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Systems of Inclusions with Unknowns in Multioperations

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Abstract. We consider systems of inclusions with unknowns and coefficients in multioperations of finite rank. An algorithm for solving such systems by the method of reduction to Boolean equations using superposition representation of multioperations by Boolean space matrices is given. Two methods for solving Boolean equations with many unknowns are described for completeness. The presentation is demonstrated by examples: the representation of the superposition of multioperations by Boolean space matrices; solving a Boolean equation by analytical and numerical methods; and finding solutions to an inclusion with one unknown. The resulting algorithm can be applied to the development of logical inference systems for multioperator logics.

Keywords: multioperation, inclusion, spatial matrix, Boolean equation, term

1. Introduction

Note that the theory of multioperations has been developing intensively lately (see, for example, the [4–6]). Functional equations are considered in almost all sections of mathematics. This paper is devoted to the development of an algorithm for finding solutions of systems of inclusions with unknowns and coefficients in multilaterals. The algorithm is based on reduction to systems of Boolean equations by means of representation of superposition of multilocations by Boolean space matrices. Note that finding a direct algorithm without complete enumeration for this problem is highly questionable due to the absence of a superassociation identity for the superposition of multilayers. The results of this paper are announced in [7].

2. Multioperations and inclusion systems with unknowns

Let A be an arbitrary set, $B(A)$ be the set of all subsets of A, n — natural number. The mapping f of Cartesian degree A^n to $B(A)$ is called a n-local multiverse operation on A. If all images are single-element, then f is called an n-local operation, and if $A = \{0, 1\}$ is a Boolean operation.

The multioperations f on an arbitrary finite set $A = \{a_0, ..., a_{k-1}\}$ can be represented as mappings

$$f : \{2^0, 2^1, ..., 2^{k-1}\}^n \to \{0, 1, ..., 2^k - 1\},$$

that you get from f when you encode

$$a_i \to 2^i; \ \ \varnothing \to 0; \ \ \{a_{i_1}, ..., a_{i_s}\} \to 2^{i_1} + ... + 2^{i_s}.$$

We say that f is a multi-operation of dimension n, of rank k. Hereafter, without further reservations, we will consider only multioperations of finite rank $k \geq 2$.

We will use the notation $\mathcal{M}_k^{(n)}$ for the set of multioperations of dimension n, rank k. Define the vector form $f = (\alpha_1, ..., \alpha_{k^n})$ of multioperations f of dimension n, of rank k, so $\alpha_i \in \{0, 1, ..., 2^k - 1\}$ and $\alpha_i = f(2^{i_1}, ..., 2^{i_n})$, and $(i_1, ..., i_n)$ is the representation of $i - 1$ in the base k calculus by a n-digit number.

Let us define the superposition operator on the set of n-local multioperations as follows: if $f \in \mathcal{M}_k^{(n)}$ and $f_1, ..., f_n \in \mathcal{M}_k^{(m)}$, then

$$f(f_1, ..., f_n)(a_1, ..., a_m) = \bigcup_{b_i \in f_i(a_1, ..., a_m)} f(b_1, ..., b_n).$$

The notion of a term with a set of variables X and a set of constants K over a set of multioperations $F \subseteq \mathcal{M}_k$ is defined as follows:

- if $y \in K \cup X$, then $t \equiv y$ is a term and $U(t) = \{y\}$;
- if $f \in F^{(n)}$ and $t_1, ..., t_n$ are terms, then $t \equiv f(t_1, ..., t_n)$ is a terms and $U(t) = U(t_1) \cup ... \cup U(t_n)$;
- if t_0 is a term, $U(t) = \{y_1, ..., y_n\}$ and $t_1, ..., t_n$ are terms, then $t \equiv t_0(t_1, ..., t_n)$ is a terms and $U(t) = U(t_1) \cup ... \cup U(t_n)$;

The general form of the system of inclusions with unknowns in multioperations is as follows:

$$\begin{cases} t_1(\tilde{c}, \tilde{z}) \subseteq q_1(\tilde{c}, \tilde{z}) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ t_m(\tilde{c}, \tilde{z}) \subseteq q_m(\tilde{c}, \tilde{z}) \end{cases} \tag{1}$$

where t_i, q_i are–terms with variables $\tilde{z}$ and constants $\tilde{c}$ of $F \subseteq \mathcal{M}_k$.

The solution of system (1) is the set of multioperations $f_1, ..., f_s$ such that for all i the inclusions are satisfied under the standard definition of the value of the terms by means of the introduced superposition operator:

$$t_i(\tilde{c}, f_1(\tilde{c}), ..., f_s(\tilde{c})) \subseteq q_i(\tilde{c}, f_1(\tilde{c}), ..., f_s(\tilde{c}))$$

Before entering the statement of the algorithm for solving a system of inclusions with unknowns in multioperations, let us give the necessary information on the representation of the superposition of multioperations by space matrices and methods for solving Boolean equations.

3. Presentation by Boolean space matrices of multioperations and superposition of multioperations

The matrix representation of the composition of binary relations as the product of Boolean square matrices is well known (see, for example, [9]). Let us generalize this result to the superposition of multioperations introduced above.

Recall some concepts from the theory of spatial matrices, which can be found, for example, in [10].

A binary n-dimensional matrix of order k is a function $\alpha : N_k^n \to \{0, 1\}$, where $N_k = \{1, ..., k\}$. Notation: $M = [\alpha_{i_1 \dots i_n}]$, where $\alpha_{i_1 \dots i_n} = \alpha(i_1, ..., i_n)$.

Given a constant value i ($i \in N_k$) of the index s, we obtain $(n-1)$-dimensional matrix of order k which is called the i-section of matrix M by index s, denoted by M^{i_s}.

Multiplication of n-dimensional matrix of $k$order M by a vector V of length k by index s gives $(n-1)$-dimensional matrix of $k$order $(M *_s V) = [\beta_{i_1 \dots i_{s-1} i_{s+1} \dots i_n}]$, with $\beta_{i_1 \dots i_{s-1} i_{s+1} \dots i_n} = M^{i_1 \dots i_{s-1} i_{s+1} \dots i_n} * V$, where $*$ is the scalar product of vectors over the Boolean semiring B.

Let $B = \langle \{0, 1\}; +, \cdot \rangle$ — a two-element Boolean semicircle. For a multioperation f of dimension n, rank k on A, let us define a Boolean spatial $(n+1)$-dimensional matrix k of order $M_f = [\alpha_{i_0 i_1 \dots i_n}]$ where:

$$\alpha_{i_0 \dots i_n} = \begin{cases} 1, & \text{if } a_{i_0} \in f(a_{i_1}, ..., a_{i_n}); \\ 0, & \text{if } a_{i_0} \notin f(a_{i_1}, ..., a_{i_n}). \end{cases}$$

Theorem 1. [8] *Let* $f_0 \in \mathcal{M}_k^{(n)}$, $f_1, ..., f_n \in \mathcal{M}_k^{(m)}$, $M_{f_0(f_1, ..., f_n)} = [\beta_{i_0 i_1 \dots i_m}]$. *Then*

$$\beta_{i_0 i_1 \dots i_m} = (...((M_{f_0}^{i_0} *_n M_{f_n}^{i_1 \dots i_m}) *_{n-1} M_{f_{n-1}}^{i_1 \dots i_m})...) *_1 M_{f_1}^{i_1 \dots i_m}$$

Proof. The proof follows from the definition of the representation of multioperations by spatial matrices and the definition of the superposition operator. □

Here is an example of superposition for k=3, n=2, m=1.

Let $f_0 = (426537011)$, $f_1 = (145)$, $f_2 = (273)$.

Note that it is more convenient to write spatial matrices of multioperations not in the form of a vector, but in the form of a two-dimensional matrix, where the rows correspond to a zero index. Let us represent certain multioperations by spatial matrices of the following form:

$$M_{f_0} = \begin{bmatrix} 000\,111\,011 \\ 011\,011\,000 \\ 101\,101\,000 \end{bmatrix},\ M_{f_1} = \begin{bmatrix} 101 \\ 000 \\ 011 \end{bmatrix}\ M_{f_2} = \begin{bmatrix} 011 \\ 111 \\ 010 \end{bmatrix}$$

let us calculate the following product of matrices:

$$M_g = \left(\begin{bmatrix} 000\,111\,011 \\ 011\,011\,000 \\ 101\,101\,000 \end{bmatrix} * \begin{bmatrix} 101 \\ 000 \\ 011 \end{bmatrix}, \begin{bmatrix} 011 \\ 111 \\ 010 \end{bmatrix} \right).$$

Let us show the calculations by columns. Multiplication of a matrix by a column is performed as a threefold scalar multiplication of the corresponding row by that column.

$$\left(\left(\begin{bmatrix} 000\,111\,011 \\ 011\,011\,000 \\ 101\,101\,000 \end{bmatrix} *_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) *_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \left(\begin{bmatrix} 011 \\ 110 \\ 000 \end{bmatrix} *_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left(\left(\begin{bmatrix} 000\,111\,011 \\ 011\,011\,000 \\ 101\,101\,000 \end{bmatrix} *_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) *_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 011 \\ 110 \\ 110 \end{bmatrix} *_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\left(\begin{bmatrix} 000\,111\,011 \\ 011\,011\,000 \\ 101\,101\,000 \end{bmatrix} *_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) *_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 011 \\ 110 \\ 110 \end{bmatrix} *_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Combining all three columns into one matrix we get:

$$M_g = \begin{bmatrix} 011 \\ 101 \\ 001 \end{bmatrix}.$$

As a result, going to the vector record of multioperations, we get:
$g = f_0(f_1, f_2) = (217)$.

4. Systems of Boolean inequalities with many unknowns

Next we will need methods for solving systems of Boolean inequalities with many unknowns. As we know, any such system reduces to a single Boolean equation. For completeness, we present two methods for solving Boolean equations with many unknowns: analytical and numerical.

A general view of a system of Boolean inequalities:

$$\begin{cases} f_1(\tilde{a},\tilde{x}) \leq g_1(\tilde{a},\tilde{x}) \\ \qquad\qquad \dots \\ f_m(\tilde{a},\tilde{x}) \leq g_m(\tilde{a},\tilde{x}) \end{cases}$$

where $\tilde{a} = a_1, ..., a_n$ set of coefficients, $\tilde{x} = x_1, ..., x_s$ set of unknowns.

Transition from inequalities to equalities with zero in the right part

$$f(\tilde{a},\tilde{x}) \leq g(\tilde{a},\tilde{x}) \Longleftrightarrow f(\tilde{a},\tilde{x}) \cdot \bar{g}(\tilde{a},\tilde{x}) = 0$$

Transition from a system of equations to a single equation

$$\begin{cases} f(\tilde{a},\tilde{x}) = 0 \\ g(\tilde{a},\tilde{x}) = 0 \end{cases} \Longleftrightarrow f(\tilde{a},\tilde{x}) \ \vee g(\tilde{a},\tilde{x}) = 0$$

The canonical form of a Boolean equation with many unknowns is as follows:

$$f(\tilde{a},\tilde{x}) = 0$$

A solution (or partial solution) of a Boolean equation is a set of Boolean operations $g_1(\tilde{a}), ..., g_s(\tilde{a})$ such that the equality is satisfied:

$$f(\tilde{a}, g_1(\tilde{a}), ..., g_s(\tilde{a})) = 0$$

We write this fact as follows: $x_i = g_i(a_1, ..., a_n), i = 1, ..., s$.
A general solution is a solution represented by formulas with parameters such that any partial solution is obtained from this formula for some values of the parameters.

Let us introduce a minimum operator and its notation as follows:

$$m_{\tilde{x}} f(\tilde{x},\tilde{a}) = f_{\tilde{x}}^{\tilde{0}}(\tilde{a}) \cdot ... \cdot f_{\tilde{x}}^{\tilde{1}}(\tilde{a})$$

The analytical method for solving Boolean equations is defined by the following statement:

Theorem 2. [1; 2] *Let equation is given*

$$f(\tilde{a},\tilde{x}) = 0 \qquad (1)$$

a) Equation (1) is solvable for the set of coefficients $\tilde{a}$ *if and only if the condition for them is satisfied:*

$$m_{\tilde{x}} f(\tilde{a}) = 0 \qquad (2)$$

b) If equation (1) is solvable for the coefficients $\tilde{a}$, then the general solution is found by recurrence formulas:

$$g_s(\tilde{a}) = \alpha_s(\tilde{a}) \cdot m_{x_1 \dots x_{s-1}} f^0_{x_s}(\tilde{a}) \vee \bar{\alpha}_s(\tilde{a}) \cdot \overline{m_{x_1 \dots x_{s-1}} f^1_{x_s}}(\tilde{a})$$

if $g_{k+1}(\tilde{a}), ..., g_s(\tilde{a})$ — found, then

$$g_k(\tilde{a}) = \alpha_k(\tilde{a}) \cdot m_{x_1 \dots x_{k-1}} f^0_{x_k}(\tilde{a}, g_{k+1}(\tilde{a}), ..., g_s(\tilde{a})) \vee \\ \vee \bar{\alpha}_1(\tilde{a}) \cdot \overline{m_{x_1 \dots x_{k-1}} f^1_{x_k}}(\tilde{a}, g_{k+1}(\tilde{a}), ..., g_s(\tilde{a})) \quad (3)$$

where $\alpha_1(\tilde{a}), ..., \alpha_s(\tilde{a})$ arbitrary Boolean operations (parameters).

Here is a demonstration example of solving an equation using the analytical method. Let an equation be given in which the Boolean operation is specified thermally:

$$f(a, b, x, y) = xy \vee ab \vee \bar{x}\bar{y}\bar{a}\bar{b} = 0 \quad (4)$$

where a, b are coefficients, x, y are unknowns.

a) Let us check the solvability condition. By condition (2) we find the values of the coefficients at which there are solutions:

$$m_{xy} f(a, b) = f^{0,0}_{x,y} \cdot f^{0,1}_{x,y} \cdot f^{1,0}_{x,y} \cdot f^{1,1}_{x,y} = (ab \vee \bar{a}\bar{b})(ab)(ab)(1) = ab = 0 \quad (5)$$

This condition breaks down into two 1) $a = 0$ or 2) $b = 0$.

b) General solution. Using formulas (3), find a general solution, given the solvability condition $ab = 0$:

$$y = g_2(a, b) = \alpha_2(a, b) \cdot m_x f^0_y(a, b) \vee \bar{\alpha}_2(a, b) \cdot \overline{m_x f^1_y}(a, b) = \\ = \alpha_2(a, b)ab \vee \bar{\alpha}_2(a, b)\overline{ab} = \bar{\alpha}_2(a, b);$$

$$x = g_1(a, b) = \alpha_1(a, b) \cdot f^0_x(a, b, g_2(a, b)) \vee \bar{\alpha}_1(a, b) \cdot \overline{f^1_x}(a, b, g_2(a, b)) = \\ = \alpha_1(a, b)(ab \vee \alpha_2(a, b)\bar{a}\bar{b}) \vee \bar{\alpha}_1(a, b)\overline{(\bar{\alpha}_2(a, b) \vee ab)} = \\ = \alpha_1(a, b)\alpha_2(a, b)\bar{a}\bar{b} \vee \bar{\alpha}_1(a, b)\alpha_2(a, b) = \\ = \alpha_2(a, b)(\bar{a}\bar{b} \vee \bar{\alpha}_1(a, b)) \quad (6)$$

where $\alpha_1(a, b), \alpha_2(a, b)$ are arbitrary Boolean operations.

c) Substituting the general form of Boolean operations $\alpha_1(a, b)$ and $\alpha_2(a, b)$ into the general solution (6), we obtain the set of all solutions of equation (4). Let

$$\alpha_1(a, b) = \sigma_1\bar{a}\bar{b} \vee \sigma_2\bar{a}b \vee \sigma_3a\bar{b} \vee \sigma_4ab;$$

$$\alpha_2(a, b) = \tau_1\bar{a}\bar{b} \vee \tau_2\bar{a}b \vee \tau_3a\bar{b} \vee \tau_4ab.$$

As a result, equation (4) has the following set of solutions:
at $a = 0$ or $b = 0$

$$\begin{cases} x = 0 \\ y = 1 \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

at $a = 0$

$$\begin{cases} x = 0 \\ y = \bar{b} \end{cases} \quad \begin{cases} x = b \\ y = \bar{b} \end{cases} \quad \begin{cases} x = \bar{b} \\ y = 0 \end{cases} \quad \begin{cases} x = \bar{b} \\ y = b \end{cases} \tag{7}$$

at $b = 0$

$$\begin{cases} x = 0 \\ y = \bar{a} \end{cases} \quad \begin{cases} x = a \\ y = \bar{a} \end{cases} \quad \begin{cases} x = \bar{a} \\ y = 0 \end{cases} \quad \begin{cases} x = \bar{a} \\ y = a \end{cases}$$

Now let us give an algorithm for solving a Boolean equation using the numerical method, which is a modification of known methods (see, for example, [1]).

Theorem 3. *Let the equation be given in vector form*

$$f(\tilde{a}, \tilde{x}) = (\sigma_{11}, .., \sigma_{1k}, \sigma_{21}, .., \sigma_{2k}, ..., \sigma_{m1}, .., \sigma_{mk}) = 0 \tag{8}$$

where $k = 2^s$, $m = 2^n$.

a) Equation 8 is solvable for the set of coefficients $\tilde{a}$ *if and only if the condition for them is satisfied:*

$$m_{\tilde{x}} f(\tilde{a}) = (\gamma_1, ..., \gamma_m) = 0$$
$$\textit{where} \;\; \gamma_i = \begin{cases} 1, \; \textit{if} \; \sigma_{i1} = \sigma_{i2} = ... = \sigma_{ik} = 1; \\ 0, \; \textit{if not.} \end{cases} \tag{9}$$

b) If equation 8 is solvable for the coefficients $\tilde{a}$*, then the set of all solutions is found by the formulas:*

$$\begin{cases} x_1 = g_1(\tilde{a}) = (\delta_{11}, ..., \delta_{1m}) \\ \dots\dots\dots\dots\dots\dots \\ x_s = g_s(\tilde{a}) = (\delta_{s1}, ..., \delta_{sm}) \end{cases} \tag{10}$$

where $\{(\delta_{1i}, ..., \delta_{si})\}$ *are all binary sets satisfying the equality*

$$f(i_1, ..., i_n, \delta_{1i}, ..., \delta_{si}) = 0$$

where the number $i = (i_1, ..., i_n)_2 + 1$ *is 1 more than the number whose binary representation corresponds to the set* $(i_1, ..., i_n)$.

Here is an example of solving a Boolean equation numerically. Let the Boolean operation in equation (4) be given by a vector:

$$f(a, b, x, y) = (1001000100011111) = 0. \tag{11}$$

We get two solvability conditions 1) $a = 0$ or 2) $b = 0$.

b) In finding the set of all solutions, to simplify the calculations we will consider each of the conditions separately.

1) If $a = 0$, then $f(0, b, x, y) = (10010001) = 0$.
The set of solutions will be found by formulae (10).

$$\{(\delta_{11}, \delta_{21})\} = \{(01), (10)\}$$

$$\{(\delta_{12}, \delta_{22})\} = \{(00), (01), (10)\}$$

Choosing all possible combinations of the first and second sets we get six solutions:

$$\begin{cases} x = (00) \\ y = (10) \end{cases} \begin{cases} x = (00) \\ y = (11) \end{cases} \begin{cases} x = (01) \\ y = (10) \end{cases} \begin{cases} x = (10) \\ y = (00) \end{cases} \begin{cases} x = (10) \\ y = (01) \end{cases} \begin{cases} x = (11) \\ y = (00) \end{cases} .$$

This answer can be written in form:

$$\begin{cases} x = 0 \\ y = \bar{b} \end{cases} \quad \begin{cases} x = 0 \\ y = 1 \end{cases} \quad \begin{cases} x = b \\ y = \bar{b} \end{cases} \quad \begin{cases} x = \bar{b} \\ y = 0 \end{cases} \quad \begin{cases} x = \bar{b} \\ y = b \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases} .$$

2) If $b = 0$, then $f(a, 0, x, y) = (10010001) = 0$.

Because of the symmetry of the equation with respect to a and b, we can immediately write out all solutions, similar to the previous point:

$$\begin{cases} x = (00) \\ y = (10) \end{cases} \begin{cases} x = (00) \\ y = (11) \end{cases} \begin{cases} x = (01) \\ y = (10) \end{cases} \begin{cases} x = (10) \\ y = (00) \end{cases} \begin{cases} x = (10) \\ y = (01) \end{cases} \begin{cases} x = (11) \\ y = (00) \end{cases} .$$

As a result, equation (11) has the following set of solutions:
at $a = 0$ or $b = 0$ we obtain:

$$\begin{cases} x = (0) \\ y = (1) \end{cases} \quad \begin{cases} x = (1) \\ y = (0) \end{cases} ,$$

at $a = 0$ we get $x = g_1(b), y = g_2(b)$:

$$\begin{cases} x = (0) \\ y = (10) \end{cases} \quad \begin{cases} x = (01) \\ y = (10) \end{cases} \quad \begin{cases} x = (10) \\ y = (0) \end{cases} \quad \begin{cases} x = (10) \\ y = (01) \end{cases} ,$$

at $b = 0$ we get $x = g_1(a), y = g_2(a)$:

$$\begin{cases} x = (0) \\ y = (10) \end{cases} \quad \begin{cases} x = (01) \\ y = (10) \end{cases} \quad \begin{cases} x = (10) \\ y = (0) \end{cases} \quad \begin{cases} x = (10) \\ y = (01) \end{cases} .$$

If this answer is written in thermal form, it will coincide with (7) for equation (4), because equation (4) and (11) are the same equation given in different forms: thermal and vector.

5. An algorithm for solving a system of inclusions with coefficients and unknowns in multioperations

Below is an algorithm for solving a system of inclusions with unknowns in multioperations by reduction to a system of Boolean inequalities.

(A1) Input: a system of inclusions in multioperations.

(A2) Transition to Boolean space matrices.

(A3) Transition to a system of Boolean inequalities, with possible transition to a single Boolean equation.

(A4) Finding a solution to the system (A3).

(A5) Output: Solution of the original system (A1), which is determined by the solution obtained in (A4).

The best way to demonstrate how the algorithm works is to use an example. For clarity, let's take an example with small parameters.

(A1) Let $A = \{0, 1\}$.

Solve a system of one inclusion with one unknown and one coefficient

$$g(h(z,c), s(c)) \subseteq g(c,z) \tag{1}$$

where multioperations are defined as follows:

$h(a,b) = \{a,b\}, s(a) = \begin{cases} \{1\}, \text{ if } a = 0; \\ \{0\}, \text{ if } a = 1, \end{cases}$

$g(a,b) = \begin{cases} \{a\}, \text{ if } a = b; \\ \varnothing, \text{ if } a \neq b. \end{cases}$ for any $a, b \in A$.

(A2) Transition to Boolean space matrices. Let us first represent the multioperations in vector form:

$$h(a,b) = (1332), s(a) = (21), g(a,b) = (1002).$$

Let us represent them by Boolean space matrices:

$$M_h = \begin{bmatrix} 11\,10 \\ 01\,11 \end{bmatrix}, M_s = \begin{bmatrix} 01 \\ 10 \end{bmatrix}, M_g = \begin{bmatrix} 10\,00 \\ 00\,01 \end{bmatrix}.$$

Let us find the representations of the left and right parts of inequality (1) by Boolean space matrices:

$M_{h(z,c)} = \begin{bmatrix} ax + bx + ay \\ bx + ay + by \end{bmatrix}, M_{s(c)} = \begin{bmatrix} b \\ a \end{bmatrix},$

$M_{g(h(z,c),s(c))} = \begin{bmatrix} bx + aby \\ ay + abx \end{bmatrix}, M_{g(c,z)} = \begin{bmatrix} ax \\ by \end{bmatrix}.$

(A3) Change to a system of Boolean inequalities and then to a single Boolean equation.

We obtain a system of Boolean inequalities with 2 unknowns and 2 coefficients:

$$\begin{cases} bx \vee aby \le ax \\ ay \vee abx \le by \end{cases} \tag{2}$$

The Boolean equation equivalent to this system (2) has the form:

$$ay \oplus bx = 0 \tag{3}$$

(A4) Find the solution of this Boolean equation, for example, by one of the methods described in the previous section.

Equation (3) is solvable for any a, b and the general solution, given that $a \neq b$, can be represented as follows:

$$x = \sigma a, y = \tau b, \text{where } \sigma, \tau \in \{0, 1\} \tag{4}$$

In total, there are three private solutions:

$$\begin{aligned} x = 0, y = 0 &\text{ at } a \neq b; \\ x = 0, y = 1 &\text{ at } a = 0, b = 1; \\ x = 1, y = 0 &\text{ at } a = 1, b = 0. \end{aligned} \tag{5}$$

(A5) As a result, we obtain 2 solutions of inequality (1) in multioperations:

$$z = f(c) = o,$$

$$z = f(c) = c.$$

Note that the coefficients in the inclusion system can be not only constants, but also any null multioperations.

6. Conclusion

Note that with a slight modification the developed algorithm can be applied to solve systems of inclusions and negations of inclusions in multioperations. The obtained algorithm can be used to develop logical inference systems of multi-operation logic. At present, on the basis of the above algorithm, a computer program for solving inclusion systems in multioperations is under development.

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Системы включений с неизвестными в мультиоперациях

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Аннотация. Рассматриваются системы включений с неизвестными и коэффициентами в мультиоперациях конечного ранга. Приводится алгоритм для решения таких систем методом сведения к булевым уравнениям с помощью представления суперпозиции мультиопераций булевыми пространственными матрицами. Для полноты изложения описаны два метода решения булевых уравнений со многими неизвестными. Изложение демонстрируется примерами: представление суперпозиции мультиопераций булевыми пространственными матрицами; решение булева уравнения аналитическим и численным методами; нахождение решений включения с одним неизвестным. Полученный алгоритм может быть применен при разработке логических систем вывода логики мультиопераций.

Ключевые слова: мультиоперация, включение, пространственная матрица, булево уравнение, терм.

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