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2-elements in an Autotopism Group of a Semifield Projective Plane

Olga V. Kravtsova[1]✉

[1] Siberian Federal University, Krasnoyarsk, Russian Federation
✉ ol71@bk.ru

Abstract. We investigate the well-known hypothesis of D.R. Hughes that the full collineation group of non-Desarguesian semifield projective plane of a finite order is solvable (the question 11.76 in Kourovka notebook was written down by N.D. Podufalov). The spread set method is used to construct the semifield projective planes with cyclic 2-subgroup of autotopisms in the case of linear space of any dimension over the field of prime order. This study completes the analogous considerations of elementary abelian 2-subgroups. We obtain the natural restriction to the order of 2-element for the semifield planes for odd and even order. It is proved that some projective linear groups can not be the autotopism subgroups for the infinite series of semifield planes. The matrix representation of Baer involution allows us to define the geometric property of autotopism of order 4. We can choose the base of a linear space such that the matrix representation of these autotopisms is convenient and uniform, it does not depend on the space dimension. The minimal counter-example is constructed to explain the restriction to the plane order. As a corollary, we proved the solvability of the full collineation group when the non-Desarguesian semifield plane has a certain even order and all its Baer subplanes are also non-Desarguesain. The main results can be used as technical for the further studies of the subgroups of even order in an autotopism group for a finite non-Desarguesian semifield plane. The results obtained are useful to investigate the semifield planes with the autotopism subgroups from J.G. Thompson's list of minimal simple groups.

Keywords: semifield plane, spread set, Baer involution, homology, autotopism

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Научная статья

2-элементы в группе автотопизмов полуполевой проективной плоскости

О. В. Кравцова[1✉]

[1] Сибирский федеральный университет, Красноярск, Российская Федерация
✉ ol71@bk.ru

Аннотация. Изучается известная гипотеза Д. Хьюза 1959 г. о разрешимости полной группы автоморфизмов недезарговой полуполевой проективной плоскости конечного порядка (также вопрос 11.76 Н. Д. Подуфалова в Коуровской тетради). Мы применяем метод регулярного множества над полем простого порядка к построению полуполевых проективных плоскостей с циклическими 2-подгруппами автотопизмов, дополняя аналогичные исследования элементарных абелевых 2-подгрупп. Естественное ограничение на порядок 2-элемента получено для полуполевых плоскостей как нечетного, так и четного порядка. Выделена бесконечная серия полуполевых плоскостей, не допускающих подгрупп автотопизмов, изоморфных определенным проективным линейным группам. На основе ранее найденного матричного представления бэровских инволюций уточнен геометрический смысл автотопизмов порядка 4, получено их унифицированное матричное представление, не зависящее от размерности пространства. Построен минимальный контрпример, поясняющий ограничение на порядок плоскости в основном результате. Доказана разрешимость полной группы коллинеаций недезарговой полуполевой плоскости четного порядка с ограничением на ранг, все бэровские подплоскости которой также недезарговы. Основные доказанные результаты являются техническими и необходимы для дальнейшего изучения подгрупп четного порядка в группе автотопизмов конечной недезарговой полуполевой плоскости. Результаты могут быть использованы для изучения полуполевых плоскостей, допускающих подгруппы автотопизмов из списка Д. Г. Томпсона минимальных простых групп.

Ключевые слова: полуполевая плоскость, регулярное множество, бэровская инволюция, гомология, автотопизм

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1. Introduction

The study of finite semifields and semifield planes started more than a century ago with the first examples constructed by L.E. Dickson [4]. A semifield (term from 1965) is called a non-associative ring $Q = (Q, +, \cdot)$ with identity where the equations $ax = b$ and $ya = b$ are uniquely solved for any $a, b \in Q$, $a \neq 0$. The abcense of an associative law in a semifield leads to a number of anomalous properties in comparison with a field or a skewfield or even a near-field. Moreover, the coordinatization of points and lines of a finite projective plane by the semifield elements provides special geometric properties.

By the mid-1950s, some classes of finite semifield planes had been found. All of them had the common property that the collineation group (automorphism group) is solvable. So D.R. Hughes conjectured in 1959 in his report that any finite projective plane coordinatized by a non-associative semifield has the solvable collineation group. This hypothesis is presented in the monography [5] (p. 178); it is proved also that the hypothesis is reduced to the solvability of an autotopism group as a group fixing a triangle. The Hughes' problem attracted the interest of a wide range of researchers who proved the collineation group solvability for an extensive list of semifield planes with certain restrictions. Nevertheless, the general approach to solving the problem has not been developed. In 1990 the problem was written down by N.D. Podufalov in the Kourovka notebook ([10], the question 11.76). Many later works implement the methods of a computer algebra where a solvability of an autotopism group is an additional result of constructing semifields and semifield planes of fixed orders.

We represent the approach to study Hughes' problem based on the classification of finite simple groups and theorem of J.G. Thompson on minimal simple groups. The spread set method allows us to identify the conditions when the semifield plane with certain autotopism subgroup exist. This method can be used also to construct examples, including computer calculations. The elimination of some groups from Thompson's list as autotopism subgroups allows us make progress in solving the problem.

It is shown by the author in [1; 7], that an autotopism of order two has the matrix representation convenient for calculations and reasoning. These marices are used further to represent the elementary abelian 2-subgroups of autotopisms [8]; it provides a natural connection of a plane order with 2-rank of autotopism group. As a corollary, we eliminate from possible autotopism subgroups of non-Desarguesian semifield plane of order p^N (p is prime, $N = 2^m \cdot s$, s is odd) the Suzuki group series $Sz(2^{2n+1})$ for $n > m$.

Here we use the spread set method to state the natural restriction to the order of 2-elements in an autotopism group. The main resulte is presented in the theorem 1. The proof is based on a concretization of a geometric sense of autotopisms of order 2 and 4, it uses also the matrix representation of autotopisms of order 4. The corollary specifies the groups $PSL(2,q)$ which cannot be the autotpism subgroups for a semifield plane of fixed order. The special case $p \equiv -1 \pmod 4$ requires another approach; it is demonstrated by examples of semifield planes of order 81.

The combination of results presented with results of [8] specifies one more class of semifield planes with solvable collineation group.

2. Main definitions and preliminary discussion

We use main definitions, according [5; 12], see also [8], for notifications.

Consider a linear space W, n-dimensional over the finite field $GF(p^s)$ (p be prime) and the subset of linear transformations $R \subset GL_n(p^s) \cup \{0\}$ such that:

1) R consists of p^{ns} square $(n \times n)$-matrices over $GF(p^s)$;
2) R contains the zero matrix 0 and the identity matrix E;
3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a nonsingular matrix.

The set R is called a *spread set* [5]. Consider a bijective mapping θ from W onto R and present the spread set as $R = \{\theta(y) \mid y \in W\}$. Determine the multiplication on W by the rule $x * y = x \cdot \theta(y)$ $(x, y \in W)$. Then $\langle W, +, * \rangle$ is a right quasifield of order p^{ns} [9; 12]. Moreover, if R is closed under addition then $\langle W, +, * \rangle$ is a semifield.

To construct and study finite semifields, we use a prime field $\mathbb{Z}_p$ as a basic field. In this case the mapping θ is presented using only linear functions; it greatly simplifies reasoning and calculations (also computer).

A semifield W coordinatizes the projective plane π of order $p^n = |W|$ such that:

1) the affine points are the elements (x, y) of the space $W \oplus W$;
2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in W\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in W\} \quad (m \in W);$$

3) the set of all cosets to the subgroup is the singular point;
4) the set of all singular points is the singular line;
5) the incidence is set-theoretical.

The solvability of a collineation group $Aut\,\pi$ for a semifield plane is reduced [5] to the solvability of an autotopism group Λ (collineations fixing a triangle). Without loss of generality, we can assume that autotopisms are determined by linear transformations of the space $W \oplus W$:

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

here the matrices A and D satisfy the condition (for instance, see [6])

$$A^{-1}\theta(m)D \in R \quad \forall\theta(m) \in R. \tag{2.1}$$

The collineations fixing a closed configuration have special properties. It is well-known [5], that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central*, or *perspectivity*, if it fixes a line pointwise (*the axis*) and a point linewise (*the center*). If the center is incident to the axis then a collineation is called *an elation*, and *a homology* in another case. The order of any elation is a factor of the order $|\pi|$ of a projective plane, and the order of any homology is a factor of $|\pi| - 1$. All the perspectivities in an autotopism group are homologies and form the cyclic subgroups [3]:

$$H_r = \left\{ \begin{pmatrix} E & 0 \\ 0 & M \end{pmatrix} \;\middle|\; M \in R_r^* \right\}, \quad H_m = \left\{ \begin{pmatrix} M & 0 \\ 0 & E \end{pmatrix} \;\middle|\; M \in R_m^* \right\},$$
$$H_l = \left\{ \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \;\middle|\; M \in R_l^* \right\}.$$

The matrix subsets R_l, R_m, R_r are defined by a spread set [3]:

$$\begin{aligned} R_l &= \{M \in GL_n(p) \cup \{0\} \mid MT = TM \ \forall T \in R\}, \\ R_m &= \{M \in R \mid MT \in R \ \forall T \in R\}, \\ R_r &= \{M \in R \mid TM \in R \ \forall T \in R\}, \end{aligned}$$

they are called *left, middle and right nuclei* of the plane π respectively. These subfields in $GL_n(p) \cup \{0\}$ are isomorphic to correspondent nuclei of the coordinatizing semifield W:

$$\begin{aligned} N_l &= \{n \in W \mid (n * a) * b = n * (a * b) \ \forall a, b \in W\}, \\ N_m &= \{n \in W \mid (a * n) * b = a * (n * b) \ \forall a, b \in W\}, \\ N_r &= \{n \in W \mid (a * b) * n = a * (b * n) \ \forall a, b \in W\}. \end{aligned} \tag{2.2}$$

The intersection $N_0 = N_r \cap N_m \cap N_l$ is the *nucleus* of the semifield W, the intersection $R_0 = R_r \cap R_m \cap R_l$ is the *nucleus* of the plane π. The nucleus order equals p^k, where $k|n$. The plane π is Desarguesian (classic) if W is a field, then $R \simeq W \simeq GF(p^n)$.

An autotopism group of a semifield plane of odd order contains three involutory homologies:

$$h_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, \quad h_2 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad h_3 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}. \tag{2.3}$$

A collineation of a projective plane π of order m is called *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|} = \sqrt{m}$ (*Baer subplane*). We use the following results on the matrix representation of a Baer involution $\tau \in \Lambda$ and of a spread set obtained earlier in [1; 7].

Let π be a non-Desarguesian semifield plane of order p^N (p be prime). If its autotopism group Λ contains the Baer involution τ then $N = 2n$ is even and we can choose the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \tag{2.4}$$

where the matrix $L \in GL_{2n}(p)$ is

$$L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \quad \text{if} \quad p > 2, \qquad L = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix} \quad \text{if} \quad p = 2. \tag{2.5}$$

We consider the Baer subplane π_τ fixed by τ as the set of points

$$\pi_\tau = \{(0, \dots, 0, x_1, \dots, x_n, 0, \dots, 0, y_1, \dots, y_n) \mid x_i, y_i \in \mathbb{Z}_p\}. \tag{2.6}$$

If $p > 2$ then the spread set R in $GL_{2n}(p) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}, \tag{2.7}$$

where $V \in Q$, $U \in K$, Q, K are the spread sets in $GL_n(p) \cup \{0\}$, K is the spread set of the Baer subplane π_τ, m, f are additive injective functions from K and Q into $GL_n(p) \cup \{0\}$, $m(E) = E$.

If $p = 2$ then the spread set R in $GL_{2n}(2) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} U + V + m(V) + w(V) & f(V) + m(U) \\ V & U + w(V) \end{pmatrix}, \tag{2.8}$$

where $U, V \in K$, K is the spread set of the Baer subplane π_τ in $GL_n(2) \cup \{0\}$. The additive functions m, f, w maps K into the ring of $(n \times n)$-matrices over $\mathbb{Z}_2$, $m(E) = 0$, the function f is injective, the lower row of the matrix $w(V)$ consists of zeros for all $V \in K$. Note that throughout the article, the blocks-submatrices have the same dimension by default.

3. Restriction to the order of 2-element

It is shown by author in [8], that the order of a semifield plane provides a natural restriction to the order of an elementary abelian 2-subgroup in an autotopism group. If a semifield plane π has an order p^N, where p is prime, $N = 2^m \cdot s$, and s is odd, then the order of an elementary abelian 2-subgroup in Λ is at most 2^m (for $p = 2$ with additional condition). We shall prove the analogous restriction to the order of 2-element in an autotopism group. Start from preliminary results.

Lemma 1. *Let π be a semifield plane of order p^N, p be prime, $p \not\equiv -1 \pmod 4$, α is an autotopism of order 4, $\tau = \alpha^2$ is the Baer involution. Then the restriction of α to the Baer subplane π_τ is a Baer involution of π_τ.*

Proof. We shall consider two cases: $p \equiv 1 \pmod 4$ and $p = 2$. Choose the base of the linear space such that the Baer involution is (2.4) and the spread set matrices are (2.7) or (2.8) respectively.

Let $p \equiv 1 \pmod 4$. Suppose that the autotopism α is an identity on the Baer subplane π_τ (2.6). Then from $\alpha\tau = \tau\alpha$ we have

$$\alpha = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & E \end{pmatrix}, \qquad A^2 = -E.$$

Since α is a collineation, then the condition (2.1) holds for $\theta(V, 0)$, so

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & f(V) \\ V & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & A^{-1}f(V) \\ VA & 0 \end{pmatrix} = \theta(VA, 0), \quad \forall V \in Q,$$

and $A \in Q_r$. Note that Q_r is a right nucleus of the semifield plane with the spread set Q. The cyclic group Q_r^* contains all scalar matrices zE, $z \in \mathbb{Z}_p^*$, and from condition $p \equiv 1 \pmod 4$ we have $A = iE$, $i^2 = -1$, $i \in \mathbb{Z}_p^*$. Then $f(iV) = -if(V)$, it contradicts $f(iV) = if(V)$ (from linearity of the function). Hence, α can not be identity on the subplane π_τ. If we suppose that the restricition of α to π_τ is a homology then we can use the conclusion proved to the product of α to homology h_1, h_2 or h_3 (2.3).

Let $p = 2$. From the condition $\alpha\tau = \tau\alpha$ we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & B_1 \end{pmatrix}, \qquad A_1^2 = B_1^2 = E.$$

If α is an identity on the Baer subplane π_τ that $A_1 = B_1 = E$ and α has an order 2. This contradiction proves the lemma. □

Lemma 2. *Let π be a semifield plane of order p^N, p be prime, $p \not\equiv -1 \pmod 4$, α is an autotopism of order 2^n, $n \geq 2$, $\tau = \alpha^{2^{n-1}}$ is the Baer involution. Then the restriction of α to the Baer subplane π_τ is an autotopism of π_τ of order 2^{n-1}.*

Proof. Consider the common proof for the cases $p \equiv 1 \pmod 4$ and $p = 2$. From the condition $\alpha\tau = \tau\alpha$ we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_3 & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & B_3 \end{pmatrix}, \qquad A_3^{2^{n-1}} = B_3^{2^{n-1}} = E$$

(we skip the conditions for other blocks). The restriction of α to the Baer subplane π_τ is defined by the matrix

$$\beta = \begin{pmatrix} A_3 & 0 \\ 0 & B_3 \end{pmatrix}.$$

If $|\beta| < 2^{n-1}$ then $A_3^{2^{n-2}} = B_3^{2^{n-2}} = E$, so the autotopism $\alpha^{2^{n-2}}$ of order 4 is an identity of π_τ; it contradicts to lemma 1. □

Theorem 1. *Let π be a semifield plane of order p^N, p be prime, $p \not\equiv -1 \pmod 4$. If α is an autotopism of order 2^n and the group $\langle\alpha\rangle$ contains no homologies then 2^n is a factor of N.*

Proof. Prove by induction. If $n = 1$ then an autotopism α is a Baer involution, so the plane π order is a square, N is even.

Let the statement is proved for $n-1$ $(n > 1)$, and let α be an autotopism of order 2^n. Then $\tau = \alpha^{2^{n-1}}$ is a Baer involution fixing pointwise a Baer subplane of order $p^{N/2}$. By the lemma 2, the restriction of α to π_τ is an autotopism of order 2^{n-1}. By inductive hypothesis, 2^{n-1} is a factor of $N/2$, so 2^n is a factor of N. The theorem is proved. □

Remark 1. The condition ≪the group $\langle\alpha\rangle$ contains no homologies≫ of the theorem 1 and the condition ≪τ is a Baer involution≫ of the lemmas 1 and 2, of course, are redundant if $p = 2$. In this case, an autotopism group of a semifield plane of even order contains no involutory homologies and elations, and any involutory autotopism is necessary Baer. Such the conditions are written down for the uniformity of statements.

Remark 2. The condition $p \not\equiv -1 \pmod 4$ in the statements is essential. The explanatory examples of the semifield planes of order 81 ($p = 3$) will be given later.

The automorphism group of a finite semifield is isomorphic to an autotopism subgroup of associated semifield plane [6], therefore the corollary is evident.

Corollary 1. *Let W be a semifield of order p^N, $p \not\equiv -1 \pmod 4$, p be prime. If its automorphism group contains an element of order 2^n then 2^n is a factor of N.*

Corollary 2. *Let π be a semifield plane of order 4^n where $n > 1$ is odd. Then the Sylow 2-subgroup in its autotopism group is elementary abelian. Moreover, if n is prime and all Baer subplanes of π are non-Desarguesian then the collineation group $Aut\,\pi$ is solvable.*

Proof. The first part is evident. To prove the second part we remind the results of [8]: if two commuting Baer involutions fixes different Baer subplanes then the plane order is 2^N, $N \equiv 0 \pmod 4$ (lemma 2). It contradicts the condition. Hence, any two different Baer involutions from a Sylow 2-subgroup fix the same Baer subplane of order 2^n. This subplane is non-Desarguesian by condition, so its nucleus is of order $4 \leq 2^k < 2^n$ (lemma 3), $k|n$ by definition of the nucleus. This contradiction shows that the Sylow 2-subgroup of an autotopism group Λ is of order 2, so Λ is solvable and $Aut\,\pi$ is solvable. $\square$

Use the theorem 1 to study the conditions when a group from Thompson's list is an autotopism subgroup. The following corollary clarifies Moorhouse's theorem [11, theorem 1.1]: if a projective palne Π of order $n < q$ admits a collineation group $G \simeq PSL(2, q)$ then Π is Desarguesian. It follows from this theorem that an autotopism group of non-Desarguesian semifield plane of order p^N cannot contain a subgroup $G \simeq PSL(2, q)$ for $q > p^N$. The theorem 1 adds the restriction to the order q of a basic field.

Corollary 3. *Let π be non-Desarguesian semifield plane of order p^N where $p = 2$ or $p \equiv 1 \pmod 4$, $N = 2^m \cdot s$ and s is odd. The autotopism group Λ of the plane π contains no subgroups isomorphic to $PSL(2, q)$ where 2^{m+2} is a factor of $q - 1$.*

Proof. It is enaugh to consider the order of the diagonal (cyclic) subgroup of $SL(2, q)$. $\square$

4. Matrix representation of order 4 autotopisms

The lemma 1 allows us to obtain unified matrix representation for the autotopism of order 4 which square is a Baer involution. Such the matrix representation is useful for the further study of semifield planes with certain autotopism subgroup and for examples constructing.

Theorem 2. *Let π be a semifield plane of order p^N, p be prime, $p \equiv 1 \pmod 4$, α is an autotopism of order 4, $\tau = \alpha^2$ is a Baer involution. Then N is divisible by 4, and the base of the linear space can be chosen such that τ is (2.4) and*

$$\alpha = \begin{pmatrix} iL & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & iL & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \qquad i^2 = -1,\ i \in \mathbb{Z}_p.$$

Proof. By the lemma 1, the restriction of α to the Baer subplane π_τ is a Baer involution. So, $|\pi_\tau|$ is a square, N is divisible by 4, and we can choose the base such that

$$\alpha = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \qquad A^2 = -E.$$

By the condition (2.1), we have

$$\begin{pmatrix} A^{-1}m(U)A & A^{-1}f(V)L \\ LVA & LUL \end{pmatrix} = \theta(LVA, LUL) \in R \qquad \forall V \in Q\ , U \in K,$$

therefore

$$LVA \in Q, \quad LUL \in K \qquad \forall V \in Q,\ U \in K.$$

The minimal polynomial of the matrix A is a factor of $\lambda^2 + 1$, so the matrix is scalar, $A = \pm iE$, or A is diagonal with diagonal elements i and $-i$. If $V = E$ we have $LA \in Q$; consider the matrix $\pm iE + LA \in Q$. If A is not $\pm iL$ then we have non-zero singular matrix in the spread set Q, it is impossible. Therefore, we can assume that $A = iL$. □

Remark 3. Carefully consider all the conditions from (2.1), we can determine the matrices $\theta(V, U)$ of the spread set if the plane admits an autotopism of order 4. Note that each of quarter-blocks $m(U)$, $f(V)$, V, U is also divided into four parts and is analogous to (2.7):

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} m_1(U_2) & m_2(V_2) & f_1(V_1) & f_2(U_1) \\ m_3(V_2) & m_4(U_2) & f_3(U_1) & f_4(V_1) \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix},$$

$V_1 \in Q_1$, $U_1 \in K_1$, $V_2 \in Q_2$, $U_2 \in K_2$. Here any block-submatrix is $(N/4 \times N/4)$-dimensional, the matrix sets Q_1, K_1, Q_2, K_2 are the spread sets of semifield planes of order $p^{N/4}$.

Theorem 3. *Let π be a semifield plane of order 2^N, α is an autotopism of order 4, $\tau = \alpha^2$ is a Baer involution. Then N is divisible by 4, and the base of the linear space can be chosen such that τ is (2.4) and*

$$\alpha = \begin{pmatrix} L & J & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L & J \\ 0 & 0 & 0 & L \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}.$$

Proof. By the lemma 1, the restriction of α to the Baer subplane π_τ is a Baer involution. So, $|\pi_\tau|$ is a square, N is divisible by 4, and we can choose the base such that

$$\alpha = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & B_1 \end{pmatrix}, \qquad \begin{array}{l} A_1^2 = B_1^2 = E, \\ A_1A_2 + A_2A_1 = E, \\ B_1B_2 + B_2B_1 = E. \end{array}$$

The restriction of α to the Baer subplane π_τ is determined by the matrix $\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$, the choice of the base allows us to assume that $A_1 = B_1 = L$. Since α is a collineation, the condition (2.1) gives us

$$\begin{pmatrix} L & LA_2L \\ 0 & L \end{pmatrix} \begin{pmatrix} L & B_2 \\ 0 & L \end{pmatrix} = \begin{pmatrix} E & LB_2 + LA_2 \\ 0 & E \end{pmatrix} \in R \Rightarrow B_2 = A_2.$$

Using the conditions $A_1A_2 + A_2A_1 = E$ and $A_1 = L$, we determine the block A_2:

$$A_2 = \begin{pmatrix} A_{21} & A_{22} \\ E & A_{21} \end{pmatrix}.$$

Change now the base of $2N$-dimensional linear space to preserve the matrix τ and to simplify the matrix α (so, A_2). We use the transition matrix

$$T = \begin{pmatrix} E & T_2 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & T_2 \\ 0 & 0 & 0 & E \end{pmatrix}, \qquad \text{where } T_2 = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}.$$

Calculate $T\alpha T^{-1}$, we have (for one block):

$$\begin{pmatrix} E & T_2 \\ 0 & E \end{pmatrix} \begin{pmatrix} L & A_2 \\ 0 & L \end{pmatrix} \begin{pmatrix} E & T_2 \\ 0 & E \end{pmatrix} = \begin{pmatrix} L & LT_2 + A_2 + T_2L \\ 0 & L \end{pmatrix} = \begin{pmatrix} L & J \\ 0 & L \end{pmatrix}.$$

The theorem is proved. □

We will not record the matrix representation of the spread set in the case of semifield plane of even order admitting the order 4 autotopism because of the very complicated form and calculations. As in the remark 3, we note that the quarter-blocks of the matrix $\theta(V, U)$ are partially similar to (2.8).

5. Examples

Consider the special case $p \equiv -1 \pmod 4$ which is eliminated from theorem 1 and from supporting lemmas. Note, that this basic field characteristic was considered as special in other investigations. For instance, G.E. Moorhouse [11, lemma 2.5] proved that for any projective plane Π of order n^2, where $n \equiv 3 \pmod 4$, and for cyclic collineation group G of order 4 the involution in G is necessarily a perspectivity.

Another feature in the case of a ≪bad≫ characteristic was noted by the author studying of 3-primitive semifield planes: the order of 2-element in an autotopism group is not restricted by the factor 2^m of the rank N.

Example 1. There are exactly eight, up to isomorphism, semifield planes of order $81 = 3^4$, admitting a Baer involution (more detail, see [2]). For each of these the autotopism group Λ is of order 2^m $(m = 8, \dots, 11)$, it is solvable and contains four or 100 (in one case) Baer involutions. We suggest paying attention to the last two columns of the following table.

Plane	$\lvert N_l\rvert$, $\lvert N_m\rvert$, $\lvert N_r\rvert$	$\lvert\Lambda\rvert$	n_2	B_2	n_4	B_4	n_8	n_{16}
A1	3,3,9	256	7	4	88	4	32	128
A2	3,3,9	512	7	4	216	4	160	128
B1	3,9,3	256	7	4	88	4	32	128
B2	3,9,3	512	7	4	216	4	160	128
C1	9,3,3	256	7	4	88	4	32	128
C2	9,3,3	512	7	4	216	4	160	128
D1	9,9,9	1024	7	4	56	8	192	768
D2	9,9,9	2048	103	100	600	8	576	768

Here n_k is the number of order k autotopisms; B_k is a number of order k Baer collineations, and $B_k = 0$ for $k > 4$, $n_k = 0$ for $k > 16$. The last columns shows that semifield planes of order $81 = 3^4$ admit order 8 and 16 autotopisms (more than 4).

Consider now the lemma 1 and suggest the example of order 4 autotopism which is trivial on a Baer subplane in the case $p \equiv -1 \pmod 4$.

Example 2. In notification of the lemma 1, we will construct a semifield plane of order 81 admitting the order 4 autotopism α, that is trivial on the Baer subplane π_τ, and the order 4 autotopism γ that is a Baer involution of π_τ. Here $\tau = \alpha^2 = \gamma^2$ is the Baer involution. We cannot record α or γ in a Jordan normal form because the polynomial $\lambda^2 + 1$ is irreducible over the field $\mathbb{Z}_p$. Let us assume that

$$\alpha = \begin{pmatrix} S & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & E \end{pmatrix}, \quad \gamma = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$S^2 = P^2 = -E$, $P \neq S$. The condition (2.1) for α and γ determines the spread set (2.7):

$$Q = K = \{-xS + yE \mid x, y \in \mathbb{Z}_p\},$$

$$m(-xS + yE) = xM + yE, \qquad f(-xS + yE) = xF - ySF,$$

$$M = \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & -p_1 \end{pmatrix}, \quad p_1^2 + p_2^2 = -1.$$

These results are obtained by simple calculations, and so we do not demonstrate them. Let the basic field be the minimal possible case $\mathbb{Z}_3$. We must choose the coefficients m_1, f_1, f_2 such that all non-zero matrices from the spread set be nonsingular. Computer calculations leads to $M = S$ and $F = \pm E$. Hence, the spread set of matrices

$$\begin{pmatrix} xS + yE & \pm(zE - tS) \\ -zS + tE & -xS + yE \end{pmatrix}, \qquad x, y, z, t \in \mathbb{Z}_3,$$

gives the example which demonstrate the necessity of condition for the field characteristic in all results of this work.

6. Conclusion

In order to study Hughes' problem on the solvability of the full collineation group of a finite non-Desarguesian semifield plane, the author considers it possible to use the obtained technical results to further investigations. These results and the method applied will probably be useful to determine the non-abelian simple groups series which cannot be the autotopism subgroups for finite semifield planes.

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Об авторах

Кравцова Ольга Вадимовна, канд. физ.-мат. наук, доц., Сибирский федеральный университет, Российская Федерация, 660041, г. Красноярск, ol71@bk.ru, https://orcid.org/0000-0002-6005-2393

About the authors

Olga V. Kravtsova, Cand. Sci. (Phys.–Math.), Assoc. Prof., Siberian Federal University, Krasnoyarsk, 660041, Russian Federation, ol71@bk.ru, https://orcid.org/0000-0002-6005-2393

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