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On Integration of the Loaded mKdV Equation in the Class of Rapidly Decreasing Functions

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Abstract. The paper is devoted to the integration of the loaded modified Korteweg-de Vries equation in the class of rapidly decreasing functions. It is well known that loaded differential equations in the literature are usually called equations containing in the coefficients or in the right-hand side any functionals of the solution, in particular, the values of the solution or its derivatives on manifolds of lower dimension. In this paper, we consider the Cauchy problem for the loaded modified Korteweg-de Vries equation. The problem is solved using the inverse scattering method, i.e. the evolution of the scattering data of a non-self-adjoint Dirac operator is derived, the potential of which is a solution to the loaded modified Korteweg-de Vries equation in the class of rapidly decreasing functions. A specific example is given to illustrate the application of the results obtained.

Keywords: loaded modified Korteweg-de Vries equation, Jost solutions, inverse scattering problem, Gelfand-Levitan-Marchenko integral equation, evolution of scattering data.

1. Introduction

The inverse scattering method (ISM) traces its origins to the work of Gardner, Greene, Kruskal and Miura [19]. They managed to find a global solution to the Cauchy problem for the Korteweg-de Vries (KdV) equation by reducing it to the inverse scattering problem for the Sturm-Liouville operator. In this direction, the following important result was obtained by V.E. Zakharov and A.B. Shabat [6]. They succeded to integrate the nonlinear Schrödinger equation (NLS). Soon M. Wadati [30], relying on the ideas of [6], proposed a method for solving the Cauchy problem for the modified Korteweg-de Vries equation (mKdV):

$$u_t + 6u^2 u_x + u_{xxx} = 0,$$

where subscripts denote the corresponding partial derivatives, u is a real scalar function.

The mKdV equation can be applied in many areas, including Alfven waves in collisionless plasma [21], hyperbolic surfaces [27], thin elastic rods [25], etc. Due to the simple expression and rich physical application of the mKdV equation, there are many results devoted to the integration of this equation [3; 8; 11; 15; 24; 31; 34]. In addition, some authors have studied in detail the more extensive form of the mKdV equation, for example, the paired mKdV [28], the multicomponent form [26; 32], and the matrix form [12; 18; 29]. In the paper [22], a deformed mKdV equation with a nonholonomic constraint in the form

$$u_t - 6u^2 u_x - u_{xxx} = \omega(x,t), \ \ \omega_x - 2u\left(c^2(t) - \omega^2\right)^{\frac{1}{2}} = 0$$

is considered. V. E. Zakharov, L. A. Takhtadzhyan, L. D. Faddeev [5], M. Ablowitz, D. Kaup, A. Newell and H. Sigur [17] showed that the ISM can also be applied to the solution of the sine-Gordon equation. The application of the ISM to the NLS equation, mKdV and sine-Gordon equations is based on the scattering problem for the Dirac operator on the entire axis. The inverse scattering problem for the Dirac operator on the whole axis was studied in the papers [13; 14].

In the works of A.M. Nakhushev [10] the most general definition of a loaded equation is given and various loaded equations are classified in detail, for example, loaded differential, loaded integral, loaded integro-differential, loaded functional equations, etc., and numerous applications are described. Among the works devoted to loaded equations, one should especially note the works of A.M. Nakhushev [9; 10], A.I. Kozhanov [7] and others.

Note that in the papers [16; 20] the loaded KdV equation was studied in the class of rapidly decreasing complex-valued and real-valued functions, respectively.

In this paper, we study the loaded mKdV equation, namely, consider the following equation

$$u_t + 6u^2 u_x + u_{xxx} + \gamma(t)u(0,t)u_x(x,t) = 0, \tag{1.1}$$

where $\gamma(t)$ is a given continuously differentiable function. The equation (1.1) is considered under the initial condition

$$u(x,0) = u_0(x) \tag{1.2}$$

where the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:
1) for some $\varepsilon > 0$

$$\int_{-\infty}^{\infty} |u_0(x)|\, e^{2\varepsilon|x|} dx < \infty; \tag{1.3}$$

2) non-self-adjoint operator $L(0) = i\begin{pmatrix} \frac{d}{dx} & -u_0(x) \\ -u_0(x) & -\frac{d}{dx} \end{pmatrix}$ has exactly $2N$ eigen-values $\xi_1(0)$, $\xi_2(0), \ldots, \xi_{2N}(0)$ with multiplicities $m_1(0), m_2(0), \ldots,$ $m_{2N}(0)$.

Suppose that the function $u(x,t)$ has the required smoothness and tends to its limits rather quickly as $x \to \pm\infty$ i.e.

$$\int_{-\infty}^{\infty} \left| \frac{\partial^j u(x,t)}{\partial x^j} \right| e^{2\varepsilon|x|} dx < \infty, \quad j = 0,1,2,3. \tag{1.4}$$

The main goal of this work is to obtain representations for the solution $u(x,t)$ of the problem (1.1) - (1.4) within the framework of the inverse scattering method for the non-self-adjoint operator

$$L(t) = i\begin{pmatrix} \frac{d}{dx} & -u(x,t) \\ -u(x,t) & -\frac{d}{dx} \end{pmatrix}.$$

2. Preliminaries

System of equations

$$L(0)Y = \xi Y, \quad -\infty < x < \infty \tag{2.1}$$

possesses Jost solutions with the following asymptotics

$$\left.\begin{aligned} \varphi(x,\xi) &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \\ \bar{\varphi}(x,\xi) &\sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \end{aligned}\right\}, \quad \mathrm{Im}\xi = 0, \quad x \to -\infty,$$
$$\left.\begin{aligned} \psi(x,\xi) &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \\ \bar{\psi}(x,\xi) &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \end{aligned}\right\}, \quad \mathrm{Im}\xi = 0, \quad x \to \infty. \tag{2.2}$$

(Note that $\bar{\varphi}$ is not complex conjugation to φ). For real ξ, pairs of vector functions $\{\varphi, \bar{\varphi}\}$ and $\{\psi, \overline{\psi}\}$ are pairs of linearly independent solutions for the system of equations (2.1). Therefore, the following relations take place:

$$\left.\begin{aligned} \varphi &= a\xi)\bar{\psi} + b(\xi)\psi, \\ \bar{\varphi} &= -\bar{a}(\xi)\psi + \bar{b}(\xi)\bar{\psi} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \psi &= -a(\xi)\bar{\varphi} + \bar{b}(\xi)\varphi, \\ \bar{\psi} &= \bar{a}(\xi)\varphi + b(\xi)\bar{\varphi} \end{aligned}\right\}, \tag{2.3}$$

where $a(\xi) = W\{\varphi, \psi\}$, $b(\xi) = W\{\bar{\varphi}, \psi\}$. It is easy to see that

$$|a(\xi)|^2 + |b(\xi)|^2 = 1, \quad \overline{a}(\xi) = a(-\xi), \quad \overline{b}(\xi) = b(-\xi).$$

The coefficients $a(\xi)$ and $b(\xi)$ are continuous functions for $\operatorname{Im}\xi = 0$ and satisfy the asymptotic equalities:

$$a(\xi) = 1 + O\Big(|\xi|^{-1}\Big), \quad b(\xi) = O\Big(|\xi|^{-1}\Big), \quad |\xi| \to \infty.$$

The function $\psi(x,\xi)$ satisfies (see [1], p. 33) the following integral representation

$$\psi(x,\xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi\, x} + \int_x^\infty \mathbf{K}\,(x,s)\, e^{i\xi\; s} ds, \tag{2.4}$$

where $\mathbf{K}\,(x,s) = \begin{pmatrix} K_1\,(x,s) \\ K_2\,(x,s) \end{pmatrix}$. In the representation (2.4), the kernel $\mathbf{K}\,(x,s)$ does not depend on ξ and the equality

$$u\,(x) = -2K_1\,(x,x) \tag{2.5}$$

holds.

Theorem 1. *([4] p. 311) If the function $u(x)$ satisfies condition* (1.3)*, then the Jost solutions $\varphi(x,\xi)$, $\psi(x,\xi)$ are analytic functions of the variable ξ for $\operatorname{Im}\xi > -\varepsilon$.*

Corollary 1. *([4] p. 314) If the function $u(x)$ satisfies condition* (1.3)*, then for each $\varepsilon > 0$ there exists a constant C_ε such that $|\mathbf{K}(x,y)| \leq C_\varepsilon e^{-\varepsilon(x+y)}$.*

Moreover, the function $a(\xi)$ admits an analytic continuation to $\operatorname{Im}\xi > -\varepsilon$ and has the asymptotics below

$$a(\xi) = 1 + O\Big(|\xi|^{-1}\Big), \quad |\xi| \to \infty,$$

and has a finite number of (in the general case, multiple) zeros there. Further, let us denote by m_k the multiplicity of the root ξ_k of the equation $a(\xi) = 0$. Nonreal zeros $\{\xi_k\}_{k=1}^N$ of the function $a(\xi)$ are eigenvalues of the operator $L(0)$ in $\operatorname{Im}\xi > 0$. The eigenvalues of the operator $L(0)$ in $\operatorname{Im}\xi < 0$ coincide with the zeros of the function $\bar{a}(\xi)$. The zeros of the function $a(\xi)$ ($\bar{a}(\xi)$) that lie in $-\varepsilon < \operatorname{Im}\xi < 0$ ($0 < \operatorname{Im}\xi < \varepsilon$) are not eigenvalues of the operator $L(0)$. So, the set $\{\xi_k, -\xi_k\}_{k=1}^N$ is the eigenvalues of the operator $L(0)$, and this operator has no other eigenvalues. The requirement that there are no spectral singularities for the non-self-adjoint operator $L(0)$ means that the function $a(\xi)$ does not have real zeros, i.e. $a(\xi) \neq 0$, $\xi \in \mathbb{R}$.

Definition 1. *Functions*

$$\overset{(s)}{\varphi}(x,\xi_k) \equiv \frac{\partial^s}{\partial \xi^s}\varphi(x,\xi)\Big|_{\xi=\xi_k}, \quad s=\overline{1,\,m_k-1},$$

are called associated functions to the eigenfunction $\varphi(x,\xi_k)$.

The associated functions to the eigenfunction $\psi(x,\xi_k)$ are defined similarly.

The eigenfunctions and associated functions satisfy the equations

$$L\overset{(s)}{\varphi}(x,\xi_k) = \xi_k\overset{(s)}{\varphi}(x,\xi_k) + s\overset{(s-1)}{\varphi}(x,\xi_k),$$
$$\overset{(0)}{\varphi}(x,\xi_k) \equiv \varphi(x,\xi_k), \quad k=\overline{1,N}, \quad s=\overline{0,\,m_k-1}\,.$$

There is such a chain of numbers $\left\{\chi_0^k,\,\chi_1^k,\dots,\,\chi_{m_k-1}^k\right\}$ that the relations ([2], [14])

$$\overset{(l)}{\varphi}(x,\xi_k) = \sum_{\nu=0}^{l}\chi_{l-\nu}^k\frac{l!}{\nu!}\overset{(\nu)}{\psi}(x,\xi_k), \quad k=\overline{1,\,N}, \quad l=\overline{0,\,m_k-1} \tag{2.6}$$

hold.

Definition 2. *Sequence of numbers* $\left\{\chi_0^k,\,\chi_1^k,\dots,\,\chi_{m_k-1}^k\right\}$ *are called the normalizing chain of the non-self-adjoint operator* $L(0)$.

The components of the kernel $\mathbf{K}(x,y)$ in the representation (2.4) for $y>x$ are solutions of the Gelfand - Levitan - Marchenko system of integral equations

$$\begin{cases} K_2(x,y) + \displaystyle\int_x^{\infty} K_1(x,s)F(s+y)ds = 0, \\ -K_1(x,y) + F(x+y) + \displaystyle\int_x^{\infty} K_2(x,s)F(s+y)ds = 0, \end{cases}$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r^+(\xi)e^{i\xi x}d\xi - i\sum_{k=1}^{N}\sum_{\nu=0}^{m_k-1}\chi_{m_k-\nu-1}^k\frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\frac{(z-\xi_k)^{m_k}}{a(z)}e^{izx}\right]\Bigg|_{z=\xi_k},$$

$r^+(\xi) \equiv \dfrac{b(\xi)}{a(\xi)}$, $(|\mathrm{Im}\,\xi|<\varepsilon)$, $a(z)$ is an analytic continuation of the function $a(\xi)$ to $\mathrm{Im}z > -\varepsilon$. Now the potential $u(x)$ is determined from the equality (2.5).

Definition 3. *A set of quantities*

$$\left\{r^{+}(\xi),\ \ \xi\in\mathbb{R};\quad \xi_k,\ \ \mathrm{Im}\,\xi_k>0;\quad \chi_j^k,\quad k=\overline{1,\,N};\ \ j=\overline{0,\,m_k-1}\right\}$$

is called the scattering data for the (2.1) *system.* .

3. Evolution of scattering data

Let the potential $u(x,t)$ in the system of equations (2.1) be a solution to the equation

$$u_t+6u^2u_x+u_{xxx}=G(x,t), \tag{3.1}$$

where $G(x,t)=-\gamma(t)u(0,t)u_x(x,t)$.

Operator

$$A=\begin{pmatrix} -4i\xi^3+2iu^2\xi & 4u\xi^2+2iu_x\xi-2u^3-u_{xx} \\ -4u\xi^2+2iu_x\xi\ \ +2u^3+u_{xx} & 4i\xi^3-2iu^2\xi \end{pmatrix} \tag{3.2}$$

satisfies the Lax relation

$$[L,A]\equiv LA-AL=i\begin{pmatrix} 0 & -6u^2u_x-u_{xxx} \\ -6u^2u_x-u_{xxx} & 0 \end{pmatrix}. \tag{3.3}$$

Therefore, the equation (3.1) can be rewritten as

$$L_t+[L,A]=iR, \tag{3.4}$$

where $R=\begin{pmatrix} 0 & -G \\ -G & 0 \end{pmatrix}$.

Differentiating the equality $L\varphi=\xi\varphi$ with respect to t, we obtain

$$L_t\varphi+L\varphi_t=\xi\varphi_t,$$

which according to (3.4) can be rewritten as

$$(L-\xi)(\varphi_t-A\varphi)=-iR\varphi. \tag{3.5}$$

Using the method of variation of constants, we can write

$$\varphi_t-A\varphi=\alpha(x)\psi+\beta(x)\varphi, \tag{3.6}$$

where the functions $\alpha(x)$ and $\beta(x)$ are to be defined. Then to determine $\alpha(x)$ and $\beta(x)$ we get

$$M\alpha_x\psi+M\beta_x\varphi=-R\varphi, \tag{3.7}$$

where $M=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

To solve the equation (3.7), it is convenient to introduce the following notation $\hat{\varphi}=\begin{pmatrix}\varphi_2\\ \varphi_1\end{pmatrix}$, $\hat{\psi}=\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}$. According to (3.3) and the definition of Wronskian, the following equalities are true

$$\hat{\psi}^T M\phi=-\hat{\varphi}^T M\psi=a,\quad \hat{\psi}^T M\psi=\hat{\varphi}^T M\varphi=0.$$

Multiplying (3.7) by $\hat{\varphi}^T$ and $\hat{\psi}^T$, we get

$$\alpha_x=\frac{\hat{\varphi}^T R\varphi}{a},\quad \beta_x=-\frac{\hat{\psi}^T R\varphi}{a}. \tag{3.8}$$

According to (3.2) for $x\to-\infty$ we have

$$\varphi_t-A\varphi\to-\begin{pmatrix}-4i\xi^3 & 0\\ 0 & 4i\xi^3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}e^{-i\xi\, x}=\begin{pmatrix}4i\xi^3\\ 0\end{pmatrix}e^{-i\xi\, x},$$

therefore, based on (3.6) for $x\to-\infty$, we have $\beta(x)\to 4i\xi^3$, $\alpha(x)\to 0$. Hence, from (3.8) we can determine

$$\beta\left(x\right)=-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^T R\varphi dx+4i\xi^3,\quad \alpha\left(x\right)=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^T R\varphi\, dx.$$

Thus, the equality (3.6) has the form

$$\varphi_t-A\varphi=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^T R\varphi\, dx\psi+\left(-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^T R\varphi\, dx+4i\xi^3\right)\varphi. \tag{3.9}$$

According to (2.3), the equality (3.9) takes the form

$$a_t\bar{\psi}+b_t\psi-A\left(a\bar{\psi}+b\psi\right)=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^T R\varphi dx\psi$$
$$+\left(-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^T R\varphi\, dx+4i\xi^3\right)\left(a\bar{\psi}+b\psi\right).$$

Using (3.2) and passing in the last equality to the limit $x\to+\infty$, we obtain

$$a_t=-\int_{-\infty}^{\infty}\hat{\psi}^T R\varphi\, dx, \tag{3.10}$$

$$b_t=-\int_{-\infty}^{\infty}\hat{\varphi}^T R\varphi\, dx-\frac{b}{a}\int_{-\infty}^{\infty}\hat{\psi}^T R\varphi\, dx+8i\xi^3 b. \tag{3.11}$$

Therefore, from the equality $\dfrac{dr^+}{dt}=\dfrac{b_t a-a_t b}{a^2}$ it follows that

$$\frac{dr^+}{dt}=8i\xi^3 r^+-\frac{1}{a^2}\int_{-\infty}^{\infty}G\left(\varphi_1^2+\varphi_2^2\right)dx. \tag{3.12}$$

Lemma 1. *If vector functions* $\varphi = \begin{pmatrix} \varphi_1(x,\xi) \\ \varphi_2(x,\xi) \end{pmatrix}$ *and* $\psi = \begin{pmatrix} \psi_1(x,\xi) \\ \psi_2(x,\xi) \end{pmatrix}$ *are solutions of the equation* (2.1), *then their components satisfy the equalities*

$$\int_{-\infty}^{\infty} G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx = 0, \tag{3.13}$$

$$\int_{-\infty}^{\infty} G\left(\varphi_1^2 + \varphi_2^2\right) dx = 2i\xi\gamma(t)u(0,t)a(\xi)b(\xi)\,. \tag{3.14}$$

Proof. Indeed, using the formulas (1.4), (2.1), (2.2) and (2.3), we have

$$\int_{-\infty}^{\infty} G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx = -\gamma(t)u(0,t)\int_{-\infty}^{\infty} u_x\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx$$

$$= -\lim_{R\to\infty}\gamma(t)u(0,t)\big[u(x,t)\left(\varphi_1\psi_1 + \varphi_2\psi_2\right)\big]\Big|_{-R}^{R}$$

$$+\gamma(t)u(0,t)\int_{-\infty}^{\infty} u\left(\varphi_1'\psi_1 + \varphi_1\psi_1' + \varphi_2'\psi_2 + \varphi_2\psi_2'\right) dx$$

$$= -\lim_{R\to\infty}\gamma(t)u(0,t)\big[u(x,t)\big(\varphi_1\psi_1 + \varphi_2\psi_2\big)\big]\Big|_{-R}^{R}$$

$$+\gamma(t)u(0,t)\int_{-\infty}^{\infty}\Big(\varphi_1'\left(-\psi_2' + i\xi\psi_2\right) + \psi_1'\left(-\varphi_2' + i\xi\varphi_2\right) + \varphi_2'\left(\psi_1' + i\xi\psi_1\right)$$

$$+\psi_2'\left(\varphi_1' + i\xi\varphi_1\right)\Big)dx = -\lim_{R\to\infty}\gamma(t)u(0,t)\big[u(x,t)\left(\varphi_1\psi_1 + \varphi_2\psi_2\right)\big]\Big|_{-R}^{R}$$

$$+i\xi\gamma(t)u(0,t)\int_{-\infty}^{\infty}\left(\varphi_1\psi_2 + \varphi_1\psi_1\right)' dx$$

$$= -\lim_{R\to\infty}\gamma(t)u(0,t)\big[u(x,t)\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) + i\xi\left(\varphi_1\psi_2 + \varphi_1\psi_1\right)\big]\Big|_{-R}^{R} = 0.$$

The following integral is calculated in the same way:

$$\int_{-\infty}^{\infty} G\left(\varphi_1^2 + \varphi_2^2\right) dx = -\gamma(t)u(0,t)\int_{-\infty}^{\infty} u_x\left(\varphi_1^2 + \varphi_2^2\right) dx$$

$$= -\gamma(t)u(0,t)\;u\left(\varphi_1^2 + \varphi_2^2\right)\Big|_{-\infty}^{\infty} + 2\gamma(t)u(0,t)\int_{-\infty}^{\infty}\left(u\varphi_1\varphi_1' + u\varphi_2\varphi_2'\right) dx$$

$$= 2\gamma(t)u(0,t)\int_{-\infty}^{\infty}\Big[\left(-\varphi_2' + i\xi\varphi_2\right)\varphi_1' + \left(\varphi_1' + i\xi\varphi_1\right)\varphi_2'\Big] dx =$$

$$= 2i\xi\gamma(t)u(0,t)\int_{-\infty}^{\infty}\left(\varphi_1\varphi_2\right)' dx = 2i\xi\gamma(t)u(0,t)\lim_{R\to\infty}\left(\varphi_1\varphi_2\right)\Big|_{-R}^{R}$$

$$= 2i\xi\gamma(t)u(0,t)a\,(\xi)\,b\,(\xi)\,.$$

□

Corollary 2. *According to Lemma 1 and the equality* (3.10)*, we have* $a_t = 0$*, therefore*

$$m_k(t) = m_k(0), \quad \xi_k(t) = \xi_k(0), \quad k = \overline{1, N}. \tag{3.15}$$

By (3.12) and (3.14) we have

$$\frac{dr^+}{dt} = \left(8i\xi^3 - 2i\xi\gamma(t)u(0,t)\right) r^+, \quad \left(|\operatorname{Im}\xi| < \varepsilon\right). \tag{3.16}$$

Let us proceed to finding the evolution of the normalizing chain $\left\{\chi_0^n,\ \chi_1^n,\ \ldots,\ \chi_{m_k-1}^n\right\}$ corresponding to the eigenvalue ξ_n, $n = \overline{1,\ N}$. To do this, we rewrite the equality (3.9) as

$$\varphi_t - A\varphi = \frac{1}{a}\left(\int_{-\infty}^{x} G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx\ \varphi - \int_{-\infty}^{x} G\left(\varphi_1^2 + \varphi_2^2\right) dx\ \psi\right) + 4i\xi^3\varphi. \tag{3.17}$$

First, using the formulas (1.4) - (2.3), we calculate the following integral:

$$\int_{-\infty}^{x} G\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx = -\gamma(t)u(0,t)\int_{-\infty}^{x} u_x\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) dx$$

$$= -\gamma(t)u(0,t)\lim_{R\to\infty} u\cdot\left(\varphi_1\psi_1 + \varphi_2\psi_2\right)\Big|_{-R}^{x}$$

$$+\gamma(t)u(0,t)\int_{-\infty}^{x} u\left(\varphi_1'\psi_1 + \varphi_1\psi_1' + \varphi_2'\psi_2 + \varphi_2\psi_2'\right) dx$$

$$= -\gamma(t)u(0,t)u(x,t)\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) + i\xi\gamma(t)u(0,t)\int_{-\infty}^{x}\left(\varphi_1\psi_2 + \varphi_2\psi_1\right)' dx$$

$$= -\gamma(t)u(0,t)u(x,t)\left(\varphi_1\psi_1 + \varphi_2\psi_2\right) + 2i\xi\gamma(t)u(0,t)\varphi_2\psi_1.$$

The following equality is shown in the same way:

$$\int_{-\infty}^{x} G\left(\varphi_1^2 + \varphi_2^2\right) dx = -\gamma(t)u(0,t)u(x,t)\left(\varphi_1^2 + \varphi_2^2\right) + 2i\xi\gamma(t)u(0,t)\varphi_1\varphi_2.$$

Based on the above, the equality (3.17) can be rewritten as

$$\varphi_t - A\varphi = \frac{1}{a}\Big[\gamma(t)u(0,t)u\left(\varphi_1^2\psi + \varphi_2^2\psi - \varphi_2\psi_2\varphi - \varphi_1\psi_1\varphi\right)$$

$$+2i\xi\gamma(t)u(0,t)\left(\varphi_2\psi_1\varphi - \varphi_1\varphi_2\psi\right)\Big] + 4i\xi^3\varphi \tag{3.18}$$

$$= \gamma(t)u(0,t)u\begin{pmatrix} -\varphi_2 \\ \varphi_1 \end{pmatrix} - 2i\xi\gamma(t)u(0,t)\begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} + 4i\xi^3\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

Differentiating the equality (3.18) (m_n-1) times by ξ and setting $\xi=\xi_n$, we get

$$\frac{\partial \overset{(m_n-1)}{\varphi_n}}{\partial t}-A_0\overset{(m_n-1)}{\varphi_n}-(m_n-1)A_1\overset{(m_n-2)}{\varphi_n}-\frac{(m_n-1)(m_n-2)}{2}A_2\overset{(m_n-3)}{\varphi_n}-$$
$$-\frac{(m_n-1)(m_n-2)(m_n-3)}{6}A_3\overset{(m_n-4)}{\varphi_n}=\gamma(t)u(0,t)u\begin{pmatrix}-\overset{(m_n-1)}{\varphi_{2n}}\\ \overset{(m_n-1)}{\varphi_{1n}}\end{pmatrix}$$
$$-2i\xi\gamma(t)u(0,t)\begin{pmatrix}0\\ \overset{(m_n-1)}{\varphi_{2n}}\end{pmatrix}-2i(m_n-1)\gamma(t)u(0,t)\begin{pmatrix}0\\ \overset{(m_n-2)}{\varphi_{2n}}\end{pmatrix}$$
$$+4i\xi_n^3\overset{(m_n-1)}{\varphi_n}+12i\xi_n^2(m_n-1)\overset{(m_n-2)}{\varphi_n}+12i\xi_n(m_n-1)(m_n-2)\overset{(m_n-3)}{\varphi_n}+$$
$$+4i(m_n-1)(m_n-2)(m_n-3)\overset{(m_n-4)}{\varphi_n}, \tag{3.19}$$

where $A_l=\frac{d^l}{d\xi^l}A\big|_{\xi=\xi_n}$, $l=0,1,2,3$. Using (1.4), (2.6), taking into account Corollary 1 of Theorem 1, passing in the equality (3.19) to a redistribution at $x\to\infty$ and equating the coefficients at $\begin{pmatrix}0\\1\end{pmatrix}(ix)^l\cdot e^{i\xi_n x}$, $l=m_n-1,\, m_n-2,\, \ldots,\, 0$, we get

$$\frac{d\chi_0^n}{dt}=\big(8i\xi_n^3-2i\xi_n\gamma(t)u(0,t)\big)\,\chi_0^n,$$
$$\frac{d\chi_1^n}{dt}=\big(8i\xi_n^3-2i\xi_n\gamma(t)u(0,t)\big)\,\chi_1^n+\big(24i\xi_n^2-2i\gamma(t)u(0,t)\big)\,\chi_0^n,$$
$$\frac{d\chi_2^n}{dt}=\big(8i\xi_n^3-2i\xi_n\gamma(t)u(0,t)\big)\,\chi_2^n+\big(24i\xi_n^2-2i\gamma(t)u(0,t)\big)\,\chi_1^n$$
$$+24i\xi_n\chi_0^n, \tag{3.20}$$
$$\frac{d\chi_3^n}{dt}=\big(8i\xi_n^3-2i\xi_n\gamma(t)u(0,t)\big)\chi_3^n+\big(24i\xi_n^2-2i\gamma(t)u(0,t)\big)\chi_2^n$$
$$+24i\xi_n\chi_1^n+8i\chi_0^n,$$
$$\frac{d\chi_l^n}{dt}=\big(8i\xi_n^3-2i\xi_n\gamma(t)u(0,t)\big)\,\chi_l^n+\big(24i\xi_n^2-2i\xi_n\gamma(t)u(0,t)\big)\,\chi_{l-1}^n$$
$$+24i\xi_n\chi_{l-2}^n+8i\chi_{l-3}^n,\quad n=1,\,2,\ldots,N,\;\; l=4,\;5,\ldots,m_n-1.$$

Thus, we have proved the following theorem.

Theorem 2. *If the function* $u(x,t)$ *is a solution to the problem* (1.1) - (1.4), *then the scattering data of the non-self-adjoint operator* $L(t)$ *with the potential* $u(x,t)$ *satisfy the differential equations* (3.15), (3.16) *and* (3.20).

The obtained equalities completely determine the evolution of the scattering data, which makes it possible to apply the inverse scattering method to solve the problem (1.1) – (1.4).

Example 1. Consider the following problem

$$u_t + 6u^2 u_x + u_{xxx} + \gamma(t)u(0,t)u_x = 0, \tag{3.21}$$

$$u(x,0) = -\frac{2}{\operatorname{ch} 2x}, \tag{3.22}$$

where $\gamma(t) = 2(t^2+1) + \dfrac{t^2+1}{2\sqrt{t^2+2}}$.

It is not difficult to find the scattering data for the operator $L(0)$:

$$N = 1, \quad r^+(0) = 0, \quad \xi_1(0) = i, \quad \chi_0(0) = 2i.$$

By Theorem 1, we have

$$\xi_1(t) = \xi_1(0) = i; \quad r^+(t) = 0, \quad \chi_0(t) = 2ie^{8t+2\beta(t)},$$

where

$$\beta(t) = \int_0^t \gamma(\tau)u(0,\tau)d\tau.$$

Consequently $F(x,t) = 2e^{(-x+8t+2\beta(t))}$. Solving the integral equation

$$K_1(x,y) - F(x+y) + \int_x^\infty \int_x^\infty K_1(x,z)F(z+s)F(s+y)dsdz = 0,$$

we can get

$$K_1(x,y) = \frac{2\exp\{-x-y+8t+2\beta(t)\}}{1+\exp\{-4x+16t+4\beta(t)\}}.$$

Whence we find the solution to the Cauchy problem (3.21) - (3.22)

$$u(x,t) = -\frac{2}{\operatorname{ch}2\left(x + \operatorname{arcsh}\frac{t}{\sqrt{2}}\right)}.$$

4. Conclusion

The article shows that the method of the inverse scattering problem can be applied to the integration of the loaded mKdV equation in the case of multiple eigenvalues of the corresponding spectral problem. Facts from the theory of inverse problems for the non-self-adjoint Dirac operator with multiple eigenvalues are presented. The evolution of the normalizing chains for the associated functions of the non-self-adjoint Dirac operator is determined.

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Об интегрировании нагруженного уравнения мКдВ в классе быстроубывающих функций

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Аннотация. Работа посвящена интегрированию нагруженного модифицированного уравнения Кортевега – де Фриза в классе быстроубывающих функций. Хорошо известно, что нагруженными дифференциальными уравнениями в литературе принято называть уравнения, содержащие в коэффициентах или в правой части какие-либо функционалы от решения, в частности значения решения или его производных на многообразиях меньшей размерности. В настоящей работе рассматривается задача Коши для нагруженного модифицированного уравнения Кортевега – де Фриза. Поставленная задача решается с помощью метода обратной задачи рассеяния, т. е. выводится эволюция данных рассеяния несамосопряженного оператора Дирака, потенциал которого является решением нагруженного модифицированного уравнения Кортевега – де Фриза в классе быстроубывающих функций. Приведен конкретный пример, иллюстрирующий применение полученных результатов.

Ключевые слова: нагруженное модифицированное уравнение Кортевега – де Фриза, решения Йоста, обратная задача теории рассеяния, интегральное уравнение Гельфанда – Левитана – Марченко, эволюция данных рассеяния.

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