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On Distributions of Trigonometric Polynomials in Gaussian Random Variables *

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Abstract. We prove new results about the inclusion of distributions of trigonometric polynomials in Gaussian random variables to Nikolskii–Besov classes. In addition, we estimate the total variance distances between distributions of trigonometric polynomials via the L^q-distances between the polynomials themselves.

Keywords: Nikolskii—Besov class, Gaussian measure, distribution of a trigonometric polynomial.

1. Introduction

We study the images of the standard Gaussian measure under trigonometric polynomials. We prove an estimate for the total variation distance between such images in terms of the L^q-distance between the polynomials themselves. Our result is a generalization of the result obtained in [9]. We also discuss the densities of such images and their properties in terms of fractional Sobolev spaces. It was proved in [16] that for any non-const trigonometric polynomial f the image measure $\gamma_n \circ f^{-1}$ has a density from the Nikolski–Besov class B^{α}. However, the proof in [16] had some gaps. Here we explain how to correct the reasoning in [16], moreover, we prove that $\gamma_n \circ f^{-1}$ has a density from the Nikolski–Besov class $B^{\widetilde{\alpha}}$ with $\widetilde{\alpha}$ greater

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than α obtained in [16], which is a stronger result. Concerning distributions of algebraic polynomials, see [3–7; 10; 11; 14; 15].

2. Measures and mappings

The standard Gaussian measure γ_n on $\mathbb{R}^n$ is a probability measure with density $\rho_{\gamma_n}(x)$ with respect to Lebesgue measure on $\mathbb{R}^n$, where

$$\rho_{\gamma_n}(x) = \frac{1}{(\sqrt{2\pi})^n} e^{-\langle x,x\rangle/2}, \quad \langle x,y\rangle = \sum_{i=1}^n x_i y_i. \tag{2.1}$$

Let $f\colon X \to \mathbb{R}$ be a measurable function on a measure space (X,μ) and $\mu \circ f^{-1}$ the image of μ under f defined by

$$\mu \circ f^{-1}(B) = \mu(f^{-1}(B)) \quad \text{where } B \subset \mathbb{R} \text{ is a Borel set.}$$

In terms of probability theory, if f is a random variable and μ is a probability measure, then $\mu \circ f^{-1}$ is the distribution of f. In this paper, we study measures $\mu = \gamma_n \circ f^{-1}$, $\nu = \gamma_n \circ g^{-1}$. The total variation distance d_{TV} between μ and ν on the $\mathbb{R}$ can be defined as follows:

$$d_{\mathrm{TV}}(\mu,\nu) := \sup\left\{ \int_X \varphi\, d(\mu-\nu),\ \varphi \in C_b^\infty(\mathbb{R}^1),\ \|\varphi\|_\infty \le 1 \right\}.$$

For $\mu = \gamma_n \circ f^{-1}$, $\nu = \gamma_n \circ g^{-1}$ the change of variables rule implies that

$$d_{\mathrm{TV}}(\mu,\nu) := \sup\left\{ \int_{\mathbb{R}^n} (\varphi\circ f - \varphi \circ g)\, d\gamma_n,\ \varphi \in C_b^\infty(\mathbb{R}^1),\ \|\varphi\|_\infty \le 1 \right\}.$$

3. Trigonometric polynomials

For a function on $\mathbb{R}^n$ we use the notation $\partial_k f := \frac{\partial}{\partial x_k} f$.

Definition 1. *A function f is a trigonometric polynomial of order d if*

$$f(x) = a_0 + \sum_{k=1}^d (a_k \cdot \cos\langle v_k, x\rangle + b_k \sin\langle v_k, x\rangle), \tag{3.1}$$

where $a_i, b_i \in \mathbb{R}$, each $v_k = (v_{k,1}, \ldots, v_{k,n}) \in \mathbb{Z}^n$ is a non-zero vector with $|v_k| = |v_{k,1}| + \ldots + |v_{k,n}| \le d$. The set of all trigonometric polynomials of order d on $\mathbb{R}^n$ will be denoted by $\mathcal{T}(d,n)$.

Remark 1. The sum in (3.1) has only d terms. This fact has many implications. For example, $\mathcal{T}(d,n)$ is a linear space only when $n=1$. Each $f \in \mathcal{T}(d,n)$ has at most d nonzero terms in (3.1). If $n=1$, then there are exactly d different non-zero elements $v_k > 0$. We can ignore $v_k < 0$ in (3.1), as cos is even and sin is odd. Thus (3.1) provides the decomposition of $f \in \mathcal{T}(d,1)$ with respect to a basis consisting of $2d+1$ functions. For $n > 1$, there are more than d possible non-zero vectors v_k even with all $v_{k,1}, \dots, v_{k,n} \geq 0$. So one can take $f, g \in \mathcal{T}(d,n)$ such that $f + g \notin \mathcal{T}(d,n)$.

Any $f \in \mathcal{T}(d,n)$ can be represented by using complex exponents.

Proposition 1. *Every $f \in \mathcal{T}(d,n)$ has a representation of the form:*

$$f(x) = a_0 + \sum_{k=1}^{d} (c_k \cdot e^{i\langle v_k, x\rangle} + d_k \cdot e^{-i\langle v_k, x\rangle}). \tag{3.2}$$

In this representation of f, the coefficients c_k and d_k belong to $\mathbb{C}$, while the real number a_0 and the vectors v_k are the same as in formula (3.1).

From formula (3.2) one can easily derive the following two results.

Corollary 1. *Let $f \in \mathcal{T}(d,n)$ and $a_1, \dots, a_n \in \mathbb{R}$. Then the functions F_k defined by the formulas*

$$F_1(t) = f(t, a_2, a_3, \dots, a_n),\ F_2(t) = f(a_1, t, a_3, \dots, a_n), \dots$$
$$F_n(t) = f(a_1, a_2, a_3, \dots, t)$$

belong to the set $\mathcal{T}(d,1)$ with respect to the variable $t \in \mathbb{R}$.

Corollary 2. *Let $f \in \mathcal{T}(d,1)$ be non-constant. Then for any $a \in \mathbb{R}$ the equation $f(t) = 0$ has at most $2d$ solutions $t \in [a, a+2\pi]$.*

We now discuss some properties of functions from $\mathcal{T}(d,1)$ related to the Gaussian measure γ_n (see (2.1)). Formula (3.1) guarantees that all functions from the class $T(d,n)$ are bounded, thus, for any $p \in [1,\infty)$ we have

$$T(d,n) \subset L^\infty(\gamma_n) \subset L^p(\gamma_n).$$

We will use $\|\cdot\|_p$ to denote the norm $\|\cdot\|_{L^p(\gamma_n)}$ of the $L^p(\gamma_n)$-space.

Proposition 2. *Let $f \in \mathcal{T}(d,n)$. Then 1) All partial derivatives $\partial_i f$ also belong to $\mathcal{T}(d,n)$. 2) There is a number $C(d)$ such that $\|\partial_i f\|_2 \leq C(d)\|f\|_2$. 3) There is a number $C(d)$ such that $\|\partial_i f + \partial_i g\|_2 \leq C(d)\|f+g\|_2$ for all $f, g \in \mathcal{T}(d,n)$.*

Proof. The fact that $\partial_i f \in \mathcal{T}(d,n)$ follows directly from (3.1).

The definition of $\mathcal{T}(d,n)$ also implies that there are $\frac{1}{2} \cdot ((2d+1)^n - 1)$ distinct non-zero vectors v_k satisfying (3.2) and (3.1). Thus, the definition of $\mathcal{T}(d,n)$ implies that $\mathcal{T}(d,n)$ is a subset of a linear space of dimension $(2d+1)^n-1$. The function $\|f\|_2$ is a norm on this linear space and $\|f\|_{2,\partial_i} = \|\partial_i f\|_2$ is a seminorm on the same space. We deal with a finite-dimensional linear space, hence there is a number $C(d,n)$ such that

$$\|\partial_i f\|_2 = \|f\|_{2,\partial_i} \leq C(d,n)\|f\|_2, \qquad i=1,\dots,n. \tag{3.3}$$

$$\|\partial_i f + \partial_i g\|_2 = \|f+g\|_{2,\partial_i} \leq C(d,n)\|f+g\|_2, \qquad i=1,\dots,n. \tag{3.4}$$

To complete the proof, we need to replace $C(d,n)$ with some $C(d)$ independent of n. Any $v_k = (v_{k,1},\dots,v_{k,n})$ in Definition 1 has at most d nonzero coordinates $v_{k,j}$. Hence, any term in (3.1) and (3.2) depends on at most d variables x_j. So f in (3.1) depends on at most d^2 variables, because the sum in (3.1) has at most d nonzero terms. This enables us to consider only d^2 variables when dealing with one $f \in \mathcal{T}(d,n)$, and $2d^2$ variables when dealing with two functions $f,g \in \mathcal{T}(d,n)$. Hence in (3.3) one has $C(d,n) \leq C(d,d^2)$ and in (3.4) one has $C(d,n) \leq C(d,2d^2)$. □

Proposition 2 implies that not only $\mathcal{T}(d,n) \in L^p(\gamma_n)$, but also $\mathcal{T}(d,n)$ is contained in the Sobolev space $W^{p,1}(\gamma_n)$ of functions $\varphi \in L^p(\gamma_n)$ such that their partial derivatives $\partial_i\varphi$ belong to $L^p(\gamma_n)$. The proposition 2 and the definition 1 imply that $\mathcal{T}(d,n) \subset C_b^\infty(\mathbb{R}^n)$. Let $F,G \in C_b^\infty(\mathbb{R}^n)$. From formula (2.1) for γ_n, we can easily derive the integration by parts formula

$$\int_{\mathbb{R}^n} \partial_k F \cdot G \, d\gamma_n = -\int_{\mathbb{R}^n} (F \cdot \partial_k G - x_k \cdot F \cdot G)\, d\gamma_n. \tag{3.5}$$

As $\mathcal{T}(d,n) \subset C_b^\infty(\mathbb{R}^n)$, the formula (3.5) holds true for $F,G \in \mathcal{T}(d,n)$.

The following theorem is proved in [12].

Theorem 1 (Turan's Lemma)**.** *Let $d \in \mathbb{N}$ and let $I \subset \mathbb{R}$ be an interval. Then there exists a number A_0 such that for each $f \in \mathcal{T}(d,1)$ and each set $E \subset I$ with Lebesgue measure $\mu(E) > 0$ one has*

$$\sup_{t\in I} |p(t)| \leq C(d,|I|) \sup_{t\in E} |p(t)| (A_0|I|)^{2d} \big(\mu(E)\big)^{-2d},$$

where $|I|$ is the length of I and $C(d,|I|)$ depends only on $d \in \mathbb{N}$ and $|I|$.

Let $f \in \mathcal{T}(d,n)$ and $f \neq$ const. If $0 < p < 1$, we define $\|f\|_p = \|f\|_{L^p(\gamma_n)}$ by the same formula as in the case $p \geq 1$. In the case $p = 0$ we set $\|f\|_0 = \lim_{p\to 0} \|f\|_p$. Note that $\|\cdot\|_p$ are not norms in the case $p \in [0,1)$.

There is a number $M(f)$ such that $\gamma_n(x \in \mathbb{R}^n \mid f(x) \geq M(f)) = e^{-1}$. Turan's Lemma yields the following bounds (see [13, Section 2]).

Proposition 3. *For every $f \in \mathcal{T}(d,n)$ one has*

$$\gamma_n(x \in \mathbb{R}^n \mid f(x) \leq (A_0\lambda)^{-2d} M(f)\,) \leq \lambda^{-1}, \tag{3.6}$$

$$\|f\|_p \leq (3A_0 \max(1,2dp))^2 d \cdot M(f), \tag{3.7}$$

$$(eA_0)^{-2d} M(f) \leq \|f\|_0 \leq (3A_0)^2 d \cdot M(f). \tag{3.8}$$

Corollary 3. *If $p, q \in [0, +\infty)$ and $d \in \mathbb{N}$, then there exist numbers $m(d,p,q) > 0$ and $M(d,p,q) > 0$ such that for every $f \in \mathcal{T}(d,n)$*

$$m(d,p,q) \cdot \|f\|_p \le \|f\|_q \le M(d,p,q) \cdot \|f\|_p. \tag{3.9}$$

Proof. If $0 \le p < q \le \infty$, we have $\|f\|_p \le \|f\|_q$ (Hölder's inequality applied to $|f|^p$ and 1). Using (3.7) and (3.8) completes our proof. □

Remark 2. Note that for a fixed number d the numbers $m = m(d,p,q)$ and $M = M(d,p,q)$ in (3.9) do not depend on n.

The following proposition is an analog of the Carbery–Wright inequality (see [8] and [13]) for functions from $\mathcal{T}(d,n)$.

Proposition 4. *For every $d \in \mathbb{N}$ there is a number $c(d)$ such that for all $n \in \mathbb{N}$ and $f \in \mathcal{T}(d,n)$ one has*

$$\gamma_n(|f| \le t) \cdot \|f\|_1^{1/2d} \le c(d) t^{1/2d}, \quad t \ge 0. \tag{3.10}$$

Proof. In (3.6) we take $\lambda = M^{1/2d}(f) \cdot A_0^{-1} \cdot t^{-1/2d}$, $t \ge 0$, apply (3.7) with $p = 1$ and get $\gamma(|f| \le t) \cdot \|f\|_1^{1/2d} \le c(A_0, d) \cdot t^{1/2d}$. □

Corollary 4. *For every $d \in \mathbb{N}$ and $p \ge 1$ there is a number $c(d,p)$ such that for all $n \in \mathbb{N}$ and $f \in \mathcal{T}(d,n)$ one has*

$$\int_{\mathbb{R}^n} \frac{1}{(f^2 + \varepsilon)^p} \, d\gamma_n \le \varepsilon^{-p+1/4d} c(d,p) \|f\|_1^{-1/2d}. \tag{3.11}$$

Proof. The function $\xi = (f^2 + \varepsilon)^{-1}$ is a random variable on $\mathbb{R}^n$ with the measure γ_n. We know that $f^2 \ge 0$, so $\xi \in (0, \varepsilon^{-1})$. Hence

$$\int_{\mathbb{R}^n} (f^2 + \varepsilon)^{-p} \, d\gamma_n = E(\xi^p) = \int_0^{\varepsilon^{-1}} p \cdot t^{p-1} \gamma_n\big((f^2 + \varepsilon)^{-1} \ge t\big) \, dt$$
$$= p \int_0^\infty \frac{\gamma_n\big(f^2 \le s\big)}{(s + \varepsilon)^{p+1}} \, ds = \frac{p}{\varepsilon^p} \int_0^\infty \frac{\gamma_n(|f| \le \sqrt{\varepsilon} \cdot \sqrt{u})}{(u+1)^{p+1}} \, du.$$

This chain of equalities along with (3.10) implies that

$$\int_{\mathbb{R}^n} (f^2 + \varepsilon)^{-p} \, d\gamma_n \le \frac{(\varepsilon)^{1/4d} \cdot c(d,p)}{\varepsilon^p \cdot \|f\|_1^{1/2d}}, \quad c(d,p) := 2c(d)p \int_0^\infty \frac{u^{1/4d}}{(u+1)^{1+p}} \, du.$$

Estimate (3.11) is proved. □

Remark 3. In order to derive the presented results from Turan's Lemma (Theorem 1), it is crucial that we define our set $\mathcal{T}(d,n)$ by formula (3.1). One might expect that in place of $f \in \mathcal{T}(d,n)$ we could consider

$$f(x) = a_0 + \sum_{k=1}^{N} (a_k \cdot \cos\langle v_k, x\rangle + b_k \sin\langle v_k, x\rangle) \tag{3.12}$$

with arbitrary $N \in \mathbb{N}$ and vectors v_k with integer coordinates similar to the ones in Definition 1. However, Theorem 1 cannot be used without the condition $N \leq d$ in (3.12). This fact is evident from Section 2 of [13], especially from the remark made there. Note that any function $f \in \mathcal{T}(d,n)$ satisfies the definition of an exponential polynomial given in the aforementioned remark in [13]. This follows from formula (3.2). Thus, the proofs of the presented results heavily employ the condition $N \leq d$ in (3.12). Notice, however, that $f(x)$ from (3.12) still belongs to $\mathcal{T}(d_1, n)$ with $d_1 = \max(d, N)$. Even in the case $N > d$ we have $f \in \mathcal{T}(\max(d,N), n)$.

Remark 4. Let us mention the following error made in our paper [16]: there we defined trigonometric polynomials in such a way that we actually allowed $N > d$ in (3.12), while still basing our proofs on Turan's Lemma and its corollaries. As explained above, this reasoning is incorrect. The main results of [16] (Theorem 3 and Theorem 4) are not made completely invalid by this error, however. The numbers α in Theorem 3 and Theorem 4 depend not only on d but also on N from (3.12). In theorem 3 it is enough to replace $\alpha = 1/(4d+\tau)$ with $\alpha = 1/(4\max(d,N)+\tau)$. This fact follows from Remark 3 above and Remark 7 below.

Moreover, replacing all $\mathcal{TR}_d(\mathbb{R}^n)$ in [16] with $\mathcal{T}(d,n)$ defined as in Definition 1 of this paper, we can fix most of the errors occurred in [16]. Nevertheless, even after this substitution, some other corrections should be made, which is discussed in Remarks 7 and 8 below.

To formulate our next theorem, we set $\|g\|_\partial = \sup_{k=1,\dots,n} \|\partial_k g\|_2$. We follow the convention that for $g = \text{const}$ one has $\|g\|_\partial^{-1} = +\infty$.

Theorem 2. *For each $d \in \mathbb{N}$ there is a number $C(d)$ such that, for each n and every pair of trigonometric polynomials $f, g \in \mathcal{T}(d,n)$, one has*

$$\|\gamma_n \circ f^{-1} - \gamma_n \circ g^{-1}\|_{\mathrm{TV}} \leq c(d)\big(\|g\|_\partial^{-1/(2d)} + 1\big)\|f-g\|_2^{1/(2d+1)}.$$

Proof. Fix a function $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$. Consider the function

$$u(t) = \int_{-\infty}^{t} \varphi(\tau)\, d\tau.$$

For any $k = 1, \dots, n$ one has $\partial_k(u(g)) = \varphi(g) \cdot \partial_k g$. Thus,

$$(\varphi(f) - \varphi(g))\partial_k g = \partial_k(u(f) - u(g)) - \varphi(f)(\partial_k f - \partial_k g), \tag{3.13}$$

where ∂_k denotes the partial derivative with respect to the variable x_k. Note that the hypotheses of our theorem say that $f(x) = f(x_1, \dots, x_n)$ and $g(x) = g(x_1, \dots, x_n)$. Thus, for each $\varepsilon > 0$ we have

$$\widehat{I} = \int_{\mathbb{R}^n} \big(\varphi(f) - \varphi(g)\big)\, d\gamma_n = \int_{\mathbb{R}^n} \big((\partial_k g)^2 + \varepsilon\big) \frac{\big(\varphi(f) - \varphi(g)\big)}{(\partial_k g)^2 + \varepsilon}\, d\gamma_n \tag{3.14}$$

We apply (3.13) to $\partial_k g$ in (3.14) and get

$$\widehat{I} = \int_{\mathbb{R}^n} \frac{\partial_k g \partial_k (u(f) - u(g))}{(\partial_k g)^2 + \varepsilon} \, d\gamma_n - \int_{\mathbb{R}^n} \frac{\partial_k g \cdot \varphi(f)(\partial_k f - \partial_k g)}{(\partial_k g)^2 + \varepsilon} \, d\gamma_n + \\ + \varepsilon \int_{\mathbb{R}^n} \big(\varphi(f) - \varphi(g)\big)\big((\partial_k g)^2 + \varepsilon\big)^{-1} \, d\gamma_n = T_1 + T_2 + T_3. \quad (3.15)$$

We now estimate each term in (3.15) separately. First, we shall prove that T_1 in (3.15) is bounded by $\widetilde{C}(d)\varepsilon^{-1/2}\|f - g\|_2$. To this end, we first consider the one-dimensional case $n = 1$. In this case $x \in \mathbb{R}$ and therefore $\partial_k g = g'(x)$, $\partial_k^2 g = g''(x)$, $x_k = x$. The integration by parts formula (3.5) and the formula (2.1) allow us to write

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^n} \frac{\partial_k g \partial_k (u(f) - u(g))}{(\partial_k g)^2 + \varepsilon} \, d\gamma_n = \int_{\mathbb{R}} \frac{g'(u(f) - u(g))'}{(g')^2 + \varepsilon} \, d\gamma_1 = \\ &= -\int_{\mathbb{R}} (u(f) - u(g)) \left(\frac{g'' - xg'}{(g')^2 + \varepsilon} - 2\frac{(g')^2 g''}{((g')^2 + \varepsilon)^2} \right) d\gamma_1 \\ &\le \frac{1}{2\varepsilon^{1/2}} \int_{\mathbb{R}} |f(x) - g(x)| \, |x| \, \gamma_1(dx) + 3 \int_{\mathbb{R}} \frac{|f - g| \cdot |g''|}{(g')^2 + \varepsilon} \rho_{\gamma_1}(x) \, dx. \end{aligned} \quad (3.16)$$

Observe that $\mathcal{T}(d, 1)$ is a $(2d + 1)$-dimensional linear space (see Remark 1) and $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ is a norm on it. Hence there is a number $C(d)$ depending only on d such that $|f(x) - g(x)| \le C(d)\|f - g\|_2$. Thus,

$$\int_{\mathbb{R}} \frac{|f - g| \cdot |g''|}{(g')^2 + \varepsilon} \rho_{\gamma_1}(x) \, dx \le C(d) \, \|f - g\|_2 \int_{\mathbb{R}} \frac{|g''(x)|}{|g'(x)|^2 + \varepsilon} \, \rho_{\gamma_1}(x) dx. \quad (3.17)$$

We take the intervals $I_j = [2\pi j; 2\pi j + 2\pi]$ and write

$$\int_{\mathbb{R}} \frac{|g''(x)|}{(g'(x))^2 + \varepsilon} \rho_{\gamma_1}(x) \, dx \le \sum_{j=-\infty}^{+\infty} \sup_{x\in I_j} \rho_{\gamma_1}(x) \int_{I_j} \frac{|g''(x)|}{(g'(x))^2 + \varepsilon} \, dx \quad (3.18)$$

Note that both g' and g'' belong to $\mathcal{T}(d, 1)$ (see Proposition 2). Let's now consider only $x \in I_j$. Each interval I_j is of length 2π. Corollary 2 implies that $g'' \in \mathcal{T}(d, 1)$ has $m \le 2d$ zeros $\tau_1 < \ldots < \tau_m$ in the interval I_j. Consider the intervals (τ_i, τ_{i+1}) with $i = 0, \ldots, m$ and $\tau_0 = 2\pi j$, $\tau_{m+1} = 2\pi j + 2\pi$. On each of the intervals (τ_i, τ_{i+1}) the function g'' has a constant sign. Thus,

$$\int_{\tau_i}^{\tau_{i+1}} \frac{|g''|}{(g')^2 + \varepsilon} \, dx = \operatorname{sign}\big(g''(x)\big) \int_{\tau_i}^{\tau_{i+1}} \frac{1}{(g')^2 + \varepsilon} \, d(g'(x)) \le \frac{\pi}{\sqrt{\varepsilon}}.$$

There are $(m + 1) \le (2d + 1)$ intervals $(\tau_i, \tau_{i+1}) \subset I_j$. Therefore,

$$\int_{I_j} \frac{|g''(x)|}{(g'(x))^2 + \varepsilon} \, dx \le \sum_{i=0}^{m} \frac{\pi}{\sqrt{\varepsilon}} \le \frac{\pi(2d + 1)}{\sqrt{\varepsilon}}$$

Applying this result to (3.18), we have

$$\int_{\mathbb{R}} \frac{|g''(x)|}{(g'(x))^2+\varepsilon}\,d\gamma_1 \leq \frac{\pi(2d+1)}{\sqrt{\varepsilon}} \sum_{j=-\infty}^{+\infty} \sup_{x\in I_j} \rho_{\gamma_1}(x) \leq \widetilde{C}(d)\varepsilon^{-1/2}. \quad (3.19)$$

The series in (3.19) converges (we substitute $\rho_{\gamma_1}(x)$ using (2.1)):

$$\sum_{j=-\infty}^{+\infty} \sup_{x\in I_j} \rho_{\gamma_1}(x) = \sum_{j=-\infty}^{+\infty} \sup_{x\in I_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{2}{\sqrt{2\pi}} \sum_{j=0}^{+\infty} e^{-2\pi^2 j^2} = S_1 < +\infty.$$

Combining (3.19), (3.16) and (3.17), we estimate T_1 from (3.15):

$$T_1 \leq \widetilde{C}_1(d)\frac{\|f-g\|_2}{\sqrt{\varepsilon}} + \frac{1}{2\sqrt{\varepsilon}} \int_{\mathbb{R}} |f(x)-g(x)|\,|x|\,\gamma_1(dx).$$

By the Cauchy–Bunyakovskii inequality the integral on the right is estimated by $\|f-g\|_2 \cdot C$ with $C = \|x\|_2$. Hence we proved the bound

$$T_1 = \int_{\mathbb{R}} \frac{g'(u(f)-u(g))'}{(g')^2+\varepsilon}\,d\gamma_1 \leq \widetilde{C}_2(d)\|f-g\|_2 \frac{1}{\sqrt{\varepsilon}} \quad (3.20)$$

for the first term in (3.15) in the case $n = 1$.

We now proceed to the case $n > 1$. The space $\mathbb{R}^n$ can be decomposed into the sum $\mathbb{R}^n = \langle e_k\rangle \oplus E_k$ with $E_k = \langle e_k\rangle^{\perp}$. If $\mathbb{R}^n = \langle e_k\rangle \oplus E_k$, then the measure γ_n can be represented as $\gamma_n = \gamma_1 \otimes \gamma_{n-1}$, where γ_1 is the standard Gaussian measure on $\langle e_k\rangle \simeq \mathbb{R}$ and γ_{n-1} is the standard Gaussian measure on $E_k = \langle e_k\rangle^{\perp} \simeq \mathbb{R}^{n-1}$. By Fubini's theorem

$$T_1 = \int_{E_k} \int_{\mathbb{R}} \frac{g_x'(t)(u(f_x(t)) - u(g_x(t)))'}{(g_x'(t))^2+\varepsilon}\,d\gamma_1 d\gamma_{n-1}. \quad (3.21)$$

In (3.21) we have $x \in E_k = \langle e_k\rangle^{\perp}$, $t \in \mathbb{R}$, $(\cdot)'$ is the derivative with respect to the variable t, and $f_x(t) := f(x+te_k)$, $g_x(t) := g(x+te_k)$. Due to Corollary 1, for any fixed $x \in E_k$, the functions $f(t) = f_x(t)$ and $g(t) = g_x(t)$ belong to $\mathcal{T}(d,1)$. We can use (3.20) to write

$$\int_{\mathbb{R}} \frac{g_x'(u(f_x)-u(g_x))'}{(g_x')^2+\varepsilon}\,d\gamma_1 \leq C'(d)\|f_x - g_x\|_2 \frac{1}{\sqrt{\varepsilon}},$$

where $f_x = f_x(t)$ and $g_x = g_x(t)$ are regarded as functions of one variable t. Substituting this into (3.21) we get

$$T_1 \leq C'(d)\frac{1}{\sqrt{\varepsilon}}\left(\int_{E_k} \|f_x - g_x\|_2^2\,\gamma_{n-1}(dx)\right)^{1/2} = c_3(d)\frac{1}{\sqrt{\varepsilon}}\|f-g\|_2, \quad (3.22)$$

where also the Cauchy–Bunyakovskii inequality $\|F\|_{L^1(P)} \leq \|F\|_{L^2(P)}$ for the probability measure $P = \gamma_{n-1}$ and $F(x) = \|f_x - g_x\|_2$ is used.

The estimate (3.22) for the term T_1 in (3.15) is valid for any $n \in \mathbb{N}$.

Now we will consider the term T_2 in (3.15). It is easily estimated as follows (here we need the Proposition 2 and the Proposition 3):

$$\begin{aligned} T_2 &= -\int_{\mathbb{R}^n} \frac{\partial_k g \varphi(f)(\partial_k f - \partial_k g)}{(\partial_k g)^2 + \varepsilon}\, d\gamma_n \le \int_{\mathbb{R}^n} \frac{|\partial_k g|\,|\partial_k f - \partial_k g|}{(\partial_k g)^2 + \varepsilon}\, d\gamma_n \\ &\le \int_{\mathbb{R}^n} \frac{|\partial_k f - \partial_k g|}{2\sqrt{\varepsilon}}\, d\gamma_n = \frac{\|\partial_k f - \partial_k g\|_1}{2\varepsilon^{1/2}} \le c_2(d)\varepsilon^{-1/2}\|f-g\|_2. \end{aligned} \tag{3.23}$$

To estimate the last term in (3.15), we apply (3.11) to the trigonometric polynomial $\partial_k g$ and obtain the bound

$$\begin{aligned} T_3 &= \varepsilon \int_{\mathbb{R}^n} \frac{\varphi(f) - \varphi(g)}{(\partial_k g)^2 + \varepsilon}\, d\gamma_n \le 2\varepsilon \int_{\mathbb{R}^n} \frac{1}{\partial_k g^2 + \varepsilon}\, d\gamma_n \\ &\le 2\varepsilon \cdot c(d,1)\|\partial_k g\|_1^{-1/2d}\varepsilon^{-1+1/4d} = c_1(d)\|\partial_k g\|_2^{-1/2d}\varepsilon^{1/4d}. \end{aligned} \tag{3.24}$$

Using (3.22), (3.23) and (3.24) together with (3.15) we get

$$\int_{\mathbb{R}^n} \big(\varphi(f) - \varphi(g)\big)\, d\gamma_n \le C(d)\|f-g\|_2 \frac{1}{\sqrt{\varepsilon}} + c_1(d)\|\partial_k g\|_2^{-1/(2d)}\varepsilon^{1/(4d)}$$

with some new constants. Taking $\varepsilon = \|f-g\|_2^{(4d)/2d+1}$ we obtain

$$\int_{\mathbb{R}^n} \big(\varphi(f) - \varphi(g)\big)\, d\gamma_n \le \Big(C(d) + c_1(d)\|\partial_k g\|_2^{-1/(d-1)}\Big)\|f-g\|_2^{1/2d+1}.$$

Finally, we replace $\|\partial_k g\|_2$ with $\|g\|_\partial = \sup_{k=1,\dots,n}\|\partial_k g\|_2$ and take the supremum over all smooth φ with $\|\varphi\|_\infty \le 1$, completing our proof. □

Remark 5. In case $g \ne$ const, Theorem 2 generalizes the result of Davydov (see [9, Section 4]) to the case of $\mathbb{R}^n$. Davydov's estimate for a pair of non-constant functions $F, G \in \mathcal{T}(d,1)$ is

$$\|\gamma_1 \circ F^{-1} - \gamma_1 \circ G^{-1}\|_{\mathrm{TV}} \le C_{F,G}(d)\|F-G\|_1^{\frac{1}{2d+1}} \le C_{F,G}(d)\|F-G\|_\infty^{\frac{1}{2d+1}},$$

where $C_{F,G}(d)$ depends on the number d and can depend on some Besov–Nikolskii norms of F and G (for a discussion of Besov–Nikolskii spaces, see the next section). Our Theorem 2 provides a similar result, but for $\mathcal{T}(d,n)$ instead of $\mathcal{T}(d,1)$: along with (3.9) it implies that for any pair of non-constant functions $f, g \in \mathcal{T}(d,n)$ with $g \ne$ const one has

$$\|\gamma_n \circ f^{-1} - \gamma_n \circ g^{-1}\|_{\mathrm{TV}} \le C_g(d)\|f-g\|_1^{\frac{1}{2d+1}},$$

where $C_g(d)$ depends on the number d and can depend on $\|g\|_\partial$.

In the next section we show that the Besov–Nikolskii norm of $g \in \mathcal{T}(d,1)$ can be bounded by a number depending on $\|g\|_\partial$. Thus, the result of Theorem 2 has even more similarity with the result of [9, Section 4].

4. Fractional Sobolev spaces

Let ν_h be the shift of a measure ν by a vector h: $\nu_h(A) = \nu(A-h)$.

Let $0 < \alpha < 1$. Let $B^\alpha(\mathbb{R}^k)$ be the Besov–Nikolskii class (see [1]) of all functions $\varrho \in L^1(\mathbb{R}^k)$ such that, for some number $C(\varrho)$, one has

$$\|\varrho(\cdot + h) - \varrho\|_{L^1} \le C(\varrho)|h|^\alpha \quad \forall\, h \in \mathbb{R}^k.$$

If $0 < \alpha < 1$, the class $B^\alpha(\mathbb{R}^k)$ coincides with the set of all densities ρ_ν of bounded Borel measures ν on $\mathbb{R}^k$ for which with some C_ν one has

$$\|\nu_h - \nu\|_{\mathrm{TV}} \le C_\nu |h|^\alpha \quad \forall\, h \in \mathbb{R}^k. \tag{4.1}$$

A measure ν belongs to $B^\alpha(\mathbb{R}^k)$ if its density is in $B^\alpha(\mathbb{R}^k)$; (4.1) implies that ν has a density (see [2, Proposition 3.4.3].) For measures (and functions) from the linear space $B^\alpha(\mathbb{R}^k)$ we use the following norm $\|\cdot\|_{B^\alpha}$:

$$\|\nu\|_{B^\alpha} := \inf\{C\colon\ \|\nu - \nu_h\|_{\mathrm{TV}} \le C|h|^\alpha\}.$$

Note that $B^\alpha(\mathbb{R}^k)$ is not a Banach space with this norm.

To check that a measure ν belongs to the class $B^\alpha(\mathbb{R}^k)$ we employ the following proposition (see [7, Proposition 3.1], [6, Theorem 1], [5, §2]).

Proposition 5. *Let $\alpha \in (0,1)$ and let ν be a Borel measure on $\mathbb{R}^1$ such that for each function $\varphi \in C_b^\infty(\mathbb{R})$ one has*

$$\int_{\mathbb{R}} \varphi'(x)\,\nu(dx) \le C\|\varphi\|_\infty^\alpha\,\|\varphi'\|_\infty^{1-\alpha}.$$

Then $\nu \in B^\alpha(\mathbb{R})$ and $\|\nu\|_{B^\alpha} \le 2^{1-\alpha}C$.

Remark 6. It is enough in Proposition 5 to use only $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \le 1$. For such φ the condition in the proposition can be written as

$$\int_{\mathbb{R}} \varphi'(x)\,\nu(dx) \le C\|\varphi'\|_\infty^{1-\alpha}.$$

We now prove that non-constant trigonometric polynomials $f \in \mathcal{T}(d,n)$ have distributions from the class $\nu \in B^\alpha(\mathbb{R}^k)$. We show that the measure $\gamma_n \circ f^{-1}$ with $f \in \mathcal{T}(d,n)$ belongs to the class $B^\alpha(\mathbb{R}^k)$ for some α.

Theorem 3. *For every $d \in \mathbb{N}$, there is a number $C(d)$ such that, for every $f \in \mathcal{T}(d,n)$ and every $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \le 1$, one has*

$$\int_{\mathbb{R}^n} \varphi'(f)\,d\gamma_n \le C(d)\|f\|_\partial^{-1/2d}\|\varphi'\|_\infty^{1-1/(2d+1)}.$$

Therefore, $\gamma_n \circ f^{-1}$ belongs to the Besov–Nikolskii class $B^{1/(2d+1)}(\mathbb{R})$ provided that f is not a constant.

Proof. We assume that $f \neq \text{const}$, since for $f = \text{const}$ we use the convention $\|f\|_\partial^{-1} = +\infty$, and the inequality in our theorem is trivial in that case.

Consider an arbitrary function $\psi \in C_b^\infty(\mathbb{R})$ with $\|\psi\|_\infty \leq 1$. Then

$$\int_{\mathbb{R}^n} \psi'(f)\, d\gamma = \int_{\mathbb{R}^n} \Big[\frac{(\partial_k f)^2}{(\partial_k f)^2 + \varepsilon}\psi'(f)\Big]\, d\gamma_n + \int_{\mathbb{R}^n} \frac{\varepsilon \cdot \psi'(f)}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n. \tag{4.2}$$

Let us write the first term in (4.2) as

$$\int \frac{(\partial_k f)^2}{(\partial_k f)^2 + \varepsilon}\psi'(f)\, d\gamma_n = \int \partial_k(\psi(f))\frac{\partial_k f}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n.$$

Integrating by parts, we obtain

$$-\int_{\mathbb{R}^n} \psi(f)\Big(\frac{\partial_k^2 f + x_k\, \partial_k f}{(\partial_k f)^2 + \varepsilon} - \frac{2(\partial_k f)^2 \partial_k^2 f}{((\partial_k f)^2 + \varepsilon)^2}\Big)\, d\gamma_n \leq \int_{\mathbb{R}^n} \frac{3\,|\partial_k^2 f|}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n + \\ + \int_{\mathbb{R}^n} \frac{|\partial_k f| \cdot |x_k|}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n \leq \widetilde{C}((6d+3)\sqrt{\pi/2} + 1)\varepsilon^{-1/2}. \tag{4.3}$$

The last step in (4.3) follows from the following two inequalities:

$$\partial_k f \cdot \sqrt{\varepsilon} \leq (\partial_k f)^2 + \varepsilon, \qquad \int_{\mathbb{R}^n} \frac{|\partial_k^2 f|}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n \leq \widetilde{C}\varepsilon^{-1/2}(3d\sqrt{\pi/2}).$$

The second inequality here is proved similarly to (3.19) .

We now use (3.11) with $p = 1$ to write

$$\int_{\mathbb{R}^n} \frac{\psi'(f)}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n \leq \|\psi'\|_\infty \cdot \varepsilon^{-1+1/4d} \cdot c(d,1)\|\partial_k f\|_1^{-1/2d}. \tag{4.4}$$

Using (4.2), (4.3) and (4.4), and taking $\varepsilon = \|\psi'\|_\infty^{-2+2/(2d+1)}$ we obtain

$$\int \psi'(f) d\gamma \leq c_1(d)\|\partial_k f\|_2^{-1/2d}\|\psi'\|_\infty \varepsilon^{1/4d} + c_2(d)\varepsilon^{-1/2} \leq \\ \leq (c_1(d)\|\partial_k f\|_2^{-1/2d} + c_2(d))\|\psi'\|_\infty^{1-1/(2d+1)}.$$

This result holds true for every $k \in \mathbb{N}$. Thus, we have

$$\int \psi'(f)\, d\gamma \leq \Big(c_1(d)\|f\|_\partial^{-1/2d} + c_2(d)\Big)\|\psi'\|_\infty^{1-1/2d+1}. \tag{4.5}$$

Since $f \neq \text{const}$ and $\|f\|_\partial > 0$, the function $f \cdot \|f\|_\partial^{-1}$ belongs to $\mathcal{T}(d,1)$. Applying (4.5) to $g = f \cdot \|f\|_\partial^{-1} \in \mathcal{T}(d,1)$, we obtain

$$\int \psi'(f\|f\|_\partial^{-1})\, d\gamma \leq (c_1(d) + c_2(d))\|\psi'\|_\infty^{1-1/(2d+1)}. \tag{4.6}$$

Let $\varphi \in C_b^\infty(\mathbb{R})$, $\|\varphi\|_\infty \leq 1$, $\psi(t) = \varphi(t\|f\|_\partial)$. Then $\varphi(t) = \psi(t \cdot \|f\|_\partial^{-1})$ and $\varphi'(f) = \|f\|_\partial^{-1} \cdot \psi'(f\|f\|_\partial^{-1})$. Hence, using (4.6) we get

$$\int \varphi'(f)\, d\gamma \leq C(d)\|f\|_\partial^{-1}\|\psi'\|_\infty^{1-1/(2d+1)} = C(d)\|f\|_\partial^{-1/(2d+1)}\|\varphi'\|_\infty^{1-1/(2d+1)},$$

with $C(d) = (c_1(d) + c_2(d))$. Having this estimate, we complete the proof by applying Proposition 5 and Remark 6. □

Remark 7. Let us complete our discussion of the error in [16], explained in Remark 4. As noted in Remark 4, replacing $\mathcal{TR}_d(\mathbb{R}^n)$ in [16] with $\mathcal{T}(d,n)$ we correct some errors in [16], but for some statements this replacement is not sufficient. In some proofs in [16] we used the following fact:

$$\text{If } f, g \in \mathcal{TR}_d(\mathbb{R}^n), \quad \text{then} \quad f \cdot g \in \mathcal{TR}_{2d}(\mathbb{R}^n). \tag{4.7}$$

This is true for $f, g \in \mathcal{TR}_d(\mathbb{R}^n)$, but it is not true for $f, g \in \mathcal{T}(d,n)$ if $n > 1$. One can easily find $f, g \in \mathcal{T}(d,n)$ such that for $h = f \cdot g$ the decomposition

$$h(x) = a_0 + \sum_{k=1}^{N}(c_k \cdot e^{i\langle v_k, x\rangle} + d_k \cdot e^{-i\langle v_k, x\rangle})$$

will have more than $N > 2d$ vectors v_k, which is not allowed for $\mathcal{T}(2d,n)$ (see Definition 1 and Remark 3). So the inclusion $f, g \in \mathcal{T}(d,n)$ does not guarantee that $f \cdot g \in \mathcal{T}(2d,n)$. We now correct the reasoning in [16] relying on (4.7). Both Theorem 3 and Theorem 4 in [16] use (4.7). Let us first discuss Theorem 3. In its proof we use (4.7) when we state that

$$\langle \nabla f; \nabla f\rangle = \sum_{i=1}^{n}(\partial_i f)^2 \quad \text{belongs to } \mathcal{TR}_{2d}(\mathbb{R}^n).$$

With $\mathcal{TR}_d(\mathbb{R}^n)$ replaced by $\mathcal{T}(d,n)$, Theorem 3 from [16] now states:

For every $d \in \mathbb{N}$ and any number $\alpha > \frac{1}{4d}$, there is a number $C(d,\alpha)$ that depends only on d and α such that, for every $f \in \mathcal{T}(d,n)$ and every function $\varphi \in C_b^\infty(\mathbb{R})$ with $\|\varphi\|_\infty \leq 1$, we have

$$\int_{\mathbb{R}^n} \varphi'(f)\, d\gamma_n \leq C(d,\alpha)\sigma_f^{-1/\alpha}\|\varphi\|_\infty^{1/\alpha}\|\varphi'\|_\infty^{1-1/\alpha}. \tag{4.8}$$

Note that the hypotheses in Theorem 3 from [16] are the same as in Theorem 3 above. Combined with Remark 6, Theorem 3 above yields

$$\int_{\mathbb{R}^n} \varphi'(f)\, d\gamma_n \leq C(d)\|f\|_\partial^{-1/2d}\|\varphi\|_\infty^{1/\alpha}|\varphi'\|_\infty^{1-1/\alpha}, \quad \alpha = \frac{1}{2d+1}.$$

So our present paper provides a result similar to (4.8), but with a larger exponent α. Here and in [16] we employ such inequalities to establish the membership of $\gamma_n \circ f^{-1}$ in $B^\alpha(\mathbb{R})$. Therefore, a larger value for α is preferable. Thus, Theorem 3 above gives a better result than Theorem 3 in [16], so in principle there is no need to explain in detail how to correct the reasoning there, nevertheless, this can be done. Indeed, in Theorem 3 in [16] the condition (4.7) is used to prove the first inequality in

$$\int_{\mathbb{R}^n} \frac{1}{\langle \nabla f, \nabla f\rangle + \varepsilon}\, d\gamma_n \le \frac{c(d)}{\|\langle \nabla f, \nabla f\rangle\|_1^{1/4d}} \varepsilon^{-1+1/4d} \le \frac{c(d)}{\sigma_f^{1/2d}} \varepsilon^{-1+1/4d}.$$

But a similar result can be derived from inequality (3.11) of this paper. Note that by (3.11) for all k we have

$$\int_{\mathbb{R}^n} \frac{1}{\langle \nabla f, \nabla f\rangle + \varepsilon}\, d\gamma_n \le \int_{\mathbb{R}^n} \frac{1}{(\partial_k f)^2 + \varepsilon}\, d\gamma_n \le \varepsilon^{-1+1/4d} c(d,1) \|\partial_k f\|_2^{-1/2d}.$$

Now observe that $\max_k \|\partial_k f\|_2 \ge \frac{\sqrt{\|\partial_1 f\|_2^2 + \ldots + \|\partial_n f\|_2^2}}{\sqrt{n}} = \frac{1}{\sqrt{n}} \|\langle \nabla f, \nabla f\rangle\|_1^{1/2}$. Taking into account that $f \in \mathcal{T}(d,n)$ has at most d^2 distinct variables x_i, we can assume that $n \le d^2$. Thus, we have

$$\int_{\mathbb{R}^n} \frac{1}{\langle \nabla f, \nabla f\rangle + \varepsilon}\, d\gamma_n \le \frac{c(d,1)\varepsilon^{-1+1/4d}}{(\max_k \|\partial_k f\|_2)^{1/2d}} \le \frac{c(d,1)\cdot d^{1/2d} \varepsilon^{-1+1/4d}}{\|\langle \nabla f, \nabla f\rangle\|_1^{1/4d}}.$$

This inequality enables us to correct the proof of Theorem 3 in [16]. However, as we have already noted earlier in Remark 7, that theorem can be now replaced with a stronger result: Theorem 3 of this paper.

Remark 8. Let us now discuss Theorem 4 in [16], which also relies on (4.7). We used (4.7) in [16] when stating that $\Delta_f = \det M_f$ belongs to $\mathcal{TR}_{2kd}(\mathbb{R}^n)$. To correct this reasoning in [16], we add the condition

$$\Delta_f = \det M_f \in \mathcal{T}(\widetilde{N}, n).$$

Observe that this condition is satisfied for $\widetilde{N} = k! \cdot (2d)^{2k}$. This follows from the fact that every element of the matrix M_f has the form $m_{ij} = \langle \nabla f_i, \nabla f_j\rangle$ with $f_i, f_j \in T(\widetilde{N}, n)$. The decomposition

$$m_{ij} = \langle \nabla f_i, \nabla f_j\rangle = a_0 + \textstyle\sum_{k=1}^N (c_k \cdot e^{i\langle v_k, x\rangle} + d_k \cdot e^{-i\langle v_k, x\rangle})$$

involves at most $(2d)^2$ distinct vectors v_k with $|v_k| \le 2d \le (2d)^2$. Hence

$$\det M_f = a_0 + \textstyle\sum_{k=1}^N (c_k \cdot e^{i\langle \widehat{v}_k, x\rangle} + d_k \cdot e^{-i\langle \widehat{v}_k, x\rangle})$$

involves at most $k! \cdot ((2d)^2)^k = k! \cdot (2d)^{2k}$ distinct vectors $\widehat{v}_k$ and for all of them one has $|\widehat{v}_k| \le 2kd \le k! \cdot (2d)^{2k}$. Thus, we always have $\Delta_f = \det M_f \in \mathcal{T}(d, \widetilde{N})$ with $\widetilde{N} = k! \cdot (2d)^{2k}$, as required by Definition 1. After all these adjustments, Theorem 4 in [16] will take the following form.

Theorem 4': *Let $k, d \in \mathbb{N}$, $a > 0$, $b > 0$, $\tau > 0$. Then there is a number $C(d,k,a,b,\tau) > 0$ such that for every map $f = (f_1, \ldots, f_k)\colon \mathbb{R}^n \to \mathbb{R}^k$ with f_i from $\mathcal{T}(d,n)$ satisfying the conditions $\|\Delta_f\|_1 \ge a$, $\max_{i\le k} \sigma_{f_i} \le b$ and $\Delta_f \in \mathcal{T}(\widetilde{N}, n)$ and for all functions $\varphi \in C_b^\infty(\mathbb{R}^k)$ and vectors $e \in \mathbb{R}^k$ with $|e| = 1$ there holds the inequality*

$$\int_{\mathbb{R}^n} \partial_e \varphi(f(x))\, \gamma_n(dx) \le C(d,k,a,b,\tau) \|\varphi\|_\infty^\alpha \|\partial_e \varphi\|_\infty^{1-\alpha}, \alpha = (2\widetilde{N} + \tau)^{-1}.$$

Consequently, $\gamma_n \circ f^{-1} \in B^{\alpha}(\mathbb{R}^k)$ *for any* $\alpha < 1/(2\widetilde{N})$.

As noted above, one can find $\widetilde{N}$ such that $\Delta_f \in \mathcal{T}(\widetilde{N}, n)$. The condition $f_i \in \mathcal{T}(d, n)$, $i \le k$, guarantees that $\Delta_f \in \mathcal{T}(N_0, n)$ with $N_0 = k! \cdot (2d)^{2k}$.

5. Conclusion

The estimate obtained by Davydov in [9] for trigonometric polynomials on $\mathbb{R}^1$ is generalized to the case of trigonometric polynomials on $\mathbb{R}^n$. We also obtain new results on the inclusion of the images of Gaussian measures under trigonometric polynomials to Nikolskii–Besov classes. In addition, some inaccuracies made in [16] are corrected.

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О распределениях тригонометрических полиномов от гауссовских случайных величин

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Аннотация. В статье доказаны новые результаты о вложении распределений тригонометрических полиномов от гауссовских случайных величин в классы Бесова – Никольского. Также получена оценка расстояния по вариации между распределениями тригонометрических полиномов через расстояние в L^q-метрике между самими полиномами.

Ключевые слова: Класс Никольского – Бесова, гауссовская мера, распределение тригонометрического полинома.

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