Серия «Математика»
2021. Т. 38. С. 36—53

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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УДК 518.517
MSC 35R30, 35R11, 34G10
DOI https://doi.org/10.26516/1997-7670.2021.38.36

Linear Inverse Problems for Multi-term Equations with Riemann — Liouville Derivatives *

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Аннотация. The issues of well-posedness of linear inverse coefficient problems for multi-term equations in Banach spaces with fractional Riemann – Liouville derivatives and with bounded operators at them are considered. Well-posedness criteria are obtained both for the equation resolved with respect to the highest fractional derivative, and in the case of a degenerate operator at the highest derivative in the equation. Two essentially different cases are investigated in the degenerate problem: when the fractional part of the order of the second-oldest derivative is equal to or different from the fractional part of the order of the highest fractional derivative. Abstract results are applied in the study of inverse problems for partial differential equations with polynomials from a self-adjoint elliptic differential operator with respect to spatial variables and with Riemann – Liouville derivatives in time.

Ключевые слова: inverse problem, Riemann – Liouville fractional derivative, degenerate evolution equation, initial-boundary value problem.

* The reported study was funded by RFBR and VAST (grants 21-51-54001 and QTRU 01-01/21-22).

1. Introduction

Problems for differential equations with unknown coefficients and overdetermination conditions, which are called inverse or inverse coefficient problems, play an important role in many applied research, when the nature of the process is known (the form of the equation of its dynamics is given), some of its parameters are not available for the direct measurement (unknown coefficients), but can be determined using additional measurements of the available parameters (setting overdetermination conditions) [21]. Among the studies of inverse problems for first order differential equations in Banach spaces in addition to the works of A. I. Prilepko, we note the works of I. V. Tikhonov, Yu. S. Eidelman [5], M. V. Falaleev [7], M. Al-Horani, A. Favini [11], S. G. Pyatkov [4]. In recent years, works on the study of inverse problems for equations with fractional derivatives [1; 8; 9; 12–14; 20] have appeared.

Multi-term fractional equations are of great interest to researchers [16; 18; 19]. For the multi-term equations with Riemann – Liouville derivatives unusual properties were revealed even in the scalar case [2; 15; 17]. In particular, the initial value problem of Cauchy type for such equations is solvable only if the initial conditions are given only for fractional derivatives of sufficiently large order [3; 10].

In this paper, for multi-term equations in Banach spaces with Riemann – Liouville derivatives and with bounded operators at them, we consider well-posedness issues for linear inverse problems with an unknown coefficient which is independent of time. In this case, the integral overdetermination condition with the Riemann – Stieltjes integral is used, which includes, as a partial case, the overdetermination condition at a fixed point of time. Both the equations resolved with respect to the highest derivative and equations with a degenerate operator at the highest derivative are considered. In the first case, conditions of the Cauchy type are specified, in the second, the initial conditions have a complex form and depend on the orders of the derivatives from the equation.

In the second section, a criterion for the well-posedness of the inverse problem for the equation in a Banach space, solved with respect to the highest Riemann – Liouville derivative, with bounded operators at the derivatives is obtained. In the third section, the inverse coefficient problem for the equation with a degenerate operator L at the highest derivative is reduced to a system of two inverse problems on subspaces for equations solved with respect to the highest derivative, under the condition that the operator at the second-highest derivative is $(L, 0)$-bounded. Two essentially different cases are considered: when the fractional part of the order of the second derivative coincides with the fractional part of the order of the highest derivative and when it differs (see [10]). For each case, a criterion for the correctness of the inverse problem is obtained. The fourth section contains applications of obtained abstract results to the study of the inverse problem for an equation with polynomials with respect to a self-adjoin elliptic differential operator in spatial variables and with Riemann – Liouville time derivatives in both the nondegenerate and degenerate cases.

2. Nondegenerate inverse problem

Let $\mathcal{Z}$ be a Banach space, by $\mathcal{L}(\mathcal{Z})$ denote the Banach space of linear bounded operators in $\mathcal{Z}$, $\mathbb{R}_+ := \{a \in \mathbb{R} : a > 0\}$, $h : \mathbb{R}_+ \to \mathcal{Z}$. For $\beta > 0$ denote the Riemann – Liouville integral

$$J_t^\beta h(t) := \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} h(s)ds.$$

By J_t^0 we denote the identity operator. Let $\alpha > 0$, $m := \lceil \alpha \rceil$ be the smallest integer, which is greater or equal to α, D_t^m is the usual derivative of the order $m \in \mathbb{N}$, D_t^α is the fractional Riemann – Liouville derivative, i. e. $D_t^\alpha h(t) := D_t^m J_t^{m-\alpha} h(t)$. At $\beta < 0$ we will use the notation $D_t^\beta h(t) := J_t^{-\beta} h(t)$.

Consider the nondegenerate equation, i. e. the equation, resolved with respect to the highest derivative

$$D_t^\alpha z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t) + \varphi(t)u, \tag{2.1}$$

with $t \in (0, T]$, where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l := \lceil \alpha_l \rceil$, $m := \lceil \alpha \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1, 2, \dots, n$, $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$, operators A_j, $j = 1, 2, \dots, m-1$, B_l, $l = 1, 2, \dots, n$, C_s, $s = 1, 2, \dots, r$, are linear and bounded in the Banach space $\mathcal{Z}$, and $\varphi \in C((0, T]; \mathbb{R}) \cap L_1(0, T; \mathbb{R})$, $u \in \mathcal{Z}$.

Let $\underline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\}, \ \alpha_l - m_l < \alpha - m\}$, $\underline{m} = \lceil \underline{\alpha} \rceil$, $\overline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\}, \ \alpha_l - m_l > \alpha - m\}$, $\overline{m} = \lceil \overline{\alpha} \rceil$. We denote by $m^* := \max\{\underline{m} - 1, \overline{m}\}$ the defect of the Cauchy type problem for the equation (2.1) [10]. Then the initial conditions of the Cauchy type have the form

$$D_t^{\alpha-m+k} z(0) = z_k, \ k = m^*, m^*+1, \dots, m-1. \tag{2.2}$$

The overdetermination condition for the inverse problem take in the form

$$\int\limits_0^T z(t) d\mu(t) = z_T, \tag{2.3}$$

where μ has a bounded variation on the segment $[0, T]$. The integral is understood as a vector integral of Riemann – Stieltjes.

By a solution of problem (2.1), (2.2), where the element $u \in \mathcal{Z}$ is known, we will call a function $z : (0,T] \to \mathcal{Z}$, such that $J_t^{m-\alpha} z \in C^m((0,T];\mathcal{Z}) \cap C^{m-1}([0,T];\mathcal{Z})$, $J_t^{m_l-\alpha_l} z \in C^{m_l}((0,T];\mathcal{Z})$, $l=1,\dots,n$, $J_t^{\beta_s} z \in C((0,T];\mathcal{Z})$, $s = 1,2,\dots,r$, and (2.2), (2.1) for $t \in (0,T]$ hold.

Theorem 1. [10]. *Let $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m = \lceil \alpha \rceil$, $m_l = \lceil \alpha_l \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1,2,\dots,n$, $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, $A_j \in \mathcal{L}(\mathcal{Z})$, $j = 1,2,\dots,m-1$, $B_l \in \mathcal{L}(\mathcal{Z})$, $l = 1,2,\dots,n$, $C_s \in \mathcal{L}(\mathcal{Z})$, $s = 1,2,\dots,r$, $z_k \in \mathcal{Z}$, $k = m^*, m^*+1,\dots,m-1$, $\varphi \in C((0,T];\mathbb{R}) \cap L_1(0,T;\mathbb{R})$, $u \in \mathcal{Z}$. Then there exists a unique solution to* (2.1), (2.2), *and it has the form*

$$z(t) = \sum_{p=m^*}^{m-1} Z_p(t) z_p + \int_0^t Z_{m-1}(t-s)\varphi(s) u ds, \qquad (2.4)$$

where at $p = m^*, m^*+1,\dots,m-1$

$$Z_p(t) := \frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha} R_\lambda \cdot \left(\lambda^{m-1-p} I - \sum_{j=p+1}^{m-1} \lambda^{j-1-p} A_j\right) e^{\lambda t} d\lambda,$$

$$R_\lambda := \left(I - \sum_{j=1}^{m-1} \lambda^{j-m} A_j - \sum_{l=1}^{n} \lambda^{\alpha_l-\alpha} B_l - \sum_{s=1}^{r} \lambda^{-\beta_s-\alpha} C_s\right)^{-1},$$

$\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_0 = \{\lambda \in \mathbb{C} : |\lambda| = r_0, \arg\lambda \in (-\pi,\pi)\}$, $\Gamma_+ = \{\lambda \in \mathbb{C} : \arg\lambda = \pi, \lambda \in [-r_0,-\infty)\}$, $\Gamma_- = \{\lambda \in \mathbb{C} : \arg\lambda = -\pi, \lambda \in (-\infty,-r_0]\}$, $r_0 > 0$ *is a large enough number.*

Now consider the inverse problem (2.1)–(2.3), assuming the element u to be unknown. By a solution of this problem we will understand a pair (z,u), where $z : (0,T] \to \mathcal{Z}$ is a solution of problem (2.1), (2.2) with the corresponding $u \in \mathcal{Z}$, which satisfies condition (2.3). For brevity, we will also often call this element $u \in \mathcal{Z}$ the solution of problem (2.1)–(2.3).

We call problem (2.1)–(2.3) well-posed if for any $z_k \in \mathcal{Z}$, $k = m^*, m^*+1,\dots,m-1$, $z_T \in \mathcal{Z}$ it has a unique solution $u \in \mathcal{Z}$, satisfying the estimate $\|u\|_\mathcal{Z} \leq C(\|z_{m^*}\|_\mathcal{Z} + \|z_{m^*+1}\|_\mathcal{Z} + \cdots + \|z_{m-1}\|_\mathcal{Z} + \|z_T\|_\mathcal{Z})$, where $C > 0$ does not depend on z_k, $k = m^*, m^*+1,\dots,m-1$, and z_T.

For given $z_k \in \mathcal{Z}$, $k = m^*, m^*+1,\dots,m-1$, $z_T \in \mathcal{Z}$ we denote

$$\psi := z_T - \int_0^T \sum_{p=m^*}^{m-1} Z_p(t) z_p d\mu(t), \quad \chi := \int_0^T d\mu(t) \int_0^t Z_{m-1}(t-s)\varphi(s) ds.$$

Theorem 2. *Let $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m = \lceil \alpha \rceil$, $m_l = \lceil \alpha_l \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1,2,\dots,n$, $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, $A_j \in \mathcal{L}(\mathcal{Z})$,*

$j = 1, 2, \dots, m-1$, $B_l \in \mathcal{L}(\mathcal{Z})$, $l = 1, 2, \dots, n$, $C_s \in \mathcal{L}(\mathcal{Z})$, $s = 1, 2, \dots, r$, $\varphi \in C((0,T];\mathbb{R}) \cap L_1(0,T;\mathbb{R})$, $\mu : [0,T] \to \mathbb{R}$ have a bounded variation, for $m^* = 0$ there exist $\varepsilon \in (0,T]$ such that $\mu \in C^1([0,\varepsilon];\mathbb{R})$. Then problem (2.1)–(2.3) is well-posed if and only if there exists the operator $\chi^{-1} \in \mathcal{L}(\mathcal{Z})$. In this case the solution has the form $u = \chi^{-1}\psi$.

Proof. By Theorem 1, a solution of Cauchy type problem (2.1), (2.2) exists for all $z_k \in \mathcal{Z}$, $k = m^*, m^*+1, \dots, m-1$, $u \in \mathcal{Z}$ and has form (2.4). Substituting solution (2.4) in condition (2.3), we obtain the equality

$$\int_0^T d\mu(t) \left(\sum_{p=m^*}^{m-1} Z_p(t) z_p + \int_0^t Z_{m-1}(t-s) f(s) u ds \right) = z_T.$$

It implies the equation $\chi u = \psi$. Therefore, problem (2.1)–(2.3) is equivalent to the last equation. Its unique solvability for any $\psi \in \mathcal{Z}$ means exactly the existence of an inverse operator $\chi^{-1} \in \mathcal{L}(\mathcal{Z})$. Then

$$\|u\|_{\mathcal{Z}} \le \|\chi^{-1}\|_{\mathcal{L}(\mathcal{Z})} \left(\|z_T\|_{\mathcal{Z}} + \sum_{p=m^*}^{m-1} \left\| \int_0^T Z_p(t) z_p d\mu(t) \right\|_{\mathcal{Z}} \right).$$

In the work [10] the inequalities

$$\|Z_0(t)\|_{\mathcal{L}(\mathcal{Z})} \le \frac{Ke^{r_0 t}}{t^{m-\alpha}}, \quad \|Z_p(t)\|_{\mathcal{L}(\mathcal{Z})} \le Ke^{r_0 t}, \quad p = 1, 2, \dots, m-1, \quad (2.5)$$

were proved. Hence, for $p = 1, 2, \dots, m-1$

$$\left\| \int_0^T Z_p(t) z_p d\mu(t) \right\|_{\mathcal{L}(\mathcal{Z})} \le Ke^{r_0 T} \|z_p\|_{\mathcal{Z}} V_0^T(\mu),$$

where $V_0^T(\mu)$ is the variation of the function μ on the segment $[0,T]$. If $m^* = 0$, then

$$\left\| \int_0^T Z_0(t) z_0 d\mu(t) \right\|_{\mathcal{L}(\mathcal{Z})} \le \left\| \int_0^\varepsilon Z_0(t) z_0 \mu'(t) dt \right\|_{\mathcal{L}(\mathcal{Z})} + \left\| \int_\varepsilon^T Z_0(t) z_0 d\mu(t) \right\|_{\mathcal{L}(\mathcal{Z})} \le$$

$$Ke^{r_0 \varepsilon} \|z_0\|_{\mathcal{Z}} \|\mu\|_{C^1([0,\varepsilon];\mathbb{R})} \int_0^\varepsilon \frac{dt}{t^{m-\alpha}} + \max_{t \in [\varepsilon,T]} \frac{Ke^{r_0 t}}{t^{m-\alpha}} \|z_0\|_{\mathcal{Z}} V_0^T(\mu).$$

Thus, the inverse problem is well-posed, if and only if $\chi^{-1} \in \mathcal{L}(\mathcal{Z})$. □

3. Degenerate inverse problems

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $L, M_1, M_2, \dots, M_{m-1}, N_1, N_2, \dots, N_n, S_1, S_2, \dots, S_r \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ (linear continuous operators from $\mathcal{X}$ into $\mathcal{Y}$), $\ker L \neq \{0\}$. All operators $N_1, N_2, \dots, N_n, S_1, S_2, \dots, S_r$ are not zero, while some of (or all) M_j are zero operators. If $M_j = 0$ for all $j = 1, 2, \dots, m-1$, then put $j_0 = 0$; otherwise there exists $j_0 \in \{1, 2, \dots, m-1\}$ such that $M_{j_0} \neq 0$, $M_j = 0$, for all $j = j_0+1, j_0+2, \dots, m-1$.

Consider the evolution equation

$$D_t^\alpha Lx(t) = \sum_{j=1}^{j_0} M_j D_t^{\alpha-m+j} x(t) + \sum_{l=1}^{n} N_l D_t^{\alpha_l} x(t) + \sum_{s=1}^{r} S_s J_t^{\beta_s} x(t) + \varphi(t)u, \tag{3.1}$$

which we call degenerate provided that $\ker L \neq \{0\}$. Here $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m := \lceil \alpha \rceil$, $m_l := \lceil \alpha_l \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1, 2, \dots, n$, $\underline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\}, \alpha_l - m_l < \alpha - m\}$, $\underline{m} := \lceil \underline{\alpha} \rceil$, $\overline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\}, \alpha_l - m_l > \alpha - m\}$, $\overline{m} := \lceil \overline{\alpha} \rceil$, $m^* := \max\{\underline{m} - 1, \overline{m}\}$, $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$, $\varphi \in C((0,T];\mathcal{Y}) \cap L_1(0,T;\mathcal{Y})$, $u \in \mathcal{Z}$. Consider the two possible cases: $\alpha_n < \alpha - m + j_0$ and $\alpha_n > \alpha - m + j_0$.

3.1. The first case

If $\alpha_n < \alpha - m + j_0$, suppose that the operator M_{j_0} is (L,σ)-bounded:

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow ((\mu L - M_{j_0})^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})).$$

In this case we define the operators $R_\mu^L(M_{j_0}) := (\mu L - M_{j_0})^{-1} L$ and $L_\mu^L(M_{j_0}) := L(\mu L - M_{j_0})^{-1}$, the projections

$$P := \frac{1}{2\pi i} \int_\gamma R_\mu^L(M_{j_0}) d\mu \in \mathcal{L}(\mathcal{X}), \quad Q := \frac{1}{2\pi i} \int_\gamma L_\mu^L(M_{j_0})\, d\mu \in \mathcal{L}(\mathcal{Y}),$$

where $\gamma := \{\mu \in \mathbb{C} : |\mu| = r > a\}$ (see [22, p. 89, 90]). Put $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \operatorname{im} P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \operatorname{im} Q$. Denote $P_0 := I - P$, $Q_0 := I - Q$ for brevity, by L_q, $M_{j,q}$, $N_{l,q}$ and $S_{s,q}$ denote the restrictions of L, M_j, $j = 1, 2, \dots, j_0$, N_l, $l = 1, 2, \dots, n$ and S_s, $s = 1, 2, \dots, r$, on $\mathcal{X}^q$, $q = 0, 1$. It is known (see [22, p. 90, 91]) that $LP = QL$, $M_{j_0} P = Q M_{j_0}$, $M_{j_0,q} \in \mathcal{L}(\mathcal{X}^q;\mathcal{Y}^q)$ and $L_q \in \mathcal{L}(\mathcal{X}^q;\mathcal{Y}^q)$, $q = 0, 1$. Moreover, in the situation under consideration we have the operators $M_{j_0,0}^{-1} \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$.

We also suppose that $L_0 = 0$; in this case the (L,σ)-bounded operator M_{j_0} is called $(L,0)$-bounded [22]. Moreover, under the additional conditions

$$\begin{gathered} QM_j = M_j P,\ j = 1, 2, \dots, j_0 - 1, \quad QN_l = N_l P,\ l = 1, 2, \dots, n, \\ QS_s = S_s P,\ s = 1, 2, \dots, r, \end{gathered} \tag{3.2}$$

equation (3.1) with the initial conditions

$$\begin{aligned} &D_t^{\alpha-m+k}x(0) = x_k,\ k = m^*, m^*+1, \dots, j_0-1, \\ &D_t^{\alpha-m+k}(Px)(0) = x_k,\ k = j_0, j_0+1, \dots, m-1, \end{aligned} \tag{3.3}$$

and with the overdetermination condition

$$\int_0^T x(t)d\mu(t) = x_T \tag{3.4}$$

can be reduced to the following problems for two equations on the mutually complemented subspaces $\mathcal{X}^1$ and $\mathcal{X}^0$:

$$\begin{aligned} D_t^\alpha v(t) = \sum_{j=1}^{j_0} L_1^{-1}M_jD_t^{\alpha-m+j}v(t) + \sum_{l=1}^{n} L_1^{-1}N_lD_t^{\alpha_l}v(t)+ \\ + \sum_{s=1}^{r} L_1^{-1}S_sJ_t^{\beta_s}v(t) + \varphi(t)L_1^{-1}u^1, \quad t \in (0,T], \end{aligned} \tag{3.5}$$

$$D_t^{\alpha-m+k}v(0) = v_k,\ k = m^*, m^*+1, \dots, m-1, \tag{3.6}$$

$$\int_0^T v(t)d\mu(t) = v_T \tag{3.7}$$

and

$$\begin{aligned} D_t^{\alpha-m+j_0}w(t) = -\sum_{j=1}^{j_0-1} M_{j_0,0}^{-1}M_jD_t^{\alpha-m+j}w(t) - \sum_{l=1}^{n} M_{j_0,0}^{-1}N_lD_t^{\alpha_l}w(t)- \\ - \sum_{s=1}^{r} M_{j_0,0}^{-1}S_sJ_t^{\beta_s}w(t) - \varphi(t)M_{j_0,0}^{-1}u^0, \quad t \in (0,T], \end{aligned} \tag{3.8}$$

$$D_t^{\alpha-m+k}w(0) = w_k,\ k = m^*, m^*+1, \dots, j_0-1, \tag{3.9}$$

$$\int_0^T w(t)d\mu(t) = w_T, \tag{3.10}$$

where $v(t) := Px(t)$, $v_k := Px_k$, $k = m^*, \dots, m-1$, $v_T := Px_T$, $u^1 = Qu$, $w(t) := P_0x(t)$, $w_k := P_0x_k$, $k = m^*, \dots, j_0-1$, $w_T := P_0x_T$ and $u^0 = Q_0u$.

Note that equations (3.5) and (3.8) are resolved with respect to the highest fractional derivative. Consequently, by theorem 2 problem (3.5)–(3.7) (problem (3.8)–(3.10)) is well-posed if and only if there exists the operator $\chi_1^{-1} \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$ ($\chi_0^{-1} \in \mathcal{L}(\mathcal{X}^0;\mathcal{Y}^0)$), wherein each of these problems has a unique solution $u^1 = \chi_1^{-1}\psi_1$ ($u^0 = \chi_0^{-1}\psi_0$). Here

$$\chi_1 := \int_0^T d\mu(t) \int_0^t Z_{m-1}^1(t-s)L_1^{-1}\varphi(s)ds,\ \psi_1 := v_T - \int_0^T \sum_{p=m^*}^{m-1} Z_p^1(t)v_pd\mu(t),$$

$$Z_p^1(t) := \frac{1}{2\pi i}\int\limits_{\Gamma_1} \lambda^{-\alpha} R_\lambda^1 \left(\lambda^{m-1-p} L_1 - \sum_{j=p+1}^{j_0} \lambda^{j-1-p} M_{j,1}\right) e^{\lambda t} d\lambda,$$

$$R_\lambda^1 := \left(L_1 - \sum_{j=1}^{j_0} \lambda^{j-m} M_{j,1} - \sum_{l=1}^{n} \lambda^{\alpha_l-\alpha} N_{l,1} - \sum_{s=1}^{r} \lambda^{-\beta_s-\alpha} S_{s,1}\right)^{-1},$$

$$\chi_0 := \int\limits_0^T d\mu(t) \int\limits_0^t Z_{m-1}^0(t-s) M_{j_0,0}^{-1}\varphi(s)ds, \ \psi_0 := w_T - \int\limits_0^T \sum_{p=m^*}^{j_0-1} Z_p^0(t) w_p d\mu(t),$$

$$Z_p^0(t) := \frac{1}{2\pi i}\int\limits_{\Gamma_0} \lambda^{-\alpha} R_\lambda^0 \cdot \left(\lambda^{m-1-p} M_{j_0,0} + \sum_{j=p+1}^{j_0-1} \lambda^{j-1-p} M_{j,0}\right) e^{\lambda t} d\lambda,$$

$$R_\lambda^0 := \left(M_{j_0,0} + \sum_{j=1}^{j_0-1} \lambda^{j-m} M_{j,0} + \sum_{l=1}^{n} \lambda^{\alpha_l-\alpha} N_{l,0} + \sum_{s=1}^{r} \lambda^{-\beta_s-\alpha} S_{s,0}\right)^{-1}.$$

Obvious transformations are used here and contours Γ_1 and Γ_0 are constructed as Γ, taking into account the norms of the operators in the problem.

Now we introduce strict definitions and formulations.

A solution to problem (3.1), (3.3) is a function $x : (0,T] \to \mathcal{X}$ such that $J_t^{m-\alpha} Lx \in C^m((0,T];\mathcal{Y}) \cap C^{m-1}([0,T];\mathcal{Y})$, $J_t^{m-\alpha} x \in C^{j_0}((0,T];\mathcal{X}) \cap C^{j_0-1}([0,T];\mathcal{X})$, $J_t^{m_l-\alpha_l} x \in C^{m_l}((0,T];\mathcal{X})$, $J_t^{\beta_s} x \in C((0,T];\mathcal{X})$, equality (3.1) for all $t \in (0,T]$ and conditions (3.3) are satisfied.

A solution to problem (3.1), (3.3), (3.4) is a pair (x,u) such that x is a solution of problem (3.1), (3.3) with the corresponding $u \in \mathcal{Y}$ and satisfies condition (3.4).

We call problem (3.1), (3.3), (3.4) well-posed if for any $x_k \in \mathcal{X}$, $k = 0,1,\dots,j_0-1$, $x_k \in \mathcal{X}^1$, $k = j_0, j_0+1,\dots,m-1$ and $x_T \in \mathcal{X}$ it has a unique solution satisfying the estimate $\|u\|_{\mathcal{Y}} \le C(\|x_{m^*}\|_{\mathcal{X}} + \|x_{m^*+1}\|_{\mathcal{X}} + \dots + \|x_{m-1}\|_{\mathcal{X}} + \|x_T\|_{\mathcal{X}})$, where C does not depend on x_k, $k = m^*,\dots,m-1$ and x_T.

Theorem 1. *Let* $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m := \lceil\alpha\rceil$, $m_l := \lceil\alpha_l\rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1,2,\dots,n$, $\beta_1 > \beta_2 > \dots > \beta_r \ge 0$, $L, M_j \in \mathcal{L}(\mathcal{Z})$, $j = 1,2,\dots,m-1$, $N_l \in \mathcal{L}(\mathcal{Z})$, $l = 1,2,\dots,n$, $S_s \in \mathcal{L}(\mathcal{Z})$, $s = 1,2,\dots,r$, *conditions* (3.2) *hold,* $\alpha_n < \alpha - m + j_0$, *operator* M_{j_0} *be* $(L,0)$*-bounded,* $\varphi \in C((0,T];\mathbb{R}) \cap L_1(0,T;\mathbb{R})$, *function* $\mu : [0,T] \to \mathbb{R}$ *have a bounded variation, for* $m^* = 0$ *there exist* $\varepsilon \in (0,T]$ *such that* $\mu \in C^1([0,\varepsilon];\mathbb{R})$. *Then problem* (3.1), (3.3), (3.4) *is well-posed if and only if there exist the operators* $\chi_1^{-1} \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, $\chi_0^{-1} \in \mathcal{L}(\mathcal{X}^0;\mathcal{Y}^0)$. *In this case the solution has the form* $u = \chi_1^{-1}\psi_1 + \chi_0^{-1}\psi_0$.

3.2. The second case

Let $\alpha_n > \alpha - m + j_0$, and let the operator N_n be $(L,0)$-bounded. Then

$$P := \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(N_n)d\mu \in \mathcal{L}(\mathcal{X}), \quad Q := \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(N_n)d\mu \in \mathcal{L}(\mathcal{Y})$$

and under the additional conditions

$$\begin{gathered} QM_j = M_jP,\ j = 1,2,\ldots,j_0, \quad QN_l = N_lP,\ l = 1,2,\ldots,n-1, \\ QS_s = S_sP,\ s = 1,2,\ldots,r, \end{gathered} \tag{3.11}$$

by analogy with the first case consider two equations on the subspaces $\mathcal{X}^1$ and $\mathcal{X}^0$: equation

$$\begin{gathered} D^\alpha_t v(t) = \sum_{j=1}^{j_0} L_1^{-1}M_jD_t^{\alpha-m+j}v(t) + \sum_{l=1}^{n} L_1^{-1}N_lD_t^{\alpha_l}v(t) + \\ + \sum_{s=1}^{r} L_1^{-1}S_sJ_t^{\beta_s}v(t) + \varphi(t)L_1^{-1}u^1, \end{gathered} \tag{3.12}$$

endowed with conditions

$$D_t^{\alpha-m+k}v(0) = v_k,\ k = m^*, m^*+1, \ldots, m-1, \tag{3.13}$$

$$\int\limits_0^T v(t)d\mu(t) = v_T, \tag{3.14}$$

and equation

$$\begin{gathered} D^{\alpha_n}_t w(t) = -\sum_{j=1}^{j_0} N_{n,0}^{-1}M_jD_t^{\alpha-m+j}w(t) - \sum_{l=1}^{n-1} N_{n,0}^{-1}N_lD_t^{\alpha_l}w(t) - \\ - \sum_{s=1}^{r} N_{n,0}^{-1}S_sJ_t^{\beta_s}w(t) - \varphi(t)N_{n,0}^{-1}u^0, \quad t \in (0,T], \end{gathered} \tag{3.15}$$

for which determine the parameters

$$\underline{m}_0 = \lceil \underline{\alpha}_0 \rceil, \overline{m}_0 = \lceil \overline{\alpha}_0 \rceil, m_0^* := \max\{\underline{m}_0 - 1, \overline{m}_0\},$$

where

$$\begin{gathered} \underline{\alpha}_0 = \max\{\alpha - m + j_0, \alpha_l : l = 1,2,\ldots,n-1, \\ \alpha_l - m_l < \alpha_n - m_n, \alpha - m < \alpha_n - m_n\}, \\ \overline{\alpha}_0 = \max\{\alpha - m + j_0, \alpha_l : l = 1,2,\ldots,n-1, \\ \alpha_l - m_l > \alpha_n - m_n, \alpha - m > \alpha_n - m_n\}. \end{gathered}$$

Now we can formulate the initial conditions for equation (3.15):

$$D_t^{\alpha_n-m_n+k}w(0) = w_k,\ k = m_0^*, m_0^*+1, \ldots, m_n-1, \tag{3.16}$$

and the overdetermination condition

$$\int_0^T w(t)d\mu(t) = w_T. \tag{3.17}$$

As before, $v(t) = Px(t)$, $u^1 = Qu$, $w(t) = P_0x(t)$, $u^0 = Q_0u$.

In this case the conditions of the inverse problem for equation (3.1) can be given in the form

$$D_t^{\alpha-m+k}(Px)(0) = v_k,\ k = m^*, m^*+1, \dots, m-1, \tag{3.18}$$

$$D_t^{\alpha_n-m_n+k}(P_0x)(0) = w_k,\ k = m_0^*, m_0^*+1, \dots, m_n-1, \tag{3.19}$$

$$\int_0^T x(t)d\mu(t) = x_T = v_T + w_T. \tag{3.20}$$

Problem (3.1), (3.18)–(3.20) is called well-posed, if problems (3.12)–(3.14) and (3.15)–(3.17) are well-posed.

By Theorem 2, as in the first case, we obtain the following result.

Theorem 3. *Let* $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m := \lceil \alpha \rceil$, $m_l := \lceil \alpha_l \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1, 2, \dots, n$, $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$, $L, M_j \in \mathcal{L}(\mathcal{Z})$, $j = 1, 2, \dots, m-1$, $N_l \in \mathcal{L}(\mathcal{Z})$, $l = 1, 2, \dots, n$, $S_s \in \mathcal{L}(\mathcal{Z})$, $s = 1, 2, \dots, r$, $\alpha_n > \alpha - m + j_0$, *conditions* (3.11) *hold, operator* N_n *be* $(L, 0)$*-bounded,* $\varphi \in C((0,T];\mathbb{R}) \cap L_1(0,T;\mathbb{R})$, *function* $\mu : [0,T] \to \mathbb{R}$ *have a bounded variation, for* $m^*m_0^* = 0$ *there exist* $\varepsilon \in (0,T]$ *such that* $\mu \in C^1([0,\varepsilon];\mathbb{R})$. *Then problem* (3.1), (3.18)–(3.20) *is well-posed if and only if there exist operators* $\chi_1^{-1} \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, $\chi_0^{-1} \in \mathcal{L}(\mathcal{X}^0;\mathcal{Y}^0)$. *In this case the solution has the form* $u = \chi_1^{-1}\psi_1 + \chi_0^{-1}\psi_0$.

Here

$$\chi_1 := \int_0^T d\mu(t) \int_0^t Z_{m-1}^1(t-s)L_1^{-1}\varphi(s)ds,\ \psi_1 := v_T - \int_0^T \sum_{p=m^*}^{m-1} Z_p^1(t)v_p d\mu(t),$$

$$Z_p^1(t) = \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^{-\alpha} R_\lambda^1 \left(\lambda^{m-1-p}L_1 - \sum_{j=p+1}^{j_0} \lambda^{j-1-p}M_{j,1} \right) e^{\lambda t} d\lambda,$$

$$R_\lambda^1 := \left(L_1 - \sum_{j=1}^{j_0} \lambda^{j-m}M_{j,1} - \sum_{l=1}^{n} \lambda^{\alpha_l-\alpha}N_{l,1} - \sum_{s=1}^{r} \lambda^{-\beta_s-\alpha}S_{s,1} \right)^{-1},$$

$$\chi_0 := \int_0^T d\mu(t) \int_0^t Z_{m-1}^0(t-s)N_{n,0}^{-1}\varphi(s)ds,\ \psi_0 := w_T - \int_0^T \sum_{p=m_0^*}^{m_n-1} Z_p^0(t)w_p d\mu(t),$$

$$Z_p^0(t)=\frac{1}{2\pi i}\int\limits_{\Gamma_0}\lambda^{-\alpha_n}R_\lambda^0\left(\lambda^{m_n-1-p}N_{n,0}+\sum_{l=p+1}^{m_n-1}\lambda^{l-1-p}T_l\right)e^{\lambda t}d\lambda,$$

$$R_\lambda^0:=\left(N_{n,0}+\sum_{j=1}^{j_0}\lambda^{j-m}M_j+\sum_{l=1}^{n-1}\lambda^{\alpha_l-\alpha}N_l+\sum_{s=1}^{r}\lambda^{-\beta_s-\alpha}S_s\right)^{-1},$$

$T_l=N_l$ for $\alpha_l-m_l=\alpha_n-m_n$, $T_l=0$ for $\alpha_l-m_l\neq\alpha_n-m_n$.

Remark 1. *In the considered cases, when the operator at the second highest derivative is* $(L,0)$*-bounded, the conditions* $D_t^{\alpha-m+k}(Px)(0)=x_k$ *are equivalent to* $D_t^{\alpha-m+k}(Lx)(0)=y_k$ *with* $y_k=Lx_k$*,* $k=m^*,m^*+1,\dots,m-1$*, since* $\ker L=\ker P$ *and there exists* $L_1^{-1}\in\mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$*.*

4. Applications

Let $P_1(\lambda)=\sum\limits_{p=0}^{\nu}a_p\lambda^p$, $P_2^j(\lambda)=\sum\limits_{p=0}^{\nu}b_p^j\lambda^p$, $P_3^l(\lambda)=\sum\limits_{p=0}^{\nu}c_p^l\lambda^p$, $P_4^s(\lambda)=\sum\limits_{p=0}^{\nu}d_p^s\lambda^p$, $a_p,b_p^j,c_p^l,d_p^s\in\mathbb{C}$, $p=0,1,\dots,\nu\in\mathbb{N}$, $j=1,2,\dots,m-1$, $l=1,2,\dots,n$, $s=1,2,\dots,r$, $a_\nu\neq0$, where $\Omega\subset\mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$,

$$(\mathcal{A}h)(\xi)=\sum_{|q|\le2\rho}a_q(\xi)\frac{\partial^{|q|}h(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}},\quad a_q\in C^\infty(\overline{\Omega}),$$

$$(\mathcal{B}_lh)(\xi)=\sum_{|q|\le\rho_l}b_{lq}(\xi)\frac{\partial^{|q|}h(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}},\quad b_{lq}\in C^\infty(\partial\Omega),\ l=1,2,\dots,\rho,$$

$q=(q_1,q_2,\dots,q_d)\in\mathbb{N}_0^d$, $|q|=q_1+\dots+q_d$, and the operator pencil $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_\rho$ is regularly elliptic [6]. Let the operator $\mathcal{A}_1\in\mathcal{C}l(L_2(\Omega))$ have the domain $D_{\mathcal{A}_1}=H^{2\rho}_{\{\mathcal{B}_l\}}(\Omega):=\{h\in H^{2\rho}(\Omega):\mathcal{B}_lh(\xi)=0,\ l=1,2,\dots,\rho,\ \xi\in\partial\Omega\}$, $\mathcal{A}_1h:=\mathcal{A}h$. Suppose that $\mathcal{A}_1$ is a self-adjoint operator; then its spectrum $\sigma(\mathcal{A}_1)$ is real and discrete [6]. Moreover, assume that the spectrum $\sigma(\mathcal{A}_1)$ is bounded from the right and does not contain zero, $\{\varphi_k:k\in\mathbb{N}\}$ is an orthonormal system of eigenfunctions of $\mathcal{A}_1$ in $L_2(\Omega)$ which is enumerated in nonincreasing order of the corresponding eigenvalues $\{\lambda_k:k\in\mathbb{N}\}$, taking into account their multiplicities.

Let $0<\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $m-1<\alpha\le m\in\mathbb{N}$, $m_l-1<\alpha_l\le m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $l=1,2,\dots,n$, $\beta_1>\beta_2>\dots>\beta_r\ge0$. Consider the equation in $\Omega\times(0,T]$

$$D_t^\alpha P_1(\mathcal{A})v(\xi,t)=\sum_{j=1}^{m-1}P_2^j(\mathcal{A})D_t^{\alpha-m+j}v(\xi,t)+\\+\sum_{l=1}^{n}P_3^l(\mathcal{A})D_t^{\alpha_l}v(\xi,t)+\sum_{s=1}^{r}P_4^s(\mathcal{A})J_t^{\beta_s}v(\xi,t)+\varphi(t)w(\xi),\tag{4.1}$$

with the boundary conditions at $(\xi, t) \in \partial\Omega \times (0, T]$

$$\mathcal{B}_l \mathcal{A}^k v(\xi, t) = 0, \quad k = 0, 1, \dots, \nu - 1, \quad l = 1, 2, \dots, \rho, \tag{4.2}$$

and the overdetermination condition

$$\int_0^T v(\xi, t) d\mu(t) = v_T(\xi), \quad \xi \in \Omega. \tag{4.3}$$

The form of initial conditions depends on situation. Put

$$\mathcal{X} = \{h \in H^{2\rho\nu}(\Omega) : \mathcal{B}_l \mathcal{A}^k h(\xi) = 0,\ k = 0, 1, \dots, \nu - 1,\ l = 1, 2, \dots, \rho,\ \xi \in \partial\Omega\}, \quad \mathcal{Y} = L_2(\Omega),$$

$L = P_1(\mathcal{A}) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $M_j = P_2^j(\mathcal{A}) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $j = 1, 2, \dots, m - 1$, $N_l = P_3^l(\mathcal{A}) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $l = 1, 2, \dots, n$, $S_s = P_4^s(\mathcal{A}) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $s = 1, 2, \dots, r$.

If $P_1(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, then there exists the inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$. Define the defect m^* for the collection of numbers $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha$ and consider the Cauchy type conditions

$$D_t^{\alpha - m + k} v(\xi, 0) = v_k(\xi),\ k = m^*, m^* + 1, \dots, m - 1, \quad \xi \in \Omega. \tag{4.4}$$

Then problem (4.1)–(4.4) can be presented in the form of (2.1)–(2.3), where $\mathcal{Z} = \mathcal{X}$, $A_j = L^{-1} M_j \in \mathcal{L}(\mathcal{Z})$, $j = 1, 2, \dots, m - 1$, $B_l = L^{-1} N_l \in \mathcal{L}(\mathcal{Z})$, $l = 1, 2, \dots, n$, $C_s = L^{-1} S_s \in \mathcal{L}(\mathcal{Z})$, $s = 1, 2, \dots, r$, $z_k = v_k(\cdot)$, $k = m^*, m^* + 1, \dots, m - 1$, $z_T = v_T(\cdot)$, $u = L^{-1} w(\cdot)$. By Theorem 2 problem (4.1)–(4.4) is well-posed if and only if for all $k \in \mathbb{N}$

$$\left| \int_0^T \int_0^t \int_\Gamma R_{\lambda,k} e^{\lambda(t-s)} d\lambda \varphi(s) ds d\mu(t) \right| \geq c, \tag{4.5}$$

$$R_{\lambda,k} := \left(\lambda^\alpha P_1(\lambda_k) - \sum_{j=1}^{m-1} \lambda^{\alpha + j - m} P_2^j(\lambda_k) - \right.$$
$$\left. - \sum_{l=1}^{n} \lambda^{\alpha_l} P_3^l(\lambda_k) - \sum_{s=1}^{r} \lambda^{-\beta_s} P_4^s(\lambda_k) \right)^{-1}.$$

It follows from the equation

$$\chi = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \langle \cdot, \varphi_k \rangle \varphi_k \int_0^T \int_0^t \int_\Gamma R_{\lambda,k} e^{\lambda(t-s)} d\lambda \varphi(s) ds d\mu(t),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L_2(\Omega)$.

EXAMPLE 1. Take $\alpha = 5/2$, $m = 3$, $n = 1$, $r = 1$, $\alpha_1 = 2/3$, $\beta_1 = 1/2$, $P_1(\lambda) = \lambda^2$, $P_2^1(\lambda) = b_1\lambda + b_2\lambda^2$, $P_2^2(\lambda) \equiv 0$, $P_3^1(\lambda) = c_0 + c_2\lambda^2$, $P_4^1(\lambda) = d_0 + d_1\lambda$, $d = 1$, $\Omega = (0,\pi)$, $\rho = 1$, $\mathcal{A}h = \frac{\partial^2 h}{\partial\xi^2}$, $\mathcal{B}_1 = I$, μ is the single jump at $t_0 \in (0,T]$ function. Then $\underline{m} = 0$, $\overline{m} = 1$, $m^* = 1$, $\lambda_k = -k^2$ for $k \in \mathbb{N}$, and (4.1)–(4.4) has the form

$$D_t^{5/2}\frac{\partial^4 v}{\partial\xi^4}(\xi,t) = \left(b_1\frac{\partial^2}{\partial\xi^2} + b_2\frac{\partial^4}{\partial\xi^4}\right)D_t^{1/2}v(s,t) + \left(c_0 + c_2\frac{\partial^4}{\partial\xi^4}\right)D_t^{2/3}v(s,t) +$$
$$+\left(d_0 + d_1\frac{\partial^2}{\partial\xi^2}\right)J_t^{1/2}v(\xi,t) + w(\xi),\ (\xi,t) \in (0,\pi)\times(0,T],$$
$$v(0,t) = v(\pi,t) = \frac{\partial^2 v}{\partial s^2}(0,t) = \frac{\partial^2 v}{\partial s^2}(\pi,t) = 0,\ \ t \in (0,T],$$
$$D^{1/2}v(\xi,0) = v_1(\xi),\quad D^{3/2}v(\xi,0) = v_2(\xi),\quad \xi \in (0,\pi),$$
$$v(\xi,t_0) = v_T(\xi),\quad \xi \in (0,\pi).$$

The necessary and sufficient condition (4.5) of its well-posedness has the form

$$\left|\int_\Gamma \Big(k^4\lambda^{3/2} + (b_1k^2 - b_2k^4)\lambda^{-1/2} - (c_0 + c_2k^4)\lambda^{-1/3} -\right.$$
$$\left. -(d_0 - d_1k^2)\lambda^{-3/2}\Big)^{-1}(e^{\lambda T} - 1)d\lambda\right| \ge c \quad \forall k \in \mathbb{N}.$$

Consider the degenerate case. Suppose that $P_1(\lambda_k) = 0$ for some $k \in \mathbb{N}$, while $P_2^{j_0} \not\equiv 0$, $P_2^j \equiv 0$, $j = j_0+1, j_0+2, \dots, m-1$, for some $j_0 \in \{0,1,\dots,m-1\}$, and $\alpha_n > \alpha - m + j_0$. Then provided that the polynomials P_1 and P_3^n do not have common roots on the set $\{\lambda_k\}$, the operator N_n $(L,0)$-bounded (see [22]), while the projections have the form

$$P = \sum_{P_1(\lambda_k)\ne 0}\langle\cdot,\varphi_k\rangle\varphi_k,\quad Q = \sum_{P_1(\lambda_k)\ne 0}\langle\cdot,\varphi_k\rangle\varphi_k.$$

By Remark 1, we take the initial conditions in the form

$$D_t^{\alpha-m+k}P_1(\mathcal{A})u(\xi,0) = y_k(\xi),\ k = m^*, m^*+1,\dots,m-1,\quad \xi\in\Omega, \tag{4.6}$$

$$\langle D_t^{\alpha_n-m_n+k}u(\cdot,0),\varphi_j\rangle = c_{kj},\ P_1(\lambda_j) = 0,\ k = m_0^*, m_0^*+1,\dots,m_n-1. \tag{4.7}$$

Here conditions (4.7) are given for $j \in \mathbb{N}$ such that $P_1(\lambda_j) = 0$; this set is finite, since P_1 is a polynomial. The finite set of n numbers c_{kj} defines the projection $D_t^{\alpha_n-m_n+k}u(\cdot,0)$ to the subspace $\mathcal{X}^0 := \ker P$, $k = m_0^*, m_0^*+1,\dots,m_n-1$. The defects m^*, m_0^* are determined as above. Now, (4.1)–(4.3), (4.6), (4.7) is representable in the form (3.1), (3.18)–(3.20) with the spaces $\mathcal{X}$ and $\mathcal{Y}$ and the operators L, M_j, N_l, S_s, $j = 1,2,\dots,m-1$, $l = 1,2,\dots,n$, $s = 1,2,\dots,r$, which are chosen above.

Theorem 3 implies that problem (4.1)–(4.3), (4.6), (4.7) is well-posed if and only if for all $k \in \mathbb{N}$ such that $P_1(\lambda_k) \ne 0$,

$$\left|\int_0^T\int_0^t\int_{\Gamma_1} R_{\lambda,k}e^{\lambda(t-s)}d\lambda\varphi(s)dsd\mu(t)\right| \ge c, \tag{4.8}$$

and for all $k \in \mathbb{N}$ such that $P_1(\lambda_k) = 0$,

$$\int_0^T \int_0^t \int_{\Gamma_0} R^0_{\lambda,k} e^{\lambda(t-s)} d\lambda \varphi(s) ds d\mu(t) \neq 0, \tag{4.9}$$

$$R^0_{\lambda,k} := \left(\sum_{j=1}^{m-1} \lambda^{\alpha+j-m} P_2^j(\lambda_k) + \sum_{l=1}^{n} \lambda^{\alpha_l} P_3^l(\lambda_k) + \sum_{s=1}^{r} \lambda^{-\beta_s} P_4^s(\lambda_k) \right)^{-1}.$$

EXAMPLE 2. Take $\alpha = 5/2$, $m = 3$, $n = 1$, $r = 1$, $\alpha_1 = 2/3$, $\beta_1 = 1/2$, $P_1(\lambda) = \lambda(\lambda+9)$, $P_2^1(\lambda) = b_1\lambda + b_2\lambda^2$, $P_2^2(\lambda) \equiv 0$, $P_3^1(\lambda) = c_0 + c_2\lambda^2$, $P_4^1(\lambda) = d_0 + d_1\lambda$, $d = 1$, $\Omega = (0, \pi)$, $\rho = 1$, $\mathcal{A}u = \frac{\partial^2 u}{\partial \xi^2}$, $\mathcal{B}_1 = I$, μ is the single jump at $t_0 \in (0, T]$ function. Then $\underline{m} = 0$, $\overline{m} = 1$, $m^* = 1$, $j_0 = 1$, $\alpha_1 > \alpha - m + j_0$, $m_1 = 1$, $\underline{m}_0 = 1$, $\overline{m}_0 = 0$ and $m_0^* = 0$, and problem (4.1)–(4.3), (4.6), (4.7) has the form

$$\begin{gathered}
D_t^{5/2}\left(\frac{\partial^4}{\partial\xi^4} + 9\frac{\partial^2}{\partial\xi^2}\right) v(\xi,t) = \left(b_1\frac{\partial^2}{\partial\xi^2} + b_2\frac{\partial^4}{\partial\xi^4}\right) D_t^{1/2} v(\xi,t) + \\
+\left(c_0 + c_2\frac{\partial^2}{\partial\xi^2}\right) D_t^{2/3} v(\xi,t) + \left(d_0 + d_1\frac{\partial^2}{\partial\xi^2}\right) J_t^{1/2} v(\xi,t) + w(\xi), \\
(\xi,t) \in (0,\pi)\times(0,T], \\
v(0,t) = v(\pi,t) = \frac{\partial^2 v}{\partial\xi^2}(0,t) = \frac{\partial^2 v}{\partial\xi^2}(\pi,t) = 0, \quad t \in (0,T],\ \xi \in (0,\pi), \\
D^{1/2}\left(\frac{\partial^4}{\partial\xi^4} + 9\frac{\partial^2}{\partial\xi^2}\right) v(\xi,0) = y_1(\xi),\ \xi \in (0,\pi), \\
D^{3/2}\left(\frac{\partial^4}{\partial\xi^4} + 9\frac{\partial^2}{\partial\xi^2}\right) v(\xi,0) = y_2(\xi),\ \xi \in (0,\pi), \\
\langle J_t^{1/3} v(\cdot,0), \sin 3\xi\rangle = c, \\
v(\xi,t_0) = v_T(\xi), \quad \xi \in (0,\pi).
\end{gathered}$$

Here $c \in \mathbb{C}$, $\langle y_k(\cdot), \sin 3\xi\rangle = 0$, $k = 1, 2$, since $\lambda_k = -k^2$, $k \in \mathbb{N}$, $P_1(\lambda_3) = 0$. Conditions (4.8), (4.9) have the form

$$\begin{gathered}
\left|\int_{\Gamma_1} \left(k^4\lambda^{3/2} + (b_1k^2 - b_2k^4)\lambda^{-1/2} - (c_0 + c_2k^4)\lambda^{-1/3} - \right.\right. \\
\left.\left. -(d_0 - d_1k^2)\lambda^{-3/2}\right)^{-1} (e^{\lambda T} - 1) d\lambda\right| \geq c, \quad k \in \mathbb{N}\setminus\{3\}, \\
\int_{\Gamma_0} \frac{(e^{\lambda T} - 1)d\lambda}{(-9b_1 + 81b_2)\lambda^{-1/2} + (c_0 + 81c_2)\lambda^{-1/3} + (d_0 - 9d_1)\lambda^{-3/2}} \neq 0.
\end{gathered}$$

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Received 29.10.2021

Линейные обратные задачи для уравнений с несколькими производными Римана – Лиувилля

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Аннотация. Рассматриваются вопросы корректности линейных обратных коэффициентных задач для уравнений в банаховых пространствах с несколькими дробными производными Римана – Лиувилля и с ограниченными операторами при них. Получены критерии корректности как для уравнения, разрешенного относительно старшей дробной производной, так и в случае вырожденного оператора при старшей производной в уравнении. В вырожденной задаче исследованы два существенно различных случая: когда дробная часть порядка второй по старшинству производной равна дробной части порядка старшей дробной производной или отличается от нее. Абстрактные результаты использованы при исследовании обратных задач для уравнений в частных производных с многочленами от самосопряженного эллиптического дифференциального по пространственным переменным оператора и с производными Римана – Лиувилля по времени.

Ключевые слова: обратная задача, дробная производная Римана – Лиувилля, вырожденные эволюционные уравнения, начально-краевая задача.

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Известия Иркутского государственного университета.
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Поступила в редакцию 29.10.2021