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Integration of the Matrix Nonlinear Schrödinger Equation with a Source

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Abstract. This paper is concerned with studying the matrix nonlinear Schrödinger equation with a self-consistent source. The source consists of the combination of the eigenfunctions of the corresponding spectral problem for the matrix Zakharov-Shabat system which has not spectral singularities. The theorem about the evolution of the scattering data of a non-self-adjoint matrix Zakharov-Shabat system which potential is a solution of the matrix nonlinear Schrödinger equation with the self-consistent source is proved. The obtained results allow us to solve the Cauchy problem for the matrix nonlinear Schrödinger equation with a self-consistent source in the class of the rapidly decreasing functions via the inverse scattering method. A one-to-one correspondence between the potential of the matrix Zakharov-Shabat system and scattering data provide the uniqueness of the solution of the considering problem. A step-by-step algorithm for finding a solution to the problem under consideration is presented.

Keywords: matrix nonlinear Schrödinger equation, self-consistent source, inverse scattering method, scattering data.

1. Introduction

The inverse scattering transform method was first proposed by Gardner, Greene, Kruskal and Miura (GGKM) [5] in 1967 for solving the Cauchy problem for the Korteweg-de Vries (KdV) equation. Their approach was based on the connection between the KdV equation and the spectral theory for the Sturm-Liouville operator on the line. Shortly thereafter, P. Lax [9] pointed out the general character of the inverse scattering method. A few years later, V.E. Zakharov and A.B. Shabat [14] managed to solve another important nonlinear evolution equation, the so-called nonlinear Schrödinger (NLS) equation, using a nontrivial extension of the methods used in [5; 9].

The inverse scattering problem for the Dirac operator on the entire line was studied by V.E. Zakharov, A.B. Shabat [14], L.A. Takhtadzhyan, L.D. Faddeev [6], A.B. Khasanov [7] and others. The work relevant to the necessity and sufficient conditions for the solvability of the inverse scattering problem for the matrix Sturm-Liouville operator on the axis was studied in [1]. In the matrix case, the inverse scattering theory for the matrix Zakharov-Shabat system [3] was investigated by F. Demontis and C. Van der Mee and applied for the integration of the matrix NLS equation [4].

The NLS equation with the self-consistent sources in various classes of functions were considered by V.K. Melnikov [10], A.B. Khasanov, A.A. Reyimberganov [8], I.D. Rakhimov [11], A.B. Yakhshimuratov [13]. In this work, we consider the matrix NLS equation with the self-consistent source in the class of rapidly decreasing matrix functions. Other matrix nonlinear evolution equations with the self-consistent sources were integrated via the inverse scattering method in the works [2; 12].

2. Scattering theory for the matrix Zakharov-Shabat system

In this section, we give well-known [3], necessary information concerning the theory of direct and inverse scattering problems for the operator

$$L = -iJ\frac{d}{dx} - V(x)$$

on the line $(-\infty < x < \infty)$ with the rapidly decreasing potential by x.

We consider the following matrix Zakharov-Shabat system

$$-iJX' - VX = \lambda X, \tag{2.1}$$

where $X(\lambda, x)$ is $2m \times m$ matrix function,

$$J = \begin{bmatrix} I_m & 0_m \\ 0_m & -I_m \end{bmatrix}, V = \begin{bmatrix} 0_m & iU(x) \\ iU^*(x) & 0_m \end{bmatrix}$$

and $U(x)$ is an $m \times m$ matrix valued function, $U^*(x)$ denotes the complex conjugate of $U(x)$, I_m and 0_m are the identity and zero matrices of order m, respectively.

We assume that the function $U(x)$ satisfies the following condition

$$\mathcal{H}_1 : \quad \int_{-\infty}^{\infty} \|U(x)\|\, dx < \infty,$$

where $\|U(x)\| = \max_j \sum_{k=1}^{m} |U_{jk}(x)|$.

For $\lambda \in \mathrm{R}$ the Jost matrices $F(\lambda, x)$ and $G(\lambda, x)$ as the $2m \times 2m$ matrix solutions of (2.1) satisfy the following asymptotic conditions:

$$\begin{array}{ll} F(\lambda, x) = [\bar{\psi}(\lambda,\, x) \;\; \psi(\lambda,\, x)] \to e^{i\lambda Jx} I_{2m}, & x \to \infty, \\ G(\lambda, x) = [\phi(\lambda,\, x) \;\; \bar{\phi}(\lambda,\, x)] \to e^{i\lambda Jx} I_{2m}, & x \to -\infty\ . \end{array} \tag{2.2}$$

Here $\psi(\lambda, x)$, $\bar{\psi}(\lambda, x)$, $\phi(\lambda, x)$ and $\bar{\phi}(\lambda, x)$ are the submatrices with $2m$ rows and m columns, which are usually called Jost solutions. Here and below bar does not mean complex conjugation.

The Jost solutions $\bar{\psi}(\lambda,\, x)$ and $\psi(\lambda,\, x)$ of the equation (2.1) at any $\lambda \in \mathrm{R}$ can be represented in the following form

$$\begin{aligned} \bar{\psi}(\lambda, x) &= e^{i\lambda x} \begin{bmatrix} I_m \\ 0_m \end{bmatrix} + \int_x^{\infty} e^{i\lambda y} \bar{K}(x, y) dy, \\ \psi(\lambda, x) &= e^{-i\lambda x} \begin{bmatrix} 0_m \\ I_m \end{bmatrix} + \int_x^{\infty} e^{-i\lambda y} K(x, y) dy, \end{aligned} \tag{2.3}$$

where

$$\left[\bar{K}(x, y) \;\; K(x, y)\right] = \begin{bmatrix} K_1(x, y) & K_2(x, y) \\ K_3(x, y) & K_4(x, y) \end{bmatrix},$$

$K_s(x, y),\ s = \overline{1, 4}$ are $m \times m$ matrices. Here the kernels have relations with the potential

$$U(x) = 2iK_2(x, x) = 2iK_3(x, x).$$

We also consider the following auxiliary equation

$$iY'J - YV = \mu Y, \tag{2.4}$$

where Y is an $m \times 2m$ matrix function.

Lemma 1. *Let $X(\lambda, x)$ and $Y(\mu, x)$ be solutions of the equations* (2.1) *and* (2.4), *respectively, then the following relation holds*

$$i(\lambda - \mu) Y(\mu, x) X(\lambda, x) = (Y J X)'. \tag{2.5}$$

For $\lambda \in \mathrm{R}$ there exists $2m \times 2m$ matrix $A(\lambda)$ such that

$$\begin{aligned} G(\lambda, x) &= F(\lambda, x) A(\lambda), \\ F(\lambda, x) &= G(\lambda, x) C(\lambda). \end{aligned} \tag{2.6}$$

Here $A(\lambda)$ and $C(\lambda)$ consist of block matrices such as

$$A(\lambda) = \begin{pmatrix} A_1(\lambda) & A_2(\lambda) \\ A_3(\lambda) & A_4(\lambda) \end{pmatrix},$$

$A_s(\lambda)$, $s=\overline{1,4}$ are $m\times m$ matrices.

Assuming that the potential $U(x)$ have entries in $\mathcal{H}_1$, we can say that for each fixed $x\in$ R the matrix functions $\bar{\psi}(\lambda,x)e^{-i\lambda x}$ and $\bar{\phi}(\lambda,x)e^{i\lambda x}$ ($\psi(\lambda,x)e^{i\lambda x}$ and $\phi(\lambda,x)e^{-i\lambda x}$) can be continued to the half-plane $Im\lambda>0$ ($Im\lambda<0$) and for all $Im\lambda>0$ ($Im\lambda<0$) the matrix functions $\bar{\psi}(\lambda,x)e^{-i\lambda x}$ and $\bar{\phi}(\lambda,x)e^{i\lambda x}$ ($\psi(\lambda,x)e^{i\lambda x}$ and $\phi(\lambda,x)e^{-i\lambda x}$) are bounded. Invertible matrix function $A_1(\lambda)$ ($A_4(\lambda)$) can be analytically continued to the half-plane $Im\lambda>0$ ($Im\lambda<0$) and there the equation $\det A_1(\lambda)=0$ ($\det A_4(\lambda)=0$) has a finite number of zeros λ_j, $j=\overline{1,N}$ ($\bar{\lambda}_j$, $j=\overline{1,\bar{N}}$) which correspond to the eigenvalues of the operator L.

Definition 1. *The real zeros of the equation* $\det A_1(\lambda)=0$ *(*$\det A_4(\lambda)=0$*) will be called the spectral singularities of the equation* (2.1)*.*

Remark 1. If $\int_{-\infty}^{\infty}\|U(x)\|\,dx<\frac{\pi}{2}$ then there do not exist neither spectral singularities of the operator L [23].

We assume that the operator L has no spectral singularities and all the eigenvalues of the operator L are simple.

The matrix functions $(A_1(\lambda))^{-1}$ and $(A_4(\lambda))^{-1}$ have simple poles on the points λ_j, $j=\overline{1,N}$ in $Im\lambda>0$ and $\bar{\lambda}_j$, $j=\overline{1,\bar{N}}$ in $Im\lambda<0$, respectively. Let $N_j=\underset{\lambda=\lambda_j}{Res}(A_1(\lambda))^{-1}$, $j=\overline{1,N}$ and $\bar{N}_j=\underset{\lambda=\bar{\lambda}_j}{Res}(A_4(\lambda))^{-1}$, $j=\overline{1,\bar{N}}$, then there are matrices R_j and $\bar{R}_j$ such that

$$\begin{aligned}\bar{\phi}(\lambda_j,x)N_j&=\bar{\psi}(\lambda_j,x)R_j,\quad j=\overline{1,N},\\ \phi(\lambda_j,x)\bar{N}_j&=\psi(\lambda_j,x)\bar{R}_j,\quad j=\overline{1,\bar{N}}.\end{aligned}\tag{2.7}$$

Definition 2. *The following matrices for* $\lambda\in$ R

$$\begin{aligned}R(\lambda)&=C_2(\lambda)C_4^{-1}(\lambda)=-A_1^{-1}(\lambda)A_2(\lambda),\\ \bar{R}(\lambda)&=C_3(\lambda)C_1^{-1}(\lambda)=-A_4^{-1}(\lambda)A_3(\lambda)\end{aligned}\tag{2.8}$$

are called reflection coefficients.

As $V^*(x)=-V(x)$, we have

$$\overline{\psi}^T(\lambda,x)=\sigma_1\psi(\lambda^*,x),\ \overline{\phi}^T(\lambda,x)=\sigma_1\ \phi(\lambda^*,x),\tag{2.9}$$

where $\sigma_1=\begin{pmatrix}0_m & I_m\\ -I_m & 0_m\end{pmatrix}$, λ^* is the complex conjugation of λ. Then, from it yields that,

$$\bar{N}=N,\quad \bar{\lambda}_j=\lambda_j^*,\quad \bar{N}_j=N_j^*,\quad \bar{R}_j=R_j^*$$

and

$$R(\lambda)=-R^*(\lambda).$$

Definition 3. *The set* $\{R(\lambda),\ \lambda_1,\ \lambda_2,\ ...,\ \lambda_N,\ R_1\ , R_2, ..., R_N\ \}$ *is called the scattering data associated with the equation* (2.1).

The direct scattering problem is to find the scattering data via the given potential $U(x)$ of the equation (2.1) and inverse scattering problem is to find the potential $U(x)$ of the equation (2.1) via the given scattering data.

The kernels of the representation (2.3) satisfy the following Gelfand-Levitan-Marchenko integral equations for $x > y$

$$\bar{K}(x,y) + \begin{bmatrix} 0_m \\ I_m \end{bmatrix} \bar{\Omega}(x+y) + \int_x^\infty K(x,z)\bar{\Omega}(z+y)dz = 0,$$
$$K(x,y) + \begin{bmatrix} I_m \\ 0_m \end{bmatrix} \Omega(x+y) + \int_x^\infty \bar{K}(x,z)\Omega(z+y)dz = 0,$$

where

$$\Omega(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\lambda)e^{i\lambda x}d\lambda + \sum_{j=1}^{N} R_j e^{i\lambda_j x},$$
$$\bar{\Omega}(x) = -\Omega^*(x).$$

Here, $\rho(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\lambda)e^{i\lambda x}d\lambda$ and $Ce^{-xA}B = \sum_{j=1}^{N} R_j e^{i\lambda_j x}$.

In the work [3], it was proven the uniquely determining of the potential $U(x)$ by the scattering data.

3. Integration of the matrix NLS equation with a self-consistent source

We consider the integration of the following problem

$$iU_t + 2UU^*U + U_{xx} = 2\sum_{n=1}^{N}(\Phi_{1,n}\Phi_{2,n}^* + \Phi_{2,n}^*\Phi_{1,n}), \tag{3.1}$$

$$-iJ\Phi_n' - V\Phi_n = \lambda_n\Phi_n,\ n = 1, 2, ..., N, \tag{3.2}$$

under the initial condition

$$U|_{t=0} = U_0(x). \tag{3.3}$$

Here $\Phi_{j,n} = \Phi_{j,n}(x,t),\ j = 1, 2$ are $m \times m$ matrix functions, columns of $\Phi_n = \begin{pmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{pmatrix}$ matrices are linearly independent eigenfunctions corresponding to the eigenvalue $\lambda_n,\ n = \overline{1, N}$ and normalized by the following conditions

$$\int_{-\infty}^{\infty} \Phi_n^*(x,t)\Phi_n(x,t)dx = a_n^2(t)\, I_m, \qquad n = 1, 2, ..., N. \tag{3.4}$$

where $\Phi_n^*, n = \overline{1,N}$ satisfy the equation (2.4), I_m is the $m \times m$ identity matrix (note that $\Phi_n^*(x,t)\Phi_n(x,t)$ are normal matrices).

Here $U = U(x,t)$ is an $m \times m$ matrix function,

$$V(x,t) = \begin{bmatrix} 0 & iU(x,t) \\ iU^*(x,t) & 0 \end{bmatrix}$$

is $2m \times 2m$ matrix, $a_n^2(t)$, $n = \overline{1,N}$ are nonzero continuous scalar functions.

The matrix function $U_0(x)$ satisfies the following properties:

1)
$$\int_{-\infty}^{\infty} \|U_0(x)\|\, dx < \infty \tag{3.5}$$

2) Operator $L(0) = -iJ\frac{d}{dx} - V_0(x)$ possesses exactly $2N$ eigenvalues $\lambda_1(0)$, $\lambda_2(0)$, ..., $\lambda_{2N}(0)$, every eigenvalue has m linearly independent corresponding eigenfunctions and linearly independent eigenfunctions corresponding to these eigenvalues don't have associated vector functions.

Our main purpose is to obtain the time evolution equations of the scattering data for finding the solution of the problem (3.1)-(3.5) which is a collection

$$\{U(x,t),\, \Phi_1(\lambda_1,x,t),\, \Phi_2(\lambda_2,x,t),\, ...,\, \Phi_N(\lambda_N,x,t)\,\}$$

under assumption of existence in the following sense:

1) for all $t > 0$,
$$\sum_{r=0}^{2} \int_{-\infty}^{\infty} \left\| \frac{\partial^r}{\partial x^r} U(x,t) \right\| dx < \infty; \tag{3.6}$$

2) the columns of the $\Phi_n(x,t)$, $n = \overline{1,2N}$ matrices belong to the domain of $L^2(R, C^{2m})$, which is the space of complex-valued vector functions of size $2m$ with components belonging to $L^2(R)$.

In the current section we will derive the representations for the evolution equations of the scattering data with which it is available to find the collection of solution of the problem (3.1)-(3.5) in the class of the rapidly decreasing functions (3.6) via the inverse scattering method for the operator $L(t) = -iJ\frac{d}{dx} - V(x,t)$.

Under the assumption $\lambda_{n+N} = \lambda_n^*$, $n = \overline{1,N}$ and $\Phi_{1,n+N} = -\Phi_{2,n}^*$, $\Phi_{2,n+N} = \Phi_{1,n}^*$ the equation (3.1) can be represented as a Lax operator equality:

$$L_t + [B,L] + \sum_{n=1}^{2N} [J, \Phi_n\Phi_n^*] = 0, \tag{3.7}$$

where $[B, L] = BL - LB$ and

$$B = \begin{pmatrix} iUU^* + 2iI_m \frac{d^2}{dx^2} & iU_x + 2iU \frac{d}{dx} \\ iU_x^* + 2iU^* \frac{d}{dx} & -iUU^* - 2iI_m \frac{d^2}{dx^2} \end{pmatrix}. \tag{3.8}$$

Here, both sides of the equality (3.7) turn out to be operators of multiplication by a matrix function.

Lemma 2. *Let $F_0(\lambda, x, t)$ be any $2m \times 2m$ matrix solution of the equation*

$$LF_0 = \lambda F_0, \quad \lambda \in \mathrm{R} \tag{3.9}$$

and let F_n, $n = 1, 2, ..., 2N$ be any matrix functions $m \times 2m$, which satisfy

$$\frac{\partial F_n}{\partial x} = i\Phi_n^* F_0, n = 1, 2, ..., 2N. \tag{3.10}$$

Then, the matrix function

$$H_0 = \dot{F}_0 + BF_0 - \sum_{n=1}^{2N} \Phi_n F_n \tag{3.11}$$

is also a matrix solution of equation (3.9)*.*

Proof. We take the derivative from equation (3.9) with respect to t

$$\dot{L}F_0 + L\dot{F}_0 = \lambda \dot{F}_0.$$

Here, we will find $L\dot{F}_0$

$$L\dot{F}_0 = \lambda\dot{F}_0 - \dot{L}F_0 = \lambda\dot{F}_0 + BLF_0 - LBF_0 + \sum_{n=1}^{2N} [J, \Phi_n\Phi_n^*] F_0.$$

Using this equality now we calculate LH_0

$$\begin{aligned} LH_0 &= \lambda H_0 + \sum_{n=1}^{2N} (\lambda - \lambda_n) \Phi_n F_n - \sum_{n=1}^{2N} \Phi_n \Phi_n^* J F_0 \\ &= \lambda H_0 + \sum_{n=1}^{2N} \Phi_n \left((\lambda - \lambda_n) F_n - \sum_{n=1}^{2N} \Phi_n^* J F_0 \right) \\ &= \lambda H_0 + \sum_{n=1}^{2N} \Phi_n H_n. \end{aligned}$$

Here, we introduce H_n, $n = 1, 2, ..., 2N$

$$H_n = (\lambda - \lambda_n)F_n - \Phi_n^* J F_0, n = 1, 2, ..., 2N. \tag{3.12}$$

According to the lemma 1, we can show that

$$\frac{\partial H_n}{\partial x} = 0 \tag{3.13}$$

and $\|H_n\| \to 0$ as $x \to \pm\infty$. Hence, it follows that $H_n \equiv 0$ for all $\lambda \in R$. This means that the matrix function H_0 for any λ satisfies the equation (3.9). The proof of lemma is complete. □

Let us take $F(\lambda, x, t)$ and $G(\lambda, x, t)$ Jost matrices as the solution F_0 of the equation (3.9) and let for $n = 1, 2, ..., 2N$ hold the following expressions

$$\begin{aligned} F_n{}^- &= i \textstyle\int_{-\infty}^{x} \Phi_n^*(x,t) G(\lambda, x, t) dx\,, \\ F_n{}^+ &= -i \textstyle\int_{x}^{\infty} \Phi_n^*(x,t) F(\lambda, x, t) dx\,. \end{aligned} \tag{3.14}$$

Here $\Phi_n(x, t)$ belong to the $L^2(R, C^{2m})$, Jost matrices F and G are bounded for $Im\lambda = 0$. Using lemma 1, we can show that these integrals are convergent:

$$i \int_{-\infty}^{x} \Phi_n^*(x,t) G(\lambda, x, t) dx = \frac{\Phi_n^*(x,t) J G(\lambda, x, t)}{(\lambda - \lambda_n)},$$

$$-i \int_{x}^{\infty} \Phi_n^*(x,t) F(\lambda, x, t) dx = \frac{\Phi_n^*(x,t) J F(\lambda, x, t)}{(\lambda - \lambda_n)}.$$

Moreover, $F_n^{\pm} \in L_2(R, C^{2m})$.

Substituting (2.2), (3.14) into the expressions (3.11) and (3.12), we define H_0^-, H_0^+, H_n^-, H_n^+. According to (3.13), it is easy to show for $n = 1, 2, ..., 2N$ that $H_n^- = H_n^+ = 0$. Therefore, we can conclude that the matrix functions

$$\begin{aligned} H_0^+ &= \dot{F} + BF - \textstyle\sum_{n=1}^{2N} \Phi_n F_n^+\,, \\ H_0^- &= \dot{G} + BG - \textstyle\sum_{n=1}^{2N} \Phi_n F_n^- \end{aligned} \tag{3.15}$$

are solutions of the equation (3.9).

Remark 2. Using the asymptotics (2.2) for the Jost solutions in (3.15) we obtain

$$H_0{}^+ \to -2i\lambda^2 \begin{pmatrix} e^{i\lambda x} I_m & 0_m \\ 0_m & e^{-i\lambda x} I_m \end{pmatrix} J, \quad x \to \infty,$$

$$H_0{}^- \to -2i\lambda^2 \begin{pmatrix} e^{i\lambda x} I_m & 0_m \\ 0_m & e^{-i\lambda x} I_m \end{pmatrix} J, \quad x \to -\infty.$$

By the uniqueness of the Jost solutions we get

$$H_0^+ = -2i\lambda^2 F J, \quad H_0^- = -2i\lambda^2 G J. \tag{3.16}$$

Lemma 3. *For all $\lambda \in \mathrm{R}$ the following equality holds*

$$\dot{R}(\lambda) = 4i\lambda^2 R(\lambda). \tag{3.17}$$

Here the dot means the derivative respect to the parameter t.

Proof. We introduce the following matrix function

$$H = H_0^- - H_0^+ A(\lambda). \tag{3.18}$$

Substituting (3.16) into the expression (3.18), we receive

$$H = -2i\lambda^2 F A(\lambda) J + 2i\lambda^2 F J A(\lambda) = 2i\lambda^2 F \left[J, A(\lambda)\right]. \tag{3.19}$$

Using the representations (3.15) and the expression (3.18), we have

$$H = F\dot{A(\lambda)} - \sum_{n=1}^{2N} \Phi_n (F_n^- - F_n^+ A(\lambda)).$$

Here,

$$F_n^- - F_n^+ A(\lambda) = i \int_{-\infty}^{+\infty} \Phi_n^* G dx.$$

Using lemma 1 and since, $\Phi_n(x,t)$ belong to the $L^2(R, C^{2m})$, Jost matrix G is bounded for $Im\lambda = 0$, we find that

$$\int_{-\infty}^{+\infty} \Phi_n^* G dx = \left. \frac{\Phi_n^* J G}{i(\lambda - \lambda_n)} \right|_{-\infty}^{\infty} = 0.$$

So, we get

$$H = F\dot{A}(\lambda). \tag{3.20}$$

Comparing (3.19) and (3.20) we find

$$2i\lambda^2 \left[J, A(\lambda)\right] = \dot{A}(\lambda). \tag{3.21}$$

Particularly,

$$\dot{A}_1(\lambda) = 0\,, \qquad \dot{A}_2(\lambda) = 4i\lambda^2 A_2(\lambda)\,, \tag{3.22}$$

$$\dot{A}_3(\lambda) = -4i\lambda^2 A_3(\lambda)\,, \qquad \dot{A}_4(\lambda) = 0\,. \tag{3.23}$$

According to $R(\lambda) = -A_1^{-1}(\lambda) A_2(\lambda)$, we can find

$$A_1(\lambda) R(\lambda) = -A_2(\lambda).$$

Taking the derivative by t from the last equality, we obtain

$$A_1(\lambda)\dot{R}(\lambda) = -4i\lambda^2 A_2(\lambda)$$

and we find that $\dot{R}(\lambda) = -4i\lambda^2 A_1^{-1}(\lambda) A_2(\lambda)$, which is (3.17). The proof of lemma is complete. □

Corollary 1. *Since, $A_1(\lambda)$ does not depend on t, its determinant* $\det A_1(\lambda)$ *and its zeros λ_j, $j = 1,\ 2,\ ...,\ N$ also do not depend on t.*

Lemma 4. *The matrix functions $R_j(t)$, $j = 1,\ 2,\ ...,\ N$ satisfy the following equations*

$$\frac{dR_j(t)}{dt} = (4i\lambda_j^2 - a_j^2(t))R_j(t). \tag{3.24}$$

Proof. For $Im\lambda_j > 0$, $j = 1,\ 2,\ ...,\ N$ we denote

$$h_0^-(\lambda_j, x, t) = \dot{\bar{\phi}}(\lambda_j, x, t) + B\bar{\phi}(\lambda_j, x, t) - \textstyle\sum_{n=1}^{2N} \Phi_n(x, t) f_n^-(\lambda_j, x, t)\,,$$

$$h_0^+(\lambda_j, x, t) = \dot{\bar{\psi}}(\lambda_j, x, t) + B\bar{\psi}(\lambda_j, x, t) - \textstyle\sum_{n=1}^{2N} \Phi_n(x, t) f_n^+(\lambda_j, x, t)\,, \tag{3.25}$$

where the vector functions $f_n^-(\lambda_j, x, t)$, $f_n^+(\lambda_j, x, t)$ are defined as follows

$$\begin{aligned} f_n^-(\lambda_j, x, t) &= i \textstyle\int_{-\infty}^{x} \Phi_n^*(x, t)\bar{\phi}(\lambda_j, x, t)dx\,, \\ f_n^+(\lambda_j, x, t) &= -i \textstyle\int_{x}^{+\infty} \Phi_n^*(x, t)\bar{\psi}(\lambda_j, x, t)dx\,. \end{aligned} \tag{3.26}$$

Here, the functions under the integrals belong to the class $L^2(R, C^{2m})$, which provide the convergence of the integrals.

We now introduce the following matrix functions

$$h_j = h_0^-(\lambda_j, x, t)N_j - ih_0^+(\lambda_j, x, t)R_j(t), \qquad j = 1, 2, ..., N. \tag{3.27}$$

Using the expressions (3.25) we can rewrite (3.27) as

$$h_j = \dot{\bar{\phi}}(\lambda_j, x, t)N_j + B\bar{\phi}(\lambda_j, x, t)N_j - i\dot{\bar{\psi}}(\lambda_j, x, t)R_j(t) +$$

$$-iB\bar{\psi}(\lambda_j, x, t)R_j(t) - \sum_{n=1}^{2N} \Phi_n(f_n^- N_j - if_n^+ R_j(t)), \quad j = 1, 2, ..., N. \tag{3.28}$$

Differentiating (2.7) with respect to t and taking account of the independence of N_j from t we obtain

$$\dot{\bar{\phi}}(\lambda_j, x, t)N_j = i\dot{\bar{\psi}}(\lambda_j, x, t)R_j(t) + i\bar{\psi}(\lambda_j, x, t)\dot{R}_j(t), \qquad j = 1, 2, ..., N. \tag{3.29}$$

Substituting (3.26) and (3.29) into (3.28) we get for $j = 1, 2, ..., N$

$$h_j = i\bar{\psi}(\lambda_j, x, t)\dot{R}_j(t) + i\sum_{n=1}^{2N} \Phi_n(x, t) \int_{-\infty}^{\infty} \Phi_n^*(x, t)\bar{\psi}(\lambda_j, x, t)dxR_j(t). \tag{3.30}$$

If $n \neq j$, according to lemma 1 we get

$$\int_{-\infty}^{\infty} \Phi_n^*(x, t)\bar{\psi}(\lambda_j, x, t)dx = 0.$$

In the case of $n = j$, we receive for $j = 1, 2, ..., N$

$$h_j = i\bar{\psi}(\lambda_j, x, t)\dot{R}_j(t) + i\Phi_j(x,t)\int_{-\infty}^{\infty}\Phi_j^*(x,t)\bar{\psi}(\lambda_j, x, t)dxR_j(t). \quad (3.31)$$

We know that

$$\Phi_j(x,t) = \bar{\psi}(\lambda_j, x, t)c_j(t), \qquad j = \overline{1, N}. \quad (3.32)$$

Here $c_j(t)$, $j = \overline{1, N}$ are $m \times m$ matrices as supposing that $a_j(t) \neq 0$, $j = \overline{1, N}$. Moreover, the columns of the matrix $\bar{\psi}(\lambda_j, x, t)R_j(t)$ are also eigenfunctions, therefore, $R_j(t)$ matrix can be represented as linear combinations of columns of $c_j(t)$, i.e., exist $m \times m$ matrices $e_j(t)$ that the following equality holds

$$R_j(t) = c_j(t)e_j(t), j = \overline{1, N}.$$

Using this representation and the relation (3.32) in the second term of the expression (3.31) we have

$$i\Phi_j(x,t)\int_{-\infty}^{\infty}\Phi_j^*(x,t)\bar{\psi}(\lambda_j, x, t)dxR_j(t)$$

$$= i\bar{\psi}(\lambda_j, x, t)c_j(t)\int_{-\infty}^{\infty}\Phi_j^*(x,t)\Phi_j(x,t)dxe_j(t) =$$

$$= i\bar{\psi}(\lambda_j, x, t)c_j(t)a_j^2(t)I_m e_j(t) = ia_j^2(t)\bar{\psi}(\lambda_j, x, t)R_j(t).$$

Hence, we get

$$h_j = i\bar{\psi}(\lambda_j, x, t)\dot{R}_j(t) + i\bar{\psi}(\lambda_j, x, t)a_j^2(t)R_j(t), \;\; j = 1, 2, ..., N. \quad (3.33)$$

According to (3.15) we get

$$h_j = -4\lambda_j^2\bar{\psi}(\lambda_j, x, t)R_j(t). \quad (3.34)$$

By comparing (3.33) and (3.34) it yields that $R_j(t)$, $j = \overline{1, N}$ satisfy the equation (3.24) for $Im\lambda_j > 0$, $j = 1, 2, ..., N$. The proof of lemma is complete. □

Thus, we have proved the following theorem.

Theorem 1. *If the collection* $\{U(x,t), \; \Phi_j(\lambda_j, x, t), j = \overline{1, N}\}$ *is a solution of the problem* (3.1)-(3.6)*, then the scattering data for the operator*

$$L(t) = -iJ\frac{d}{dx} - V(x,t),$$

satisfy the following relations

$$\dot{R}(\lambda) = 4i\lambda^2 R(\lambda), \lambda \in \mathrm{R},$$

$$\frac{d\lambda_j}{dt} = 0, \quad \frac{dR_j(t)}{dt} = (4i\lambda_j^2 - a_j^2(t))R_j(t), \;\; j = \overline{1, N}.$$

Using the following algorithm we can find the solution. Let us given the functions $U_0(x)$ and $a_n^2(t)$, $n = \overline{1, N}$.

- Solving the direct scattering problem for the initial matrix $U_0(x)$, obtain the scattering data $\{R(\lambda),\ \lambda_1,\ \lambda_2,\ ...,\ \lambda_N,\ R_1\,, R_2, ..., R_N\}$ of the operator $L(0) = -iJ\dfrac{d}{dx} - V(x)$.
- Using the results of the Theorem 1, find the scattering data for $t > 0$ $\{R(\lambda,t),\ \lambda_1(t),\ \lambda_2(t),\ ...,\ \lambda_N(t),\ R_1(t)\,, R_2(t), ..., R_N(t)\}$.
- Using the method based on the Gelfand-Levitan-Marchenko integral equation, solve the inverse scattering problem, i.e. from the scattering data $\{R(\lambda,t),\ \lambda_1(t),\ \lambda_2(t),\ ...,\ \lambda_{2N}(t),\ R_1(t)\,, R_2(t), ..., R_N(t)\}$ determine $U(x,t)$.
- Find the Jost solutions of the operator $L(t)$ with the potential $U(x,t)$ and then using (2.3), construct the matrix $\Phi_n(x,t)$.

4. Conclusion

In this work, we have deduced the evolution of the scattering data of a non-self-adjoint matrix Zakharov-Shabat system. The obtained results completely specify the time evolution of the scattering data for $L(t)$ and satisfy the condition of the one-to-one correspondence between the potential of the matrix Zakharov-Shabat system and scattering data. This allows to find solution of the problem (3.1)-(3.6) in the class of the rapidly decreasing functions via the inverse scattering method.

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Интегрирование матричного нелинейного уравнения Шредингера с источником

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Аннотация. Работа посвящена исследованию матричного нелинейного уравнения Шредингера с самосогласованным источником, состоящим из комбинаций собственных функций соответствующей спектральной задачи для матричной системы Захарова – Шабата и не имеющей спектральных особенностей. Доказана теорема об эволюции данных рассеяния несамосопряженной матричной системы Захарова – Шабата, потенциал которой является решением матричного нелинейного уравнения Шредингера с самосогласованным источником. Полученные результаты позволяют решить задачу Коши для матричного нелинейного уравнения Шредингера с самосогласованным источником в классе быстроубывающих функций методом обратной задачи 1.1 — соответствие между потенциалом матричной системы Захарова – Шабата и данными рассеяния обеспечивает однозначность решения рассматриваемой задачи. Приведен пошаговый алгоритм поиска решения рассматриваемой задачи.

Ключевые слова: матричное нелинейное уравнения Шредингера, метод обратной задачи рассеяния, самосогласованный источник, данных рассеяния.

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