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Investigation of the Amplitude-Frequency Response of a Dam-Type Viscoelastic Body With Steady-State Forced Harmonic Vibrations

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Abstract. The article discusses the amplitude-frequency response of a viscoelastic body of the dam type, under steady-state forced harmonic vibrations. An important factor is the determination of the number of frequencies and resonance peaks that arise in the process of harmonic effect of the water body on the dam. The use of the finite element method (FEM) for the numerical solution of dynamic problems allows, by expanding the solution in terms of eigenmodes and frequencies, to reduce the original problem to a system of separated linear integro-differential equations with respect to the sought parameters of generalized functions. The process of the influence of viscoelastic properties of the dam material on the resonance curves that arise under harmonic loads of different frequencies at different dimensions of the foot of the dam is investigated. The analysis of the curves of the amplitude-frequency responses of a dam-type viscoelastic body, under steady-state forced harmonic oscillations, showed that the occurrence of resonance peaks depends on the viscoelastic properties of the dam body and the dimensions of the foot of the dam. The main resonance peaks occur at frequencies less than the sixth eigenfrequency, as a result of which a further increase in the number of eigenmodes in the expansion does not introduce any significant changes in the amplitude of the distribution of resonance curves of the amplitude-frequency response of the dam.

Keywords: dam, viscoelasticity, harmonic vibrations, amplitude, resonance peaks, FEM

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Научная статья

Исследование амплитудно-частотных характеристик вязкоупругого тела типа плотины при установившихся вынужденных гармонических колебаниях

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Аннотация. Рассматриваются амплитудно-частотные характеристики вязкоупругого тела типа плотины при установившихся вынужденных гармонических колебаниях. Важным фактором является определение количества частот и резонансных пиков, возникающих в процессе гармонических воздействий водного массива на плотину. Использование метода конечных элементов для численного решения динамических задач позволяет посредством разложения решения по собственным формам и частотам свести исходную задачу к системе разделенных линейных интегро-дифференциальных уравнений, относительно искомых параметров обобщенных функций. Исследуется процесс влияния вязкоупругих свойств материала плотины на резонансные кривые, возникающих при воздействии гармонических нагрузок различной частоты при различных размерах подошвы плотины. Проведенный анализ кривых амплитудно-частотных характеристик вязкоупругого тела типа плотины при установившихся вынужденных гармонических колебаниях показал, что возникновение резонансных пиков зависят от вязкоупругих свойств тела плотины и размеров подошвы плотины. Основные резонансные пики происходят при частотах меньших шестой собственной частоты, вследствие этого дальнейшее увеличение количества собственных форм в разложении не вносит каких-либо существенных изменений в амплитуду распределения резонансных кривых амплитудно-частотных характеристик плотины.

Ключевые слова: плотина, вязкоупругость, гармонические колебания, амплитуда, резонансные пики, МКЭм

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1. Introduction

When calculating the dynamic effect on the dam, it is necessary to determine the influence of the viscoelastic properties of the dam material on the amplitude-frequency response. The use of the finite element method for the numerical solution of dynamic problems allows one to reduce the original problem to a system of separated linear integro-differential equations by expanding the solution body on the dam.

In [9] presents a comprehensive review on different theoretical elastic and viscoelastic foundation models in oscillatory systems. The review covers the simplest foundation models to the most complicated one and fully tracks the recent theories on the topic of mechanical foundations. It is fully discussed why each theory has been developed, what limitations each one contains, and which approaches have been applied to remove these limitations. Moreover, corresponding theories about structures supported by such foundations are briefly reviewed. Subsequently, an introduction to popular solution methods is presented. Finally, several important practical applications related to the linear and nonlinear foundations are reviewed. This paper provides a detailed theoretical background and also physical understanding from different types of foundations with applications in structural mechanics, nanosystems, bio-devices, composite structures, and aerospace-based mechanical systems.

In [5] a fractional viscoelastic model is presented for analyzing the vibrations of a completely free-supported plate, excited by supports movement. To control the steady-state condition of the plate vibration, the lower limit of the integral in the fractional Riemann-Liouville derivative is taken equal to minus infinity. The Voigt viscoelastic model is used to describe the damping effect. The governing equation for the viscoelastic plate is derived from the Kirchhoff hypothesis. An analytical solution is proposed for the dynamic response of a fractional viscoelastic plate excited by supports. The effect of the order of fractional derivative and the damping coefficient on the amplitude-frequency response of the plate is investigated.

The dampers consisting of clusters of viscoelastic particles, subjected to vibrations are considered in [2]; the amplitude of vibrations is such that the slip between the particles is negligible. Energy dissipation occurs mainly due to viscoelastic processes inside each particle and is maximized when a standing wave is generated in the granular medium. In that article, the medium is presented as an equivalent viscoelastic rigid body, and the models built using standard finite element software are used to predict their performance. Two numerical approaches were considered: one uses a Forward Frequency Response and the other uses standard modal analysis combined with analytic expressions for energy dissipation based on a wave equation. The performance of these prediction methods is compared with the behavior calculated in experiments with box-type structures and hollow-core composite pipe assemblies. For the considered systems, it was shown that the methods suit for acceleration amplitudes almost up to the amplitude of gravity.

In [8] dynamic model of a sandwich beam with viscoelastic coating based on a fractional constitutive equation was developed. The fractional model is a generalized model of sandwich beams that can be modified into integer models. The wave propagation method was extended to analyze vibrations of fractional sandwich beams. Further, the influence of fractional order and size parameters on the dynamic characteristics of the sandwich beam was investigated, and the steady-state frequency response of the sandwich beam was calculated. The results can provide a theoretical basis for calculating the size of sandwich beams with damping requirements.

An analysis of the dynamic behavior of constrained layer damping beams with a thick viscoelastic layer is given in [1]. A homogenized model for the bending stiffness is formulated using Reddy-Bickford's quadratic shear in each layer, and it is compared with Ross-Kerwin-Ungar classical model, which accounts for a uniform shear deformation of a viscoelastic core. In order to analyze the efficiency of both models, a numerical application is performed and the results obtained are compared with the results of a 2D model using finite elements, which accounts for extension and shear stresses and longitudinal, transverse, and rotational inertia. A fractional derivative model, with a frequency-dependent complex modulus of elasticity, characterizes the intermediate viscoelastic material. Eigenvalues and eigenvectors are obtained using an iterative method, which permits avoiding computational problems related to the frequency dependence of the stiffness matrices.

A new method for the analysis of the dynamic properties of structures with viscoelastic dampers was proposed in [4]. The method proposed for the first time enabled analyses that take into account the influence of temperature on the dynamic characteristics of systems with viscoelastic dampers. The frequency-temperature accordance principle was accepted (i.e., a thermo-rheologically simple material of dampers was assumed). The problem was reduced to an appropriately formulated nonlinear eigenvalue problem with parameters. The eigenvalue problem was solved using the homotopy method and the incremental-iterative procedure. The results of several examples were presented and discussed in detail.

In [7], laboratory and field tests, and a numerical simulation were conducted to assess the effect of a fragment in the rock mass on the ground vibration propagation. In order to predict vibrations in different sites under different blasting and geological condition, the authors investigated the differences in vibration behaviors due to the nature of blasting works, and the difference in geological conditions. The result of a series of tests showed that the fragments in the rock mass are related to the damping effect of soil caused by the vibration explosion. Accurate prediction of ground vibration can be performed taking into account the state of the fragment in the rock mass.

In [3] the method of the solution of free vibrations problems of viscoelastic elements of structures built from a linear viscoelastic material was developed for any hereditary kernels. Expressions of frequency and damping coefficient of viscoelastic vibrations were specified and approximately obtained for the first time by Ilyyushin A. A. and his employees by an averaging method. The method is based to an original approach to the calculation of poles of subintegral function in Mellin's formula for any hereditary kernel of a relaxation, not set analytically. It opens a way to the application of integrated Laplace transform in the solution of non-stationary dynamic problems of a viscoelastic material with real rheological properties.

2. Description of the method for solving the problem

A stationary dynamic problem for a dam-type body, which is acted upon by a uniformly distributed harmonic force

$$P = P_0 \sin(\omega_0 t),$$

where P_0 is the amplitude of the force, the direction of which coincides with the direction of the hydrostatic force acting on the dam in the form of water mass pressure; ω_0 are the oscillation frequencies set for the research [6]. Applying the procedure of the finite element method to a three-dimensional dynamic problem, we obtain a system of differential equations of the following form [10]:

$$M\ddot{U} + KU - R^*(U) = P, \tag{2.1}$$

where

M – is the matrix of inertia,

K – is the stiffness matrix,

U – is the vector of nodal displacements,

R^* – is the Volterra relaxation operator for viscoelastic bodies:

$$R^*(U) = \int_0^t R(t-\tau)U(\tau)d\tau.$$

Initial conditions are:

$$U\,|_{t=0} = U_0; \qquad \dot{U}\,|_{t=0} = V_0. \tag{2.2}$$

The solution to equation (2.1) is constructed in terms of its eigenmodes and eigenfrequencies in the following form:

$$U = SQ(t), \tag{2.3}$$

where

S – is the matrix of eigenmodes;

$Q(t) = (q_1, q_2, \dots, q_m)$ – is the vector of the sought generalized functions.

The eigenmodes and eigenfrequencies are determined from the matrix equation

$$(M - \Lambda K)\, Z = 0,$$

where $\Lambda = diag(v_1, v_2, ..., v_m)$.

To determine the largest eigenvalues and orthonormalized eigenvectors, the method of iterating by the Krylov subspace using the Rayleigh-Ritz procedure is used.

Substituting expression (2.3) into relations (2.1) and (2.2), and performing some transformations considering the orthogonality of the eigenmodes, we obtain a system of separated ordinary differential equations with respect to the sought q_i – parameters of generalized functions:

$$\ddot{q}_i(t) + \omega_i^2 \left(q_i - \int_0^t R(t-\tau)q_i(\tau)d\tau \right) = \omega_i^2 r_i, \tag{2.4}$$

$$q_i\,|_{t=0} = S_i^T U_0; \qquad \dot{q}_i\,|_{t=0} = S_i^T V_0, \tag{2.5}$$

where

$r_i = S_i^T P_0 \sin(\omega_0 t)$,

i – is the number of eigenfrequency,

S_i – is the natural mode of the dam, corresponding to the i – theigenfrequency,

$R(t)$ – is the Rzhanitsyn relaxation kernel, $R(t) = \bar{A}e^{-\beta\, t}t^{\alpha-1}$,

$\bar{A},\ \beta,\ \alpha$ – are the parameters of viscoelastic properties of the body, determined experimentally.

The solution to equation (2.4) is represented as:

$$q_i(t) = q_i^s \sin(\omega_0 t) + q_i^c \cos(\omega_0 t). \tag{2.6}$$

Substituting relation (2.6) into equation (2.4), and solving the system with respect to unknown coefficients, we have:

$$q_i^s = \frac{r_i \omega_i^2 \left[\omega_i^2 (1 - R_c) - \omega_0^2\right]}{\left[\omega_i^2 (1 - R_c) - \omega_0^2\right]^2 + \omega_i^4 R_s^2}, \tag{2.7}$$

$$q_i^c = \frac{-r_i^2 \omega_i^4 R_s}{\left[\omega_i^2 (1 - R_c) - \omega_0^2\right]^2 + \omega_i^4 R_s^2}, \tag{2.8}$$

where

$$R_c = \int_0^\infty R(\tau) \cdot \cos(\omega_0 \tau) d\tau = \bar{A} \cdot \frac{\Gamma(\alpha) \cdot \sin(\alpha\theta)}{\sqrt{\left(\beta^2 + \omega_0^2\right)^\alpha}},$$

$$R_s = \int_0^\infty R(\tau) \cdot \sin(\omega_0 \tau) d\tau = \bar{A} \cdot \frac{\Gamma(\alpha) \cdot \cos(\alpha\theta)}{\sqrt{\left(\beta^2 + \omega_0^2\right)^\alpha}},$$

$$\theta = \arcsin \frac{\omega_0}{\sqrt{\left(\beta^2 + \omega_0^2\right)}}.$$

Substituting expressions (2.7) and (2.8) into relation (2.6), and then into expression (2.3), we have:

$$U = A^s \sin(\omega_0 t) + A^c \cos(\omega_0 t), \tag{2.9}$$

where

$A^s = S^T Q^s, \quad A^c = S^T Q^c,$

Q^s and Q^c – are the vectors of amplitudes of generalized functions.

The construction of resonance curves for different values of the forced frequency ω_0 in different areas of the dam is considered.

The dependence of the resonance curves on the thickness of the foot of the dam, the viscosity of the material, and the number of eigenmodes in the expansion is investigated by (2.3).

Computational experiments have shown that:

- resonance peaks at different points of the dam occur under the same frequencies of the driving force and differ only in the degree of resonance;
- the main resonance peaks occur at frequencies less than the sixth eigenfrequency, as a result of which a further increase in the number of eigenmodes in the expansion does not introduce any significant changes in the amplitude.

Considering the above, this article presents the results of calculations of the dam body for the midpoint at the dam head, and the number of eigenmodes participating in the expansion is taken to be six.

3. Example of calculation

A dam considered as an example is determined by the coordinates of the vertices of $ABCDA'B'C'D'$ hexagon (Fig. 1(a) shows a general view of the dam). The coordinates of the hexagon vertices have the following values (in meters): $A\,(l_x/2;\ -20.0;\ 0.0)$, $B\,(l_x/2;\ 10.0;\ 20.0)$, $C\,(-l_x/2;\ 10.0;\ -20.0)$, $D\ (-l_x/2;\ 10.0;\ 20.0)$, $A'\ (5.0;\ -100.0;\ 80.0)$, $B'\ (5.0;\ 100.0;\ 80.0)$, $C'\ (-5.0;\ -100.0;\ 80.0)$, $D'\ (-5.0;\ 100.0;\ 80.0)$, where $l_x > 0$ will be defined as specific values, later.

Characteristics of the dam material are the density of a dam-type body $\rho = 0.2\ T/m^3$, modulus of elasticity $E = 4.0 \cdot 10^6\ T/m^2$ and Poisson's ratio $\nu = 0.16$. The computational experiment is conducted for the following values of the width of the dam base $l_x = 10, 20, 40\ m$. The amplitude of the given harmonic load on the crest of the dam is $P_0 = 1000\ T/m^2$. Rzhanitsyn kernel parameters are $\beta = 0.05$ and $\alpha = 0.1$. The viscosity parameter $\bar{A}$ takes values of 0.01, 0.05 and 0.1, which correspond to low, medium and high viscosity values. The dam is fixed along the sides and base.

The values of eigenfrequencies of the dam obtained in the research for various widths of the base are given in the table 1.

Table 1

Values of eigenfrequencies

$l_x \backslash \omega$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
10	27.030	63.632	83.963	114.41	118.02	118.05
20	35.238	72.974	104.27	124.52	125.38	128.35
40	52.883	92.502	131.22	131.32	135.44	149.54

Figs. 1(b)-(d) shows the amplitude-frequency response for various values of the base thickness l_x and the viscosity parameter $\bar{A} = 0.01$ (solid lines), 0.05 (dash-dotted lines) and 0.1 (dashed lines) and for dimensions of the dam base thickness: $l_x = 10\,m$ (Fig. 1(b)), $l_x = 20\,m$ (Fig. 1(c)) and $l_x = 40\,m$ (Fig. 1(d)). The horizontal axis shows the values of the frequencies of harmonic vibrations w_0, and the vertical axis shows the values of the maximum displacements (amplitudes).

Analysis of the calculation results showed that not all eigenfrequencies create peak resonances, which is explained by the orthogonality of the external load to the corresponding modes. At low viscosity (Fig. 1 (b)), additional resonance peaks appear on the resonance curve, which correspond to the working modes. With an increase in viscosity (Fig. 1 (c)), these resonance peaks significantly decrease and shift to the left, that is, they appear at a frequency lower than the eigenfrequency. At higher viscosity (Fig. 1 (d)), the resonance curves change their pattern dramatically. A single resonance peak remains on the curve, which corresponds to the first eigenfrequency. With an increase in the dimension of the dam foot thickness, the amplitude of the resonance peaks decreases, and their number changes, which is related to the redistribution of eigenmodes.

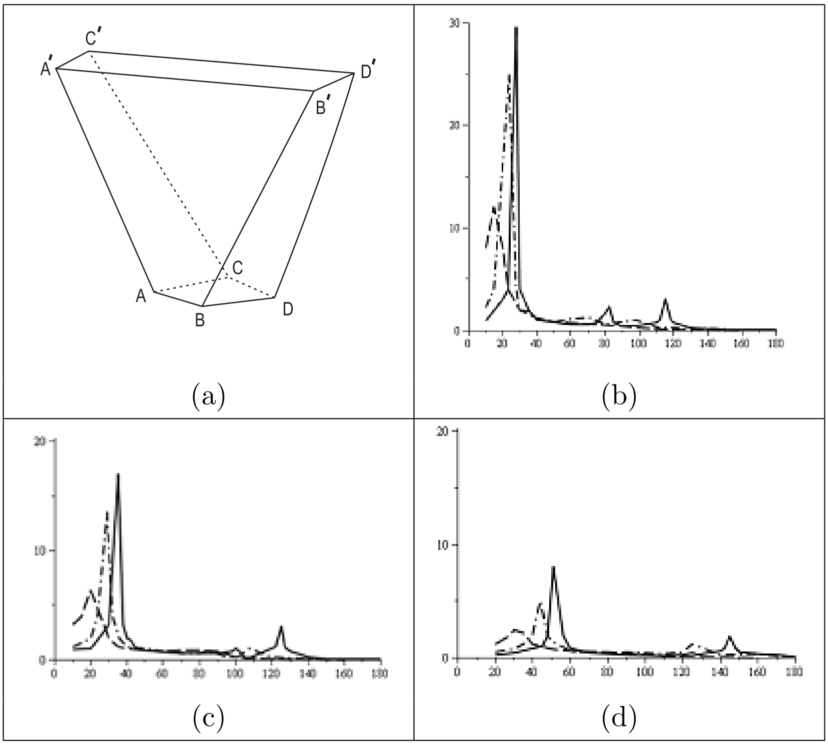

Figure 1. General view and curves of the amplitude-frequency response of the dam

4. Conclusions

The analysis of the curves of the amplitude-frequency responses of a dam-type viscoelastic body under steady-state forced harmonic oscillations, showed that the occurrence of resonance peaks depends on the viscoelastic properties of the dam body and the dimensions of the foot of the dam. The main resonance peaks occur at frequencies less than the sixth eigenfrequency, as a result of which a further increase in the number of eigenmodes in the expansion does not introduce any significant changes in the amplitude of the distribution of resonance curves of the amplitude-frequency response of the dam.

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