35, 08A62, 68Q06, 94C11
DOI https://doi.org/10.26516/1997-7670.2022.39.111

On Endomorphisms of the Additive Monoid of Subnets of a Two-layer Neural Network

Andrey V. Litavrin[1]✉

[1] Siberian Federal University, Krasnoyarsk, Russian Federation
✉ anm11@rambler.ru

Abstract. Previously, for each multilayer neural network of direct signal propagation (hereinafter, simply a neural network), finite commutative groupoids were introduced, which were called additive subnet groupoids. These groupoids are closely related to the subnets of the neural network over which they are built. A grupoid is a monoid if and only if it is built over a two-layer neural network. Earlier, endomorphisms and their properties were studied for these groupoids. Some endomorphisms were constructed, but an exhaustive element-by-element description was not received. It was shown that every finite monoid is isomorphic to some submonoid of the monoid of all endomorphisms of a suitable additive subnet groupoid for some suitable neural network.

In this paper, we study endomorphisms of additive groupoids of subnets of two-layer neural networks. The main result of the work is an element-wise description of the monoid of all endomorphisms of additive monoids of subnets built over a two-layer neural network. The item-by-item description is obtained by constructing a general form of endomorphism. The general view of an endomorphism is parameterized by the endomorphisms of suitable booleans with respect to the union operation. Therefore, endomorphisms of these Booleans were studied in this work. In particular, the semirings of endomorphisms of these Booleans with respect to the union were studied. In addition, to describe the general form of the endomorphism of the additive monoid of subnets, homomorphisms of one Boalean into another (with respect to union) were used.

Keywords: groupoid endomorphism, feedforward multilayer neural network, multilayer neural network subnet

Acknowledgements: This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2022-876).

For citation: Litavrin A. V.On Endomorphisms of the Additive Monoid of Subnets of a Two-layer Neural Network. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 39, pp. 111–126.
https://doi.org/10.26516/1997-7670.2022.39.111

Научная статья

Об эндоморфизмах аддитивного моноида подсетей двухслойной нейронной сети

А. В. Литаврин[1]✉

[1] Сибирский федеральный университет, Красноярск, Российская Федерация
✉ anm11@rambler.ru

Аннотация. Ранее для каждой многослойной нейронной сети прямого распространения сигнала (далее нейронная сеть) вводились конечные коммутативные группоиды, которые получили название аддитивные группоиды подсетей. Данные группоиды тесно связаны с подсетями нейронной сети, над которыми они построены. Группоид является моноидом тогда и только тогда, когда он построен над двухслойной нейронной сетью. Ранее для данных группоидов изучались эндоморфизмы и их свойства, а также были построены некоторые эндоморфизмы, но исчерпывающего поэлементного описания не получено. Было показано, что всякий конечный моноид изоморфен некоторому подмоноиду моноида всех эндоморфизмов подходящего аддитивного группоида подсетей для некоторой подходящей нейронной сети.
В работе рассмотрены эндоморфизмы аддитивных группоидов подсетей двухслойных нейронных сетей. Основным результатом исследования является поэлементное описание моноида всех эндоморфизмов аддитивных моноидов подсетей, построенных над двухслойной нейронной сетью. Поэлементное описание получено за счет построения общего вида эндоморфизма. Общий вид эндоморфизма параметризуется эндоморфизмами подходящих булеанов относительно операции объединения. Поэтому изучены эндоморфизмы данных булеанов, в том числе полукольца эндоморфизмов данных булеанов относительно объединения. Кроме того, для описания общего вида эндоморфизма аддитивного моноида подсетей использованы гомоморфизмы одного буалеана в другой (относительно объединения).

Ключевые слова: эндоморфизм группоида, многослойная нейронная сеть прямого распространения сигнала, подсеть многослойной нейронной сети

Благодарности: Работа выполнена при поддержке Красноярского математического центра и финансировании Министерства науки и высшего образования Российской Федерации (Проект № 075-02-2022-876).

Ссылка для цитирования: Литаврин А. В. Об эндоморфизмах аддитивного моноида подсетей двухслойной нейронной сети // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 39. С. 111–126.
https://doi.org/10.26516/1997-7670.2022.39.111

1. Introduction

This paper is a continuation of the study [4] in which the algebraic properties of some finite commutative groupoids $\mathrm{AGS}(\mathcal{N})$ are studied.

Groupoids $\mathrm{AGS}(\mathcal{N})$ are built over a given multilayer neural network $\mathcal{N}$ with direct signal distribution. The elements of this groupoid model the subnets of the neural network $\mathcal{N}$ in the sense of Definition 4 from [4].

In [4], the groupoids $\mathrm{AGS}(\mathcal{N})$ are called it additive subnet groupoids of multilayer neural network $\mathcal{N}$.

Among the main problems considered in the work [4] the problem was of element-wise description of the monoid of all endomorphisms of the groupoid $\mathrm{AGS}(\mathcal{N})$. It was shown that every finite monoid can be isomorphically embeddable into the monoid of all endomorphisms of the groupoid $\mathrm{AGS}(\mathcal{N})$ for a suitable neural network $\mathcal{N}$. Some endomorphisms of the groupoid $\mathrm{AGS}(\mathcal{N})$ have been described, but an exhaustive description of the elements $\mathrm{End}(\mathrm{AGS}(\mathcal{N}))$ was not received.

It turned out that the groupoid $\mathrm{AGS}(\mathcal{N})$ is a monoid if and only if $\mathcal{N}$ is a two-layer neural network ($n(\mathcal{N}) = 2$). Moreover, if $\mathcal{N}$ is a two-layer neural network and M_1 and M_2 are the set of all neurons lying in the first and second layers, and $B(X) := (2^X, \cup)$, then the equality $\mathrm{AGS}(\mathcal{N}) = B(M_1) \times B(M_2)$ holds (equality of sets of supports and equality operations; a stronger condition than isomorphism).

The main result of this work is the element-wise description of the monoid of all endomorphisms $\mathrm{End}(\mathrm{AGS}(\mathcal{N}))$, when $n(\mathcal{N}) = 2$. To describe endomorphisms from $\mathrm{End}(\mathrm{AGS}(\mathcal{N}))$ are used homomorphisms from $B(X)$ to $B(Y)$ and endomorphisms from $\mathrm{End}(B(X))$ for special X. The paper considers ways of describing such homomorphisms and endomorphisms. It is well known that the set of all endomorphisms of a commutative monoid forms a semiring with respect to the standard addition of endomorphisms and the composition of endomorphisms. In this paper, a special matrix representation of the endomorphism semiring $\mathrm{End}(B(X))$ is obtained for an arbitrary finite set X.

There are many studies on the properties of endomorphisms (in particular, automorphisms) of algebraic systems (see, for example, [9; 11; 12]). In particular, their element-wise descriptions. The properties of automorphisms of geometric objects are studied (see, for example, [8]).

Basic information about neural networks (in particular about multilayer neural networks) can be found in [2–5; 10]. It should be noted that the approach to determining the subnet of a multilayer neural network differs from the approach to determining the subsystem of a given algebraic system. In the theory of abstract automata (see, for example, the survey [1;6]), an abstract automaton is identified with a three-base algebraic system. The work [7] introduces the concept *abstract neural network*. This concept is similar to the concept of an abstract automaton, but differs in some specificity that is convenient for applying this abstraction to the study of issues specific to neural networks (in particular, training). There also arises the concept of an abstract neural network *subnet*, built as a subsystem of the corresponding three-base algebraic system. This approach is fundamentally different from the approach of introducing the concept of *subnet* in [4].

It should be noted that, in essence, it is impossible to study the internal structure of a neural network from the standpoint of abstract automata, therefore, from the standpoint of abstract neural networks. This detail is well known and was noted by V.M. Glushkov in the review [1, p.59,conclusion].

2. Basic definitions related to neural networks

This section will define the notions of a multilayer neural network, its subnet and groupoid $\mathrm{AGS}(\mathcal{N})$.

In this paper, sets will be denoted in capital Latin letters, and tuples composed of sets, in capital Latin letters with a bar. A tuple of empty sets will be denoted by the symbol $\overline{\varnothing} := (\varnothing, ..., \varnothing)$ (the length of such a tuple will always be clear from the context).

By default, $\mathbb{R}$ is the set of all real numbers. By $F(\mathbb{R})$ we denote the set of all functions $h: \mathbb{R} \to \mathbb{R}$ (here it is understood that the domain of the function h coincides with the set $\mathbb{R}$).

Next, we give definition 3 from [4].

Definition 1. *Let the following objects be given:*

1) a tuple $(M_1, ..., M_n)$ *of length* $n > 1$ *of finite non-empty sets, where* $M_i \cap M_j = \varnothing$ *is true for* $i \neq j$*;*

2) the set $S := (M_1 \times M_2) \cup (M_2 \times M_3) \cup ... \cup (M_{n-1} \times M_n)$*;*

3) the mapping $f: S \to \mathbb{R}$*, which assigns a real number to each pair from* S*;*

4) the set $A := M_1 \cup ... \cup M_n$*;*

5) the mapping $g: A \to F(\mathbb{R})$*, which assigns to each element from* A *a function from* $F(\mathbb{R})$*;*

6) the mapping $l: A \to \mathbb{R}$*, which assigns to each element from* A *some number from* $\mathbb{R}$*.*

Then the tuple $\mathcal{N} = (M_1, ..., M_n, f, g, l)$ *will be called a multilayer neural network of direct distribution (in the framework of this work, just neural networks).*

The tuple $(M_1, ..., M_n)$ is interpreted as the main tuple of neurons in the neural network $\mathcal{N}$, S is interpreted as a set of synoptic connections. The f function defines the synoptic connection weights, and the g function defines the functions activation in each neuron. The l function defines the threshold values of neurons. The input layer will be called the set of neurons M_1.

Information about the standard operation of a neural network as a computational circuit can be found in [2–4] and others.

Let two tuples $\overline{X} = (X_1, ..., X_n)$ and $\overline{Y} = (Y_1, ..., Y_n)$ of finite non-empty sets be given. Then by $\overline{X} \cup \overline{Y}$ we will denote the componentwise union $\overline{X} \cup \overline{Y} := (X_1 \cup Y_1, ..., X_n \cup Y_n)$.

If $\overline{X} = (X_1, ..., X_n)$ and $\overline{M} = (M_1, ..., M_n)$ are two tuples whose components are sets, then we say that the condition $\overline{X} \subseteq \overline{M}$ if all inclusions $X_1 \subseteq M_1, ..., X_n \subseteq M_n$ are true (componentwise inclusion).

Let $(X_1, ..., X_n)$ be some tuple composed of finite sets, we say that the tuple *is continuous* if for all different $i, j \in \{1, ..., n\}$ the following implication holds: if $X_i \neq \varnothing$ and $X_j \neq \varnothing$ and $i < j$, then for all $s \in \{i, ..., j\}$ the inequality $X_s \neq \varnothing$ holds. The tuple $\overline{\varnothing}$ is assumed to be continuous by definition. For a tuple of sets to be continuous, it should not have alternation of a non-empty set with an interval of empty sets, and then again with a non-empty set.

Let us give definition 4 from [4].

Definition 2. *Let the neural network be defined $\mathcal{N} = (M_1, ..., M_n, f, g, l)$ and a continuous tuple $\overline{X} = (X_1, ..., X_n)$ is given such that it contains more than one component other than the empty set, and*

$$(X_1, ..., X_n) \subseteq (M_1, ..., M_n).$$

We assume that $\overline{Y} = (Y_1, ..., Y_m)$ is a tuple obtained from a tuple $\overline{X}$ by deleting components equal to the empty set, where $m \leq n$.

If f' is a restriction of the function f on the set

$$S' := (Y_1 \times Y_2) \cup (Y_2 \times Y_3) \cup ... \cup (Y_{m-1} \times Y_m)$$

and g', l' is the restriction of the functions g and l on the set $A' := Y_1 \cup ... \cup Y_m$, then object

$$\mathcal{N}' := (Y_1, ..., Y_m, f', g', l')$$

will be called subnet of the network $\mathcal{N}$. We say that the tuple $\overline{X}$ induces the subnet $\mathcal{N}'$. The $\overline{Y}$ tuple is the main tuple of neurons in the $\mathcal{N}'$ subnet. In general, the tuples $\overline{X}$ and $\overline{Y}$ can be different.

More information about neural network subnets can be found in [4]. Note that the proposed approach to defining the subnetwork of a neural network corresponds to works studying the applied aspects of neural networks.

Construction of groupoids $\mathrm{AGS}(\mathcal{N})$**.** Next, we formulate Definition 1 from [4] groupoid $\mathrm{AGS}(\mathcal{N})$.

Definition 3. *Let the neural network $\mathcal{N}$ be defined with the main tuple of neurons $\overline{M}$. The set of all possible continuous tuples $\overline{X} \subseteq \overline{M}$ will be denoted by the symbol* $\mathrm{AGS}(\mathcal{N})$*.*

We assume that $\overline{X}$ *and* $\overline{Y}$ *are two arbitrary element from* $\mathrm{AGS}(\mathcal{N})$. *Let's define a binary algebraic operation* $(+)$*:*

$$\overline{X}+\overline{Y}:=\begin{cases}\overline{X}\cup\overline{Y}, & if\ \overline{X}\cup\overline{Y}\in\mathrm{AGS}(\mathcal{N})\\ \overline{\varnothing}, & if\ \overline{X}\cup\overline{Y}\notin\mathrm{AGS}(\mathcal{N}).\end{cases}$$

Then the groupoid $\mathrm{AGS}(\mathcal{N}) := (\mathrm{AGS}(\mathcal{N}), +)$ *will be called the additive groupoid of subnets of neural network* $\mathcal{N}$*.*

If $\mathcal{N}$ is a two-layer neural network, then for all $\overline{X},\overline{Y}\in\mathrm{AGS}(\mathcal{N})$ the equality holds $\overline{X}+\overline{Y}=\overline{X}\cup\overline{Y}$ and $\mathrm{AGS}(\mathcal{N})=B(M_1)\times B(M_2)$ (equality of sets).

3. Some definitions and formulation of the main result

Let us formulate the necessary definitions. Let $G=(G,\circ)$ be a monoid and $1\in G$ is a neutral element of this monoid. Then the mapping $\phi: G\to G$ is called an *endomorphism* of the monoid G if $1^{\phi}=1$ and for all $x,y\in G$ the equality is true

$$(x\circ y)^{\phi}=x^{\phi}\circ y^{\phi}. \tag{3.1}$$

A semiring is a non-empty set S with two binary algebraic operations $(+)$ and $(*)$ such that $(S,+)$ is a commutative monoid, $(S,\cdot)$ is a semigroup, addition and multiplication are related by left and right distributivity with respect to addition, and a neutral element o of the monoid $(S,+)$ satisfies the identity $o\cdot x=x\cdot o=o$ (multiplicative property of zero). It is well known that the set of all endomorphisms of a commutative monoid with respect to the standard operation of addition of two endomorphisms and the composition of two endomorphisms forms a semiring. Let G be a commutative monoid. *Notation related to composition of endomorphisms.* We assume that $x\in G$ and $\phi\in\mathrm{End}(G)$. Then x^{ϕ} is the image of the element x under the action of the endomorphism ϕ. The composition of two endomorphisms will be denoted by the symbol $(\cdot)$. If $\phi_1,\phi_2\in\mathrm{End}(G)$ and $x\in G$, then $x^{\phi_1\cdot\phi_2}:=(x^{\phi_2})^{\phi_1}$.

Notation related to the sum of endomorphisms. Let be $\phi_1,\phi_2\in\mathrm{End}(G)$ and $x\in G$. Then, as usual, the sum $(+)$ of two endomorphisms will denote the mapping $\phi_1+\phi_2$, which acts on G according to the rule

$$x^{\phi_1+\phi_2}:=x^{\phi_1}+x^{\phi_2}.$$

It is well known that the sum of two endomorphisms of a commutative monoid is again an endomorphism of this monoid.

Let X be some finite set, 2^X a Boolean of the set X. We will use the notation $B(X)=(2^X,\cup)$. In the framework of this paper, we consider the

Boolean of some set only with respect to the operation $(\cup)$. It is well known that $B(X) = (2^X, \cup)$ is a commutative monoid consisting of idempotents.

We assume that $\mathcal{N}$ is a two-layer neural network with the main tuple of neurons (M_1, M_2). As noted in the introduction, the equality is true $\mathrm{AGS}(\mathcal{N}) = B(M_1) \times B(M_2)$.

For any endomorphism $\tau_1 \in \mathrm{End}(B(M_1))$ and any homomorphism τ_2 of the monoid $B(M_2)$ into the monoid $B(M_1)$ we introduce the mapping

$$\alpha_{\tau_1,\tau_2}(\overline{U}) = U_1^{\tau_1} \cup U_2^{\tau_2} \qquad (\overline{U} = (U_1, U_2) \in \mathrm{AGS}(\mathcal{N})).$$

The mapping α_{τ_1,τ_2} is a homomorphism from $\mathrm{AGS}(\mathcal{N})) = B(M_1) \times B(M_2)$ to $B(M_1)$. Indeed, let $\overline{U} = (U_1, U_2)$ and $\overline{V} = (V_1, V_2)$ be two arbitrary elements from $\mathrm{AGS}(\mathcal{N}))$. We have the equalities

$$\alpha_{\tau_1,\tau_2}(\overline{U} \cup \overline{V}) = (U_1 \cup V_1)^{\tau_1} \cup (U_2 \cup V_2)^{\tau_2} = U_1^{\tau_1} \cup V_1^{\tau_1} \cup U_2^{\tau_2} \cup V_2^{\tau_2} =$$

$$[U_1^{\tau_1} \cup U_2^{\tau_2}] \cup [V_1^{\tau_1} \cup V_2^{\tau_2}] = \alpha_{\tau_1,\tau_2}(\overline{U}) \cup \alpha_{\tau_1,\tau_2}(\overline{V}).$$

Thus, we have shown that α_{τ_1,τ_2} is a homomorphism.

For every homomorphism ζ_1 of the monoid $B(M_1)$ into the monoid $B(M_2)$ and every endomorphism $\zeta_2 \in \mathrm{End}(B(M_2))$ we introduce the mapping

$$\beta_{\zeta_1,\zeta_2}(\overline{U}) = U_1^{\zeta_1} \cup U_2^{\zeta_2} \qquad (\overline{U} = (U_1, U_2) \in \mathrm{AGS}(\mathcal{N})).$$

The mapping β_{ζ_1,ζ_2} is a homomorphism of the monoid $B(M_1) \times B(M_2)$ into $B(M_2)$. Indeed, let $\overline{U} = (U_1, U_2)$ and $\overline{V} = (V_1, V_2)$ be two arbitrary elements from $\mathrm{AGS}(\mathcal{N}))$. We have the equalities

$$\beta_{\zeta_1,\zeta_2}(\overline{U} \cup \overline{V}) = (U_1 \cup V_1)^{\zeta_1} \cup (U_2 \cup V_2)^{\zeta_2} = U_1^{\zeta_1} \cup V_1^{\zeta_1} \cup U_2^{\zeta_2} \cup V_2^{\zeta_2} =$$

$$[U_1^{\zeta_1} \cup U_2^{\zeta_2}] \cup [V_1^{\zeta_1} \cup V_2^{\zeta_2}] = \beta_{\zeta_1,\zeta_2}(\overline{U}) \cup \beta_{\zeta_1,\zeta_2}(\overline{V}).$$

Thus, we have shown that β_{ζ_1,ζ_2} is a homomorphism.

For any $\tau_1 \in \mathrm{End}(B(M_1))$, $\zeta_2 \in \mathrm{End}(B(M_2))$, arbitrary homomorphisms τ_2 of the monoid $B(M_2)$ into the monoid $B(M_1)$ and ζ_1 of the monoid $B(M_1)$ into the monoid $B(M_2)$ we introduce the mapping $\rho : B(M_1) \times B(M_2) \to B(M_1) \times B(M_2)$ given by the rule

$$\overline{U}^{\rho} = (\alpha_{\tau_1,\tau_2}(\overline{U}), \beta_{\zeta_1,\zeta_2}(\overline{U})) \quad (\overline{U} \in B(X) \times B(X)). \tag{3.2}$$

Let us show that the mapping ρ introduced by the rule (3.2) is an endomorphism of the monoid $\mathrm{AGS}(\mathcal{N})$. Let $\overline{U} = (U_1, U_2)$ and $\overline{V} = (V_1, V_2)$ - two arbitrary elements from $\mathrm{AGS}(\mathcal{N})$. We get equalities

$$(\overline{U} + \overline{V})^{\rho} = (\alpha_{\tau_1,\tau_2}(\overline{U} + \overline{V}), \beta_{\zeta_1,\zeta_2}(\overline{U} + \overline{V})) = (\alpha_{\tau_1,\tau_2}(\overline{U} \cup \overline{V}), \beta_{\zeta_1,\zeta_2}(\overline{U} \cup \overline{V})) =$$

$$(\alpha_{\tau_1,\tau_2}(\overline{U}) \cup \alpha_{\tau_1,\tau_2}(\overline{V}), \beta_{\zeta_1,\zeta_2}(\overline{U}) \cup \beta_{\zeta_1,\zeta_2}(\overline{V})) = \overline{U}^{\rho} + \overline{V}^{\rho}.$$

The main theorem in this work is the theorem

Theorem 1. *The set of all endomorphisms of the monoid* $\mathrm{AGS}(\mathcal{N})$ *for* $n(\mathcal{N}) = 2$ *is bounded by all kinds of endomorphisms* ρ.

Thus, an arbitrary endomorphism of the monoid $\mathrm{AGS}(\mathcal{N})$ is parameterized by homomorphisms from $\mathrm{AGS}(\mathcal{N})$ $(n(\mathcal{N}) = 2)$ to $B(M_1)$, $B(M_2)$. These homomorphisms are parameterized by homomorphisms (in particular, endomorphisms) from $B(X)$ to $B(Y)$, when $X = M_1, M_2$ and $Y = M_1, M_2$. Therefore, in this paper we prove Proposition 1 (see the next section).

Proposition 1 gives an element-wise description of all homomorphisms of the Boolean $B(X)$ into $B(A)$. A consequence of Proposition 1 (see Corollary 1, next section) is an element-wise description of all endomorphisms of the Boolean $B(X)$ for an arbitrary finite set X.

For the monoid of all endomorphisms $\mathrm{End}(B(X))$ for an arbitrary finite set X one can establish a matrix representation over a special semiring. As noted above, $\mathrm{End}(B(X))$ is a semiring under addition and composition of two endomorphisms.

Next, we need basic binary logic functions: *conjunction* (we will denote $(\wedge)$) and *disjunction* (we will denote $(\vee)$). We will use the logical semiring $B = (\{0, 1\}, \vee, \wedge)$. The set of all possible square matrices of order n with elements from the ring B will be denoted by $M_{n\times n}(B)$.

Theorem 2. *For each finite set* X *consisting of* n *elements, the semiring* $\mathrm{End}(B(X))$ *of all endomorphisms of the monoid* $B(X) = (B(X), \cup)$ *is isomorphic to the semiring of matrices* $M_{n\times n}(B)$ *with elements from the logical semiring* B.

4. Homomorphisms from $B(A)$ to $B(C)$

Consider homomorphisms from the Boolean $B(A)$ to the Boolean $B(C)$.

It is easy to show that the set $Q = \{\varnothing\} \cup \{\{x\} \mid x \in A\}$ is a generating set of the monoid $B(A)$.

General view of the homomorphism from $B(A)$ *to* $B(C)$. For each family $\mathcal{L} = \{L_x\}_{x\in A}$ of sets from 2^C define the mapping $\phi_{\mathcal{L}}$ given by the rule

$$U^{\phi_{\mathcal{L}}} = \bigcup_{u\in U} L_u, \quad \varnothing^{\phi_{\mathcal{L}}} = \varnothing$$

for any non-empty set $U \in B(A)$. Since the inclusions $L_x \in B(C)$ and $U \in B(A)$ holds, then the inclusion $U^{\phi_{\mathcal{L}}} \in B(C)$ holds.

Lemma 1. *The mapping* $\phi_{\mathcal{L}}$ *is a homomorphism of the monoid* $B(A) = (2^A, \cup)$ *into the monoid* $B(C) = (2^C, \cup)$.

Proof. Next, $\phi := \phi_{\mathcal{L}}$. Let U and V be two arbitrary elements from $B(A)$. Since the monoid $B(A)$ is commutative, associative, and idempotent, the equality is true

$$(U \cup V)^{\phi} = \bigcup_{m \in U \cup V} L_m = \left(\bigcup_{u \in U} L_u \right) \cup \left(\bigcup_{v \in V} L_v \right).$$

On the other hand, the equality is true $U^{\phi} \cup V^{\phi} = \left(\bigcup_{u \in U} L_u \right) \cup \left(\bigcup_{v \in V} L_v \right)$. Thus, we have shown that for any $U, V \in B(A)$ the equality is true

$$(U \cup V)^{\phi} = U^{\phi} \cup V^{\phi}.$$

The lemmae is proved. □

Obviously, if equality $A = C$ is true and the family $\mathcal{L} = \{L_x\}_{x \in A}$ of sets from 2^C is defined, then $\phi_{\mathcal{L}}$ is an endomorphism of the monoid $B(A)$.

Proposition 1. *Any homomorphism of the monoid $B(A)$ into the monoid $B(C)$ is a homomorphism $\phi_{\mathcal{L}}$ for a suitable family $\mathcal{L}$ of subsets from $B(C)$.*

Proof. Let ϕ be an arbitrary monoid homomorphism $B(A)$ to $B(C)$.

1. It is clear that $\varnothing^{\phi} = \varnothing$. Consider the action ϕ on the generating set Q. We assume that the image of the element $\{x\}, x \in A$ under the action of ϕ is equal to the set $W_x \in B(C)$.

We introduce a family $\mathcal{L} = \{L_x\}_{x \in A}$ of sets L_x from $B(C)$ such that for all $x \in A$ the equality is true $L_x = W_x$.

The family $\mathcal{L}$ is defined so that for all $x \in A$ the equalities are true

$$\{x\}^{\phi_{\mathcal{L}}} = L_x = W_x = \{x\}^{\phi} \quad (\{x\} \in Q).$$

Thus, we have shown that every homomorphism ϕ acts on the set Q as a homomorphism $\phi_{\mathcal{L}}$.

It remains for us to show that ϕ acts on $B(A) \setminus Q$ as endomorphism $\phi_{\mathcal{L}}$.

2. Suppose that $U \in B(A) \setminus Q$. For this element, the decomposition $U = \cup_{x \in U} \{x\}$ is valid and the equalities are true

$$U^{\phi} = \left(\bigcup_{x \in U} \{x\} \right)^{\phi} = \bigcup_{x \in U} \{x\}^{\phi} = \bigcup_{x \in U} \{x\}^{\phi_{\mathcal{L}}} = \left(\bigcup_{x \in U} \{x\} \right)^{\phi_{\mathcal{L}}} = U^{\phi_{\mathcal{L}}}.$$

Thus, an arbitrary homomorphism ϕ acts on $B(A)$ as a homomorphism $\phi_{\mathcal{L}}$ for a suitable family $\mathcal{L}$. □

If we assume in Proposition 1 (and its proof) that $A = C = X$, then we obtain

Corollary 1. *Any endomorphism of the monoid $B(X)$ is an endomorphism $\phi_{\mathcal{L}}$ for a suitable family $\mathcal{L} = \{L_x\}_{x \in X}$ of subsets from $B(X)$.*

5. Composition and sum of two endomorphisms $B(X)$

Let there be given two families of sets $\mathcal{L} = \{L_x\}_{x\in X}$, $\mathcal{D} = \{D_x\}_{x\in X}$ from 2^X. The mappings $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{D}}$ are endomorphisms of $B(X)$. Then the equalities are true

$$\phi_{\mathcal{L}} \cdot \phi_{\mathcal{D}} = \phi_{\mathcal{Z}}, \quad \phi_{\mathcal{L}} + \phi_{\mathcal{D}} = \phi_{\mathcal{V}},$$

where family members $\mathcal{Z} = \{Z_x\}_{x\in X}$ and $\mathcal{V} = \{V_x\}_{x\in X}$ satisfy the equalities

$$Z_x = \begin{cases} \bigcup_{y\in D_x} L_y, & if\ D_x \neq \varnothing \\ \varnothing, & if\ D_x = \varnothing, \end{cases} \tag{5.1}$$

$$V_x = L_x \cup D_x. \tag{5.2}$$

Indeed, let U be an arbitrary element from 2^X. Then the equalities hold

$$U^{\phi_{\mathcal{L}}\cdot\phi_{\mathcal{D}}} = (U^{\phi_{\mathcal{D}}})^{\phi_{\mathcal{L}}} = \left(\bigcup_{x\in U} D_x\right)^{\phi_{\mathcal{L}}} = \bigcup_{x\in U} D_x^{\phi_{\mathcal{L}}} = \bigcup_{x\in U} \left(\bigcup_{y\in D_x} L_y\right) =$$

$$= \left[Z_x := \bigcup_{y\in D_x} L_y\right] = \bigcup_{x\in U} Z_x = U^{\phi_{\mathcal{Z}}},$$

where $\mathcal{Z} = \{Z_x\}_{x\in X}$. Equality (5.1) is proved.

On the other hand, the equalities are true

$$U^{\phi_{\mathcal{L}}+\phi_{\mathcal{D}}} = U^{\phi_{\mathcal{L}}} \cup U^{\phi_{\mathcal{D}}} = \left(\bigcup_{u\in U} L_u\right) \cup \left(\bigcup_{u\in U} D_u\right) = \bigcup_{u\in U} (L_u \cup D_u) =$$

$$[V_x := L_x \cup D_x] = \bigcup_{u\in U} V_u = U^{\phi_{\mathcal{V}}}.$$

Equality (5.2) is proved.

6. Proof of Theorem 2

Further, we need the basic binary logical functions: *conjunction* (we will denote $(\wedge)$) and *disjunction* (we will denote $(\vee)$). We will use the logical semiring $B = (\{0,1\}, \vee, \wedge)$.

Further, we assume that the set X is finite ($|X| = n$) and ordered. In accordance with this ordering, we will denote the elements of the set $X = \{x_1, ..., x_n\}$.

Let a family of sets $\mathcal{L} = \{L_x\}_{x\in X}$ be given. Then the endomorphism $\phi_{\mathcal{L}}$ is defined. Since the order is defined on the elements of the set X, specifying the family $\mathcal{L}$ is equivalent to specifying the tuple $\overline{L} = (L_1, ..., L_n)$, where $L_i := L_{x_i}$.

For each endomorphism $\phi_{\overline{L}}$, where $\overline{L} = (L_1, ..., L_n)$ we define a square matrix $A_{\overline{L}}$ with elements from B of order n as follows:

1) $A_{\overline{L}} = (a_{ij})$;
2) $a_{ij} = 1$ if and only if $x_i \in L_j$;
3) $a_{ij} = 0$ if and only if $x_i \notin L_j$.

This way of assignment can be reformulated in words. If the component L_j contains the element x_i, then the element in the j-th column and in the i-th row is equal to one, otherwise zero. Thus, the matrix $A_{\overline{L}}$ is compiled column by column.

The set of all possible square matrices of order n with elements from the ring B will be denoted by $M_{n\times n}(B)$.

Next, consider the mapping $\alpha : \mathrm{End}(B(X)) \to M_{n\times n}(B)$ defined by rule

$$\alpha(\phi_{\overline{L}}) = A_{\overline{L}}.$$

From the way of constructing the matrix $A_{\overline{L}}$ it can be seen that the mapping α is a bijection. Let us show that α is an isomorphism between the semirings $\mathrm{End}(B(X))$ and $M_{n\times n}(B)$.

To show that α is an isomorphism, we show that for any tuples $\overline{L} = (L_1, ..., L_n)$ and $\overline{D} = (D_1, ..., D_n)$ from $(2^X)^n$ equalities are true

$$\alpha(\phi_{\overline{L}} \cdot \phi_{\overline{D}}) = A_{\overline{L}} \cdot A_{\overline{D}}, \tag{6.1}$$

$$\alpha(\phi_{\overline{L}} + \phi_{\overline{D}}) = A_{\overline{L}} + A_{\overline{D}}, \tag{6.2}$$

where on the right stand the usual matrix multiplication and matrix addition.

Isomorphism of multiplicative semigroups of semirings. Next, we will show that the equality (6.1). Let be

$$\phi_{\overline{Z}} := \phi_{\overline{L}} \cdot \phi_{\overline{D}},$$

where, by virtue of (5.1), the equalities

$$Z_j = \begin{cases} \bigcup_{x_i \in D_j} L_i, & \text{if } D_j \neq \varnothing \\ \varnothing, & \text{if } D_j = \varnothing. \end{cases}$$

We have the equality $\alpha(\phi_{\overline{Z}}) = A_{\overline{Z}}$. We assume that $A_{\overline{Z}} = (z_{ij})$, $A_{\overline{L}} = (a_{ij})$, $A_{\overline{D}} = (b_{ij})$ and $A_{\overline{L}} \cdot A_{\overline{D}} = C = (c_{ij})$ where

$$c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj}).$$

Let $c_{ij} = 1$. This means that there are elements $a_{ik'}$ and $b_{k'j}$ equal to one. Which in turn means that D_j contains the element $x_{k'}$, and the set

$L_{k'}$ contains the element x_i. Hence, the set Z_j contains the element x_i, therefore, the matrix $A_{\overline{Z}}$ contains the element $z_{ij} = 1 = c_{ij}$.

Let $c_{ij} = 0$. This is possible in one (and only one) of the cases:

1. all b_{kj} are equal to zero for any $1 \leq k \leq n$;

2. among the elements of b_{kj} there are nonzero elements, denote them by

$$\{b_{k_1,j}, b_{k_2,j}, ..., b_{k_s,j}\},$$

but all elements $\{a_{i,k_1}, a_{i,k_2}, ..., a_{i,k_s}\}$ are equal to zero.

In the first case, we get that the set D_j is empty, therefore, the set Z_j is also empty. Hence, $z_{ij} = 0 = c_{ij}$ (in this case, for any i). In the second case, we get that the set D_j contains elements $\{x_{k_1}, x_{k_2}, ..., x_{k_s}\}$ and sets $\{L_{k_1}, L_{k_2}, ..., L_{k_s}\}$ do not contain element x_i, therefore, the following conditions are true

$$Z_j = \bigcup_{h=1}^{h=s} L_{k_h}, \quad x_i \notin Z_j.$$

This means that $z_{ij} = 0 = c_{ij}$ (in this case, for specific i and j).

Thus, we have shown that the matrices $A_{\overline{Z}} = (z_{ij})$ and $A_{\overline{L}} \cdot A_{\overline{D}} = C$ are equal. Since $\alpha(\phi_{\overline{L}} \cdot \phi_{\overline{D}}) = \alpha(\phi_{\overline{Z}}) = A_{\overline{Z}} = A_{\overline{L}} \cdot A_{\overline{D}}$, then the identity (6.1) also holds.

Isomorphism of additive commutative monoids of semirings. Let be

$$\phi_{\overline{V}} := \phi_{\overline{L}} + \phi_{\overline{D}},$$

where by virtue of (5.2) the relations $V_j = L_j \cup D_j$.

We have the equality $\alpha(\phi_{\overline{V}}) = A_{\overline{V}}$. We assume that $A_{\overline{V}} = (v_{ij})$ and $A_{\overline{L}} + A_{\overline{D}} = W = (w_{ij})$ where $w_{ij} = a_{ij} \vee b_{ij}$.

Let be $w_{ij} = 1$. Hence, a_{ij} or b_{ij} equal to one. Means what $x_i \in L_j$ or $x_i \in D_j$, hence, $x_i \in L_j \cup D_j = V_j$. Hence we obtain the equality $w_{ij} = v_{ij} = 1$.

Let be $w_{ij} = 0$. Therefore, $a_{ij} = 0$ and $b_{ij} = 0$. Hence, $x_i \notin L_j$ and $x_i \notin D_j$, hence, $x_i \notin L_j \cup D_j = V_j$. Hence we obtain the equality $w_{ij} = v_{ij} = 0$.

Thus, we have shown that the matrices $A_{\overline{V}}$ and $W = A_{\overline{L}} + A_{\overline{D}}$ are equal. Therefore, the equality (6.2) is also true.

Thus, we have proved Theorem 2.

7. Proof of the main theorem 1

1. Let ϕ be an endomorphism of the monoid $\mathrm{AGS}(\mathcal{N})$ for $n(\mathcal{N}) = 2$ and $\overline{U}$ from $\mathrm{AGS}(\mathcal{N})$. Then the equalities are true

$$\overline{U}^{\phi} = (R_1(\overline{U}), R_2(\overline{U})),$$

where $R_1 : \mathrm{AGS}(\mathcal{N}) \to B(M_1)$ and $R_2 : \mathrm{AGS}(\mathcal{N}) \to B(M_2)$. It is trivially established that R_1 is a homomorphism from $\mathrm{AGS}(\mathcal{N})$ to $B(M_1)$ and R_2 is a homomorphism from $\mathrm{AGS}(\mathcal{N})$ to $B(M_2)$. Indeed, since $n(\mathcal{N}) = 2$, then $\overline{U} + \overline{V} = \overline{U} \cup \overline{V}$ and for arbitrary $\overline{U}$ and $\overline{V}$ equalities hold

$$(\overline{U} + \overline{V})^{\phi} = (R_1(\overline{U} + \overline{V}), R_2(\overline{U} + \overline{V}));$$

$$(\overline{U} + \overline{V})^{\phi} = \overline{U}^{\phi} + \overline{V}^{\phi} = (R_1(\overline{U}), R_2(\overline{U})) + (R_1(\overline{V}), R_2(\overline{V})) =$$
$$(R_1(\overline{U}) \cup R_1(\overline{V}), R_2(\overline{U}) \cup R_2(\overline{V})).$$

Thus, the equalities are true

$$R_1(\overline{U} + \overline{V}) = R_1(\overline{U}) \cup R_1(\overline{V}), \quad R_2(\overline{U} + \overline{V}) = R_2(\overline{U}) \cup R_2(\overline{V}).$$

Next, we need a description of all endomorphisms from $\mathrm{AGS}(\mathcal{N})$ to $B(M_1)$ and $B(M_2)$.

2. *Let us show that all homomorphisms from* $\mathrm{AGS}(\mathcal{N})$ *to* $B(M_1)$ *are exhausted by the homomorphisms* α_{τ_1,τ_2}.

Let φ be an arbitrary homomorphism from $\mathrm{AGS}(\mathcal{N})$ to $B(M_1)$. The monoid $\mathrm{AGS}(\mathcal{N})$ will be generated by the set $T_l \cup T_r \cup \{(\varnothing, \varnothing)\}$, where

$$T_l := \{(\varnothing, \{x\}) \mid x \in M_2\}, \quad T_r := \{(\{x\}, \varnothing) \mid x \in M_1\}.$$

Let $\overline{U}_x := (\varnothing, \{x\}) \in T_l$ and $\overline{V}_y := (\{y\}, \varnothing) \in T_r$. Then the equalities hold

$$\varphi(\overline{U}_x) = H_x \in B(M_1), \quad \varphi(\overline{V}_y) = B_y \in B(M_1), \quad \varphi((\varnothing, \varnothing)) = \varnothing.$$

We define two families of sets

$$\mathcal{L} = \{L_x\}_{x \in M_2} \subseteq B(M_1), \quad \mathcal{D} = \{D_x\}_{x \in M_1} \subseteq B(M_1)$$

such that $L_x = H_x$ and $D_y = B_y$. Next, consider the endomorphism $\tau_1 = \phi_{\mathcal{D}}$ and the homomorphism $\tau_2 = \phi_{\mathcal{L}}$. It is easy to show that $\alpha_{\tau_1,\tau_2}((\varnothing, \varnothing)) = \varnothing$. Mapping α_{τ_1,τ_2} will satisfy the equalities

$$\varphi(\overline{U}_x) = \alpha_{\tau_1,\tau_2}(\overline{U}_x), \quad \varphi(\overline{V}_y) = \alpha_{\tau_1,\tau_2}(\overline{V}_y), \quad \varphi((\varnothing, \varnothing)) = \alpha_{\tau_1,\tau_2}((\varnothing, \varnothing))$$

for all $x \in M_2$, $y \in M_1$.

Further, let $\overline{U} = (U, V)$ be an arbitrary element from $B(M_1) \times B(M_2)$. Consider the action φ on this element (the tuples $\overline{U}_x$ and $\overline{V}_y$ are defined above)

$$\varphi(\overline{U}) = \varphi\left(\bigcup_{x \in U} \overline{U}_x \cup \bigcup_{y \in V} \overline{V}_y\right) = \bigcup_{x \in U} \varphi(\overline{U}_x) \cup \bigcup_{y \in V} \varphi(\overline{V}_y) =$$

$$\bigcup_{x\in U}\alpha_{\tau_1,\tau_2}(\overline{U}_x)\cup\bigcup_{y\in V}\alpha_{\tau_1,\tau_2}(\overline{V}_y)=\alpha_{\tau_1,\tau_2}\left(\bigcup_{x\in U}\overline{U}_x\cup\bigcup_{y\in V}\overline{V}_y\right)=\alpha_{\tau_1,\tau_2}(\overline{U}).$$

Thus, we have shown that on an arbitrary element $\overline{U}$ from AGS($\mathcal{N}$), the homomorphism φ acts as a homomorphism α_{τ_1,τ_2}.

Similarly, one can show that every homomorphism ϕ from AGS($\mathcal{N}$) to $B(M_2)$ acts as a homomorphism β_{ζ_1,ζ_2}.

3. Thus, the homomorphism R_1 is a suitable homomorphism α_{τ_1,τ_2}, and the homomorphism R_2 is a suitable homomorphism β_{ζ_1,ζ_2}. And an arbitrary endomorphism of the monoid AGS($\mathcal{N}$) is an endomorphism of ρ given by the rule (3.2). Theorem 1 is proved.

8. Conclusion

Thus, an arbitrary endomorphism of the monoid AGS($\mathcal{N}$) is parameterized by homomorphisms from AGS($\mathcal{N}$) $(n(\mathcal{N})=2)$ to $B(M_1)$, $B(M_2)$. These homomorphisms are parameterized by homomorphisms (in particular, endomorphisms) from $B(X)$ to $B(Y)$, when $X=M_1,M_2$ and $Y=M_1,M_2$. Proposition 1 and its Corollary 1 give an element-wise description of these homomorphisms and endomorphisms. And Theorem 2 gives more detailed information about the structure of the semiring End(X), for an arbitrary finite set X.

These results are of theoretical and practical interest. These results can be used to carry out calculations in the construction or theoretical study of multilayer neural networks of direct signal propagation.

References

1. Glushkov V.M. Abstract theory of automata. *UMN,* 1961. vol. 16, no. 5, pp. 3–62. (in Russian)
2. Golovko V.A., Krasnoproshin V.V. *Neural network technologies for data processing.* Minsk, Publ. house Belarus State University, 2017, 263 p. (in Russian)
3. Gorban' A.N. Generalized approximation theorem and computational capabilities of neural networks. *Sib. zhurn. calculated mathematics*, 1998, vol. 1, no. 1, pp. 11–24. (in Russian)
4. Litavrin A.V. Endomorphisms of finite commutative groupoids related with multilayer feedforward neural networks. *Trudy IMM UrO RAN*, 2021, vol. 27, no. 1, pp. 130–145. doi.org/10.21538/0134-4889-2021-27-1-130-145 (in Russian)
5. Litinskii L.B. On the Problem of Decomposition of a Neural Network into Several Subnets. *Mat. modeling,* 1996, vol. 8, no. 11, pp. 119–127. (in Russian)
6. Plotkin B.I., Greenglaz L.Ya., Gvaramia A.A. *Elements of the algebraic theory of automata.* Moscow, Higher School Publ., 1994, 192 p. (in Russian)
7. Slepovichev I.I. Algebraic properties of abstract neural networks. *Izvestiya Saratov Univ. New series. Series Mathematics. Mechanics. Informatics,* 2016, vol. 16, no. 1, pp. 96–103. doi.org/10.18500/1816-9791-2016-16-1-96-103(in Russian)

8. Kravtsova O.V. Elementary Abelian 2-subgroups in an Autotopism Group of a Semifield Projective Plane. *The Bulletin of Irkutsk State University. Series Mathematics,* 2020, vol. 32, pp. 49-63. doi.org/10.26516/1997-7670.2020.32.49
9. Litavrin A.V. Endomorphisms of some groupoids of order $k + k^2$. *The Bulletin of the Irkutsk State University. Series Mathematics*, 2020, vol. 32, pp. 64-78. https://doi.org/10.26516/1997-7670.2020.32.64
10. McCCulloh W., Pitts W. A logical calculus of the ideas immanent in nervous activity. *Bulletin Math. Biophysics*, 1943, no. 5, pp. 115–133.
11. Tsarkov O.I. Endomorphisms of the semigroup $G_2(r)$ over partially ordered commutative rings without zero divisors and with 1/2. *J. Math. Sc.,* 2014, vol. 201, no. 4, pp. 534–551.
12. Zhuchok Yu.V. Endomorphism semigroups of some free products. *J. Math. Sci.*, 2012. vol. 187, no. 2, pp. 146–152.

Список источников

1. Глушков В. М. Абстрактная теория автоматов // Успехи математических наук. 1961. Т. 16, № 5. С. 3–62
2. Головко В. А., Краснопрошин В. В. Нейросетевые технологии обработки данных : учеб. Пособие. Минск : Изд-во Беларус. гос. ун-та, 2017. 263 с.
3. Горбань А. Н. Обобщенная аппроксимационная теорема и вычислительные возможности нейронных сетей // Сибирский журнал вычислительной математики. 1998. Т. 1, № 1. С. 11–24.
4. Литаврин А. В. Эндоморфизмы конечных коммутативных группоидов, связанных с многослойными нейронными сетями прямого распределения // Труды ИММ УрО РАН. 2021. Т. 27, № 1, С. 130–145. https://doi.org/10.21538/0134-4889-2021-27-1-130-145
5. Литинский Л. Б. О задаче декомпозиции нейронной сети на несколько подсетей // Математическое моделирование. 1996. Т. 8, № 11. С. 119–127.
6. Плоткин Б. И., Гринглаз Л. Я., Гварамия А. А. Элементы алгебраической теории автоматов. М. : Высшая школа, 1994. 192 с.
7. Слеповичев И. И. Алгебраические свойства абстрактных нейронных сетей // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2016. Т. 16, № 1. С. 96–103. https://doi.org/10.18500/1816-9791-2016-16-1-96-103
8. Kravtsova O.V. Elementary Abelian 2-subgroups in an Autotopism Group of a Semifield Projective Plane // The Bulletin of the Irkutsk State University. Series Mathematics. 2020. Vol. 32. P. 49–63. doi.org/10.26516/1997-7670.2020.32.49
9. Litavrin A.V. Endomorphisms of some groupoids of order $k + k^2$ // TheBulletin of the Irkutsk State University. Series Mathematics. 2020. Vol. 32. P. 64-78. doi.org/10.26516/1997-7670.2020.32.64
10. McCulloh W., Pitts W. A logical calculus of the ideas immanent in nervous activity // Bulletin Math. Biophysics. 1943. N 5. P. 115–133.
11. Tsarkov O.I. Endomorphisms of the semigroup $G_2(r)$ over partially ordered commutative rings without zero divisors and with 1/2 // J. Math. Sci. 2014. Vol. 201, N 4. P. 534–551.
12. Zhuchok Yu.V. Endomorphism semigroups of some free products // J. Math. Sc. 2012. Vol. 187, N 2. P. 146–152.

Об авторах

Литаврин Андрей Викторович, канд. физ.-мат. наук, доц., Сибирский федеральный университет, Российская Федерация, 660041, г. Красноярск, anm11@rambler.ru, https://orcid.org/0000-0001-6285-0201

About the authors

Andrey V. Litavrin, Cand. Sci. (Phys.–Math.), Assoc. Prof., Siberian Federal University, Krasnoyarsk, 660041, Russian Federation, anm11@rambler.ru, https://orcid.org/0000-0001-6285-0201

Поступила в редакцию / Received 13.12.2021
Поступила после рецензирования / Revised 19.01.2022
Принята к публикации / Accepted 27.01.2022