Existence and Uniqueness of Weak Solutions for the Model Representing Motions of Curves Made of Elastic Materials

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Abstract. We consider the initial boundary value problem for the beam equation with the nonlinear strain. In our previous work this problem was proposed as a mathematical model for stretching and shrinking motions of the curve made of the elastic material on the plane. The aim of this paper is to establish uniqueness and existence of weak solutions. In particular, the uniqueness is proved by applying the approximate dual equation method.

Keywords: Beam equation, nonlinear strain, dual equation method.

1. Introduction

In this paper, we consider the following initial and boundary value problem for the partial differential equation: The problem is to find a function $u : Q(T) \to \mathbb{R}^2$, where $Q(T) := (0, T) \times (0, 1)$, $T > 0$, satisfying

$$\rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\partial u}{\partial x}\right) = 0, \quad \varepsilon = \left|\frac{\partial u}{\partial x}\right| - 1 \text{ on } Q(T), \text{ (1.1)}$$

$$\frac{\partial^i u}{\partial x^i}(t, 0) = \frac{\partial^i u}{\partial x^i}(t, 1) \quad \text{for all } t \in [0, T] \text{ and } i = 0, 1, 2, 3, \quad \text{(1.2)}$$

$$u(0, x) = u_0(x), \quad \frac{\partial u}{\partial x}(0, x) = v_0(x) \quad \text{for all } x \in [0, 1], \quad \text{(1.3)}$$

where ρ is a positive constant denoting the density, γ is also a positive constant, ε is the strain of the elastic material, f is a continuous function on $\mathbb{R}$, u_0 is the initial position and v_0 is the initial velocity. We call the system $(1.1) - (1.3)$ the problem P.

The problem P is a mathematical model for stretching and shrinking motions of the one-dimensional elastic material on the plane $\mathbb{R}^2$ as in Figure 1. In [2] we proposed, an ordinary differential equation system as a model describing the motion of a polygon having N vertices, and proved existence and uniqueness of solutions to the ODE system. Also, we showed some theorems concerned with the numerical scheme developed by applying the structure preserving numerical method (see [4; 12]). Here, by letting $N \to \infty$ in the ODE system and adding the fourth derivative term γu_{xxxx}, we can obtain the problem P. This limiting process and numerical results for the ODE system and P will be discussed in our forthcoming paper. We note that the boundary condition (1.2) means that the material is connected smoothly. Now, we emphasize that our problem P has the following four features.
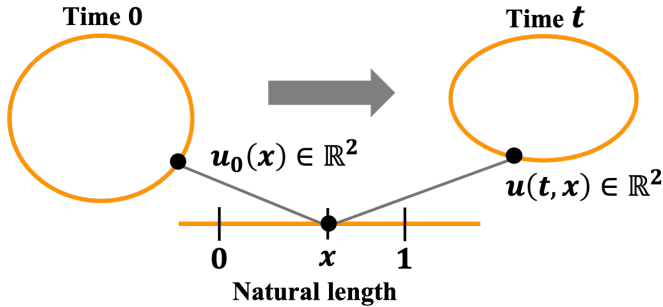

Figure 1.

i) (Unknown function) Usually, the kinetic equation for elastic materials is described with the displacement as an unknown function (see Figure 1). In our argument the unknown function of the system is the position u, since we would like to represent the motions, directly.

ii) (Nonlinear strain) In this paper we define the strain ε by $\varepsilon = |u_x| - 1$. This strain expresses the ratio between the length of stretching and its original length. Since we describe the motion of the one dimensional material on $\mathbb{R}^2$, such nonlinear strain appears. Here, we note that $|u_x|$ may vanish in general and in this case it is impossible to calculate the derivative of ε with respect to x. Hence, in this paper we consider only weak solutions such that the differentiability of ε is not necessary.

iii) (Stress function) In [2] the magnitude of the stress is given by the function $f(\varepsilon)$ having a singularity such that $f(\varepsilon) \to -\infty$ as $\varepsilon \downarrow -1$. This type of the singularity for the stress function was already studied

in material science for the compressible elastic body (see [5], [9], [10]). However, it is not easy to handle this singularity, mathematically. Therefore, we suppose that the stress function $f = f(\varepsilon)$ is continuous on $\mathbb{R}$ in this paper.

iv) (Fourth derivative term) The equation (1.1) is called a beam equation which contains the fourth derivative term γu_{xxxx} and appears when we approximate the motion of a three-dimensional material by the one-dimensional model. This kind of equations is a part of the Falk model dealing with shape memory alloys and is well studied, mathematically. Due to [3], this term is regarded as a description of non-local effect induced by interfacial energy. In order to investigate the role of this term we will observe the numerical results for solutions to P.

The aim of this paper is to establish existence and uniqueness of weak solutions of P under the following conditions for f:

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, monotone increasing and $f(0) = 0$.

Here, we give a remark for the proof of the uniqueness. From the assumption for f, the regularity of the solution is not enough to apply the standard method for the uniqueness. Namely, we can get no good estimates by multiplying (1.1) with the time derivative of the difference of solutions. Therefore, we prove the uniqueness by using the approximate of the dual equation.

The idea using the dual equation is found in [6] for proofs of uniqueness of weak solutions to parabolic and hyperbolic equations. Niezgódka and Pawlow had proved the uniqueness of weak solutions to the multi-dimensional Stefan problem by approximating the dual equation of the original equation in [8]. Also, by applying their method Aiki [1] proved uniqueness of weak solutions to the Falk model. Moreover, Yoshikawa [11] established uniqueness of solutions in a wider class than that in [1].

In this paper, since the stress function f satisfies only Lipschitz continuity, it is also not easy to obtain uniform estimates for solutions of approximate dual problems. In order to overcome this difficulty, we multiply the approximate dual problem by $(-\Delta + I)^{-1}\eta_n$, where η_n is a solution of the approximate dual problem, $-\Delta$ is the Laplace operator and I is the identity. By this idea, we can obtain the useful estimate in Lemmas 2 and 3, and prove the uniqueness in the similar class to that of [11]. Moreover, we can weaken the regularity conditions for the stress function f discussed in [1] and [11].

We define a weak solution for our problem and give a statement of our theorem in the next section. In Section 3, we prove the uniqueness of a solution to P. Finally, we show the existence of a solution by applying the standard Galerkin method.

2. Main result

Throughout this paper, we use the spaces

$$H := \left(L^2(0,1)\right)^2, V := \left\{z \in \left(W^{2,2}(0,1)\right)^2 | z(0) = z(1), z_x(0) = z_x(1)\right\}$$

with standard norms denoted by $|\cdot|_H$, $|\cdot|_V$, respectively, and $\mathbb{Z}_{>0} := \{n \in \mathbb{Z} | n > 0\}$.

First, we give a definition for a weak solution of P.

Definition 1. *A function u from $Q(T)$ to $\mathbb{R}^2$ is called a weak solution of P on $Q(T)$ if u has the following properties: $u \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)$, $u(0) = u_0$ and satisfying*

$$-\rho \int_{Q(T)} u_t \cdot \eta_t dxdt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dxdt + \int_{Q(T)} f(\varepsilon)u_x \cdot \eta_x dxdt$$

$$= \rho \int_0^1 v_0 \cdot \eta(0)dx \text{ for } \eta \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \text{ with } \eta(T) = 0.$$

We note that $u \cdot v = u_1v_1 + u_2v_2$ for $u = (u_1, v_1)$, $v = (v_1, v_2) \in \mathbb{R}^2$. The main result of this paper is as follows:

Theorem 1. *Let $T > 0$. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, monotone increasing and $f(0) = 0$, $u_0 \in V$ and $v_0 \in H$, then P has a unique weak solution on $Q(T)$.*

The proof of the uniqueness is given in the next section. In Section 4 we prove the existence of solutions.

3. Uniqueness of the solution

In this section we give a proof of the uniqueness for a solution to P and suppose that all assumptions of Theorem 1, satisfy.

Let u_1 and u_2 be solutions of P, namely, u_1 and u_2 satisfies the properties of Definition 1. Also, we put $u = u_1 - u_2$, and $W = \left\{\eta \in W^{2,2}(0,T;H)\right.$

$\cap L^2(0,T;W^{4,2}(0,1)^2) \Big| \eta(T) = \eta_t(T) = 0, \dfrac{\partial^i \eta}{\partial x^i}(t,0) = \dfrac{\partial^i \eta}{\partial x^i}(t,1)$ for $t \in [0,T]$

and $i = 0,1,2,3 \Big\}$. For any $\eta \in W$, we have

$$-\rho \int_{Q(T)} u_t \cdot \eta_t dxdt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dxdt$$

$$+ \int_{Q(T)} \{f(\varepsilon_1)u_{1x} - f(\varepsilon_2)u_{2x}\} \cdot \eta_x dxdt = 0.$$

By integrating by parts in this equation, we have

$$\rho \int_{Q(T)} u \cdot \eta_{tt} dx dt + \gamma \int_{Q(T)} u \cdot \eta_{xxxx} dx dt$$

$$= - \int_{Q(T)} \{(f(\varepsilon_1) - f(\varepsilon_2)) \, u_{1x} + f(\varepsilon_2) \, (u_{1x} - u_{2x})\} \cdot \eta_x dx dt$$

$$= - \int_{Q(T)} \{F_0 a \cdot u_{1x} + f(\varepsilon_2)\} \, \eta_x \cdot u_x dx dt, \qquad (3.1)$$

where $\varepsilon = \varepsilon_1 - \varepsilon_2$, $a = \left(a^{(1)}, a^{(2)} \right)$,

$$F_0 = \begin{cases} \dfrac{f(\varepsilon_1) - f(\varepsilon_2)}{\varepsilon_1 - \varepsilon_2} & \text{if } \varepsilon_1 \neq \varepsilon_2, \\ 0 & \text{if } \varepsilon_1 = \varepsilon_2, \end{cases} \quad a^{(i)} = \begin{cases} \dfrac{u_{1x}^{(i)} + u_{2x}^{(i)}}{|u_{1x}| + |u_{2x}|} & \text{if } |u_{1x}| + |u_{2x}| \neq 0, \\ 0 & \text{if } |u_{1x}| + |u_{2x}| = 0, \end{cases}$$

for $i = 1, 2$, and $u_{jx} = \left(u_{jx}^{(1)}, u_{jx}^{(2)} \right)$ for $j = 1, 2$. Recall that $\varepsilon_j = |u_{jx}| - 1$ for $j = 1, 2$. Also, we put $F = F_0 a \cdot u_{1x} + f(\varepsilon_2)$, and then (3.1) is represented by F as follows:

$$\int_{Q(T)} u \cdot (\rho \eta_{tt} + \gamma \eta_{xxxx}) \, dx dt + \int_{Q(T)} u_x \cdot (F \eta_x) \, dx dt = 0 \text{ for } \eta \in W. (3.2)$$

Since f is Lipschitz continuous and $u_{1x} \in L^\infty(Q(T))$, we have $F \in L^\infty(Q(T))$ and can approximate it by $\{F_n\} \subset C_0^\infty(Q(T))$ satisfying

$$\{F_n\} \text{ is uniformly bounded in } L^\infty(Q(T)) \text{ and}$$
$$F_n \to F \text{ in } L^2(Q(T)) \text{ as } n \to \infty. \qquad (3.3)$$

The first lemma is concerned with the existence of a solution of the approximate dual problem.

Lemma 1. *Let $\varphi \in C_0^\infty(Q(T))$. For $n \in \mathbb{Z}_{>0}$, there exists a unique solution $\eta_n \in W^{2,\infty}(0, T; H) \cap L^\infty \left(0, T; \left(W^{4,2}(0, 1) \right)^2 \right)$ of the following approximate dual problem:*

$$\rho \eta_{ntt} + \gamma \eta_{nxxxx} - (F_n \eta_{nx})_x = \varphi \text{ in } Q(T), \qquad (3.4)$$

$$\eta_n(T) = \eta_{nt}(T) = 0 \text{ on } (0, 1), \qquad (3.5)$$

$$\frac{\partial^i \eta_n}{\partial x^i}(t, 0) = \frac{\partial^i \eta_n}{\partial x^i}(t, 1) \text{ for } t \in [0, T] \text{ and } i = 0, 1, 2, 3. \qquad (3.6)$$

We can easily prove Lemma 1 by the standard discretization method, see Section 5.2 in [3], since (3.4) is linear. So, we omit its proof. The following Lemmas 2 and 3 are keys in the proof of the uniqueness.

Lemma 2. *For each $t \in [0, T]$ there exists a unique solution $\xi_n(t) \in V$ such that*

$$\begin{cases} -\xi_{nxx}(t) + \xi_n(t) = \eta_n(t) \ on \ (0, 1), \\ \xi_n(t, 0) = \xi_n(t, 1) \ and \ \xi_{nx}(t, 0) = \xi_{nx}(t, 1). \end{cases} \tag{3.7}$$

Moreover, it holds that $\xi_n \in W^{2,2}(0, T; , (W^{2,2}(0, 1))^2)$ and

$$-\xi_{nttxx}(t) + \xi_{ntt}(t) = \eta_{ntt}(t) \ on \ (0, 1),$$
$$\xi_{ntt}(t, 0) = \xi_{ntt}(t, 1), \xi_{nttx}(t, 0) = \xi_{nttx}(t, 1) \ for \ a.e. \ t \in [0, T].$$

This lemma is a direct consequence of the Riesz representation theorem to the Hilbert space $X = \left\{ z \in \left(W^{1,2}(0, 1) \right)^2 \Big| z(0) = z(1) \right\}$. In fact, we define a weak solution of the problem (3.7), if ξ_n satisfies

$$\xi_n \in X \ and \ (\xi_n, z)_X = \int_0^1 \eta_n(t, x) \cdot z(x) dx \ for \ z \in X, \tag{3.8}$$

where $(\cdot, \cdot)_X$ is the standard inner product of X. Thanks to the Riesz representation theorem, there exists a unique weak solution $\xi_n(t)$ of (3.7) for each $t \in [0, T]$, since $\eta_n(t) \in H$ for $t \in [0, T]$. Moreover, it is easily seen that $\xi_n(t)$ is a strong solution of (3.7).

From Lemma 2 we can get the following uniform estimate for η_{nx} with respect to n.

Lemma 3. *There exists $\alpha > 0$ such that*

$$|\eta_{nx}|_H \le \alpha \ on \ [0, T] \ for \ n \in Z_{>0}.$$

Proof of Lemma 3. For $n \in \mathbb{Z}_{>0}$, let η_n be a solution for the approximate dual problem (3.4) - (3.6), and ξ_n be a solution of (3.7). By putting $\widehat{\eta}_n(t) = \eta_n(T - t)$, $\widehat{\xi}_n(t) = \xi_n(T - t)$, $\widehat{F}_n(t) = F_n(T - t)$ and $\widehat{\varphi}(t) = \varphi(T - t)$ for $t \in (0, T)$, and $n \in \mathbb{Z}_{>0}$, we have

$$\rho \widehat{\eta}_{ntt} + \gamma \widehat{\eta}_{nxxxx} - \left(\widehat{F}_n \widehat{\eta}_{nx} \right)_x = \widehat{\varphi} \ in \ Q(T), \tag{3.9}$$
$$\widehat{\eta}_n(0) = \widehat{\eta}_{nt}(0) = 0 \ on \ (0, 1),$$
$$\frac{\partial^i \widehat{\eta}_n}{\partial x^i}(t, 0) = \frac{\partial^i \widehat{\eta}_n}{\partial x^i}(t, 1) \ on \ (0, T) \ for \ i = 0, 1, 2, 3,$$

and

$$-\widehat{\xi}_{nttxx}(t) + \widehat{\xi}_{ntt}(t) = \widehat{\eta}_{ntt}(t) \ in \ (0, 1),$$
$$\widehat{\xi}_{ntt}(t, 0) = \widehat{\xi}_{ntt}(t, 1) \ and \ \widehat{\xi}_{nttx}(t, 0) = \widehat{\xi}_{nttx}(t, 1) \ for \ a.e. \ t \in (0, T). \ (3.10)$$

We multiply both sides of (3.9) and (3.10) by $\widehat{\xi}_{nt}$, and then we have

$$\rho\widehat{\eta}_{ntt} \cdot \widehat{\xi}_{nt} + \gamma\widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt} = \left(\widehat{F}_n\widehat{\eta}_{nx}\right)_x \cdot \widehat{\xi}_{nt} + \widehat{\varphi} \cdot \widehat{\xi}_{nt} \text{ in } Q(T), (3.11)$$

$$-\widehat{\xi}_{nttxx} \cdot \widehat{\xi}_{nt} + \widehat{\xi}_{ntt} \cdot \widehat{\xi}_{nt} = \widehat{\eta}_{ntt} \cdot \widehat{\xi}_{nt} \text{ in } Q(T). \tag{3.12}$$

By substituting (3.12) into (3.11) we have

$$-\rho\int_0^1 \widehat{\xi}_{nttxx} \cdot \widehat{\xi}_{nt}dx + \rho\int_0^1 \widehat{\xi}_{ntt} \cdot \widehat{\xi}_{nt}dx + \gamma\int_0^1 \widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt}dx$$

$$= \int_0^1 \left(\widehat{F}_n\widehat{\eta}_{nx}\right)_x \cdot \widehat{\xi}_{nt}dx + \int_0^1 \widehat{\varphi} \cdot \widehat{\xi}_{nt}dx \quad \text{a.e. on } [0,T].$$

Here, we note that

$$\int_0^1 \widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt}dx = \int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\xi}_{ntxx}dx$$

$$= -\int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\eta}_{nt}dx + \int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\xi}_{nt}dx \quad \text{on } [0,T].$$

Accordingly, we have

$$\frac{1}{2}\frac{d}{dt}\left(\rho\left|\widehat{\xi}_{nxt}\right|_H^2 + \rho\left|\widehat{\xi}_{nt}\right|_H^2 + \gamma\left|\widehat{\eta}_{nx}\right|_H^2\right)$$

$$= \gamma\int_0^1 \widehat{\eta}_{nx} \cdot \widehat{\xi}_{nxt}dx$$

$$- \int_0^1 \widehat{F}_n\widehat{\eta}_{nx} \cdot \widehat{\xi}_{nxt}dx + \int_0^1 \widehat{\varphi} \cdot \widehat{\xi}_{nt}dx \quad \text{a.e. on } [0,T].$$

Since $\varphi \in C_0^\infty(Q(T))$, there exists a positive constant C_1 such that $|\varphi(t,x)| \le C_1$ for $(t,x) \in Q(T)$, and then we have

$$\frac{d}{dt}\left(\rho\left|\widehat{\xi}_{nxt}\right|_H^2 + \rho\left|\widehat{\xi}_{nt}\right|_H^2 + \gamma\left|\widehat{\eta}_{nx}\right|_H^2\right)$$

$$\le C_2\left\{\rho\left|\widehat{\xi}_{nxt}\right|_H^2 + \rho\left|\widehat{\xi}_{nt}\right|_H^2 + \gamma\left|\widehat{\eta}_{nx}\right|_H^2\right\} + C_1^2 \text{ a.e. on } [0,T],$$

where C_2 is a positive constant depending only on ρ, γ and

$$\max_{n\in\mathbb{Z}_{>0}}\left|\widehat{F}_n\right|_{L^\infty(Q(T))}.$$

By applying Gronwall's inequality, we obtain

$$\rho\left|\widehat{\xi}_{nxt}\right|_H^2 + \rho\left|\widehat{\xi}_{nt}\right|_H^2 + \gamma\left|\widehat{\eta}_{nx}\right|_H^2$$

$$\le e^{C_2T}\left(\rho\left|\widehat{\xi}_{nxt}(0)\right|_H^2 + \rho\left|\widehat{\xi}_{nt}(0)\right|_H^2 + \gamma\left|\widehat{\eta}_{nx}(0)\right|_H^2 + C_1^2T\right) \text{ on } [0,T].$$

Hence, in view of (3.7) and (3.9), this lemma is proved.   □

Proof of the uniqueness. Let $n \in \mathbb{Z}_{>0}$ and $\varphi \in C_0^\infty(Q(T))^2$. By Lemma 1, there exists a solution $\eta_n \in W$ of (3.4) - (3.6). From (3.4), integration by parts and (3.2), it follows

$$\left| \int_{Q(T)} u \cdot \varphi \, dx dt \right| = \left| \int_{Q(T)} u \left\{ \rho \eta_{ntt} + \gamma \eta_{nxxxx} \right\} dx dt, - \int_{Q(T)} u \left(F_n \eta_{nx} \right)_x dx dt \right|$$

$$= \left| \int_{Q(T)} (F_n - F) \, u_x \cdot \eta_{nx} dx dt \right| \text{ for each } n \in \mathbb{Z}_{>0}.$$

Thanks to Lemma 3, we have

$$\left| \int_{Q(T)} u \cdot \varphi \, dx dt \right| \leq \alpha \left| u_x \right|_{L^\infty(Q(T))} \int_0^T \left| F_n - F \right|_H dt$$

$$\leq \alpha \sqrt{T} \left| u_x \right|_{L^\infty(Q(T))} \left| F_n - F \right|_{L^2(Q(T))} \text{ for } \varphi \in C_0^\infty(Q(T)).$$

Thus, (3.3) implies that

$$\int_{Q(T)} u \cdot \varphi \, dx dt = 0 \text{ for } \varphi \in C_0^\infty(Q(T)),$$

and then $u = 0$ on $Q(T)$. Hence, we have proved the uniqueness of the solution for P.   □

4. Existence of the solutions

In this section we prove existence of a solution to P. Since V is a separable Hilbert space, we can choose a complete orthonormal system $\{\psi_n\}_{n=1}^\infty$ of V normalized in H. Also, we shall use the closed linear space V_n generated by $\psi_1, \psi_2, \ldots, \psi_n$ for $n \in \mathbb{Z}_{>0}$. Moreover, since $u_0 \in V$, $v_0 \in H$ and V is dense in H, there exist $\{u_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V$, $\{v_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V$, and $\{m_n\}_n \in \mathbb{Z}_{>0}$ such that

$$u_{0n}, v_{0n} \in V_{m_n} \text{ for } n \in \mathbb{Z}_{>0},$$
$$u_{0n} \to u_0 \text{ in } V \text{ and } v_{0n} \to v_0 \text{ in } H \text{ and } m_n \to \infty \text{ as } n \to \infty.$$

We prove the existence by the Galerkin method, namely, first for $n \in \mathbb{Z}_{>0}$ we find $u_n(t) = \sum_{k=1}^{m_n} a_k^{(n)}(t)\psi_k$ satisfying

$$\rho \int_0^1 u_{ntt}(t) \cdot \psi_j dx + \gamma \int_0^1 u_{ntt}(t) \cdot \psi_{jxx} dx$$

$$+ \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot \psi_{jx} dx = 0 \text{ for } t \in [0, T] \text{ and } j = 1, 2, \ldots, m_n, \quad (4.1)$$

$$u_n(0) = u_{0n}, u_{nt}(0) = v_{0n} \text{ and } \varepsilon_n = |u_{nx}| - 1 \text{ on } Q(T). \quad (4.2)$$

We denote by P_n the problem (4.1) and (4.2) for each $n \in \mathbb{Z}_{>0}$. For proving the existence of a solution u_n of P_n for $n \in \mathbb{Z}_{>0}$, we solve the following initial value problem I_n for the ordinary differential equations:

Find $a^{(n)} = \left(a_1^{(n)}, a_2^{(n)}, \ldots, a_{m_n}^{(n)}\right) \in C^2([0, T])^{m_n}$ such that

$$\rho \frac{d^2 a^{(n)}}{dt^2} = -F\left(a^{(n)}\right) - G\left(a^{(n)}\right) \text{ on } [0, T],$$

$$a^{(n)}(0) = a_0^{(n)}, \frac{da^{(n)}}{dt} = b_0^{(n)},$$

where $a_0^{(n)} = \left(a_{01}^{(n)}, a_{02}^{(n)}, \ldots, a_{0m_n}^{(n)}\right) \in \mathbb{R}^{m_n}$, $b_0^{(n)} = \left(b_{01}^{(n)}, b_{02}^{(n)}, \ldots, b_{0m_n}^{(n)}\right) \in \mathbb{R}^{m_n}$, $F = (F_1, F_2, \ldots, F_{m_n})$, $G = (G_1, G_2, \ldots, G_{m_n})$,

$$F_j\left(a^{(n)}\right) = \gamma \sum_{k=1}^{m_n} a_k^{(n)}(t) \int_0^1 \psi_{kxx} \cdot \psi_{jxx} dx,$$

$$G_j\left(a^{(n)}\right) = \int_0^1 f\left(\left|\sum_{k=1}^{m_n} a_k^{(n)}(t)\psi_{kx}(x)\right| - 1\right)\left(\sum_{k=1}^{m_n} a_k^{(n)}(t)\psi_{kx}(x) \cdot \psi_{jx}(x)\right) dx$$

for $j = 1, 2, \ldots, m_n$.

Here, we note that Picard's theorem for ordinary differential equations guarantees the existence and uniqueness of the solution for I_n, since F and G are locally Lipschitz continuous on $\mathbb{R}^{m_n}$, and the uniform estimate (4.4) for u_n holds. Thus, we have:

Lemma 4. *Let $n \in \mathbb{Z}_{>0}$. If $a_0^{(n)} = \left(a_{01}^{(n)}, a_{02}^{(n)}, \ldots, a_{0m_n}^{(n)}\right) \in \mathbb{R}^{m_n}$,*

$b_0^{(n)} = \left(b_{01}^{(n)}, b_{02}^{(n)}, \ldots, b_{0m_n}^{(n)}\right) \in \mathbb{R}^{m_n}$ *satisfying* $u_{0n} = \sum_{k=1}^{m_n} a_{0k}^{(n)} \psi_k,$

$v_{0n} = \sum_{k=1}^{m_n} b_{0k}^{(n)} \psi_k,$ *then there exists one and only one* $u_n \in C^2([0, T]; V_{m_n})$

satisfying (4.1) *and* (4.2). *Also, it holds that*

$$\rho \int_0^1 u_{ntt}(t) \cdot \eta dx + \gamma \int_0^1 u_{nxx}(t) \cdot \eta_{xx} dx + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot \eta_x dx = 0$$

$$for \ \eta \in V_{m_n}. \ (4.3)$$

Now, we give a lemma dealing with the uniform estimate of u_n.

Lemma 5. *If u_n is a solution of P_n on $[0, T]$ for $n \in \mathbb{Z}_{>0}$, the following energy G_n is conserved:*

$$G_n = \frac{\rho}{2} \int_0^1 |u_{nt}|^2 \, dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 \, dx + \frac{1}{2} \int_0^1 \widehat{g}(|u_{nx}|^2) dx,$$

$$\frac{d}{dt} G_n = 0 \ on \ [0, T],$$

where $\widehat{g}$ is a primitive of f, and satisfies $\widehat{g}(1) = 0$. Moreover, it holds that

$$\frac{\rho}{2} \int_0^1 |u_{nt}|^2 \, dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 \, dx \le G_n(0) \ on \ [0, T]. \tag{4.4}$$

Proof.   Let u_n be a solution of P_n on $[0, T]$, namely, it is represented by $u_n = \sum_{k=1}^{m_n} a_k^{(n)} \psi_k$ on $Q(T)$ for $n \in \mathbb{Z}_{>0}$. Since $u_{nt} = \sum_{k=1}^{m_n} \dfrac{da_k^{(n)}}{dt} \psi_k \in V_{m_n}$ on $Q(T)$, we can substitute $\eta = u_{nt}$ into (4.3) and have

$$\rho \int_0^1 u_{ntt}(t) \cdot u_{nt}(t) dx + \gamma \int_0^1 u_{nxx}(t) \cdot u_{ntxx}(t) dx$$

$$+ \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot u_{ntx}(t) dx = 0 \text{ for } t \in [0, T].$$

Here, we put $z = |u_{nx}|^2$ and $g(z) = f(\sqrt{z} - 1)$ for $z \in \mathbb{R}$, and then we have

$$\int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot u_{ntx}(t) dx = \frac{1}{2} \int_0^1 \frac{\partial}{\partial t} \widehat{g}(z) dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_0^1 \widehat{g}\left(|u_{nx}|^2\right) dx \text{ for } t \in [0, T],$$

where $\widehat{g}(r) = \int_1^r g(\xi) d\xi$ for $r \in \mathbb{R}$. Hence, we obtain

$$\frac{d}{dt} \left(\frac{\rho}{2} \int_0^1 |u_{nt}|^2 \, dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 \, dx + \frac{1}{2} \int_0^1 \widehat{g}\left(|u_{nx}^2|\right) dx \right) = 0 \text{ on } [0, T].$$

Clearly, $\widehat{g}$ is a primitive of g and satisfies $\widehat{g}(1) = 0$. Since f is monotone increasing, we see that $\widehat{g}(r) \geq 0$ for any $r \in \mathbb{R}$. Thus, Lemma 4.4 has been proved. $\square$

Lemma 6. *It holds that $\{u_n\}_{n \in \mathbb{Z}_{>0}}$ is bounded in $L^\infty(0, T; V)$ and $W^{1,\infty}(0, T; H)$.*

Proof. First, by the boundedness of $\{u_{0n}\}_{n \in \mathbb{Z}_{>0}}$ in V, $\{u_{0nx}\}_{n \in \mathbb{Z}_{>0}}$ is bounded in $L^\infty(0, 1)$. This shows that $\{G_n(0)\}_{n \in \mathbb{Z}_{>0}}$ is bounded and $\{u_{nt}\}_{n \in \mathbb{Z}_{>0}}$ and $\{u_{nxx}\}_{n \in \mathbb{Z}_{>0}}$ are bounded in $L^\infty(0, T; H)$. Hence, it is clear that the assertion of this lemma is true. $\square$

Next, we show existence of a convergence subsequense. Here, we put

$$X = \left\{ z \in W^{1,2}(0,1)^2 \,\middle|\, z(0) = z(1) \right\}, \text{ again.}$$

Lemma 7. *There exist a subsequence $\{n_k\} \subset \{n\}$ and a function u on $Q(T)$ such that $u \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H)$,*

$$u_{n_k} \to u \quad \text{weakly* in } L^\infty(0, T; V), \text{ in } L^2(0, T; X),$$
$$\text{and weakly* in } W^{1,\infty}(0, T; H) \text{ as } k \to \infty.$$

Proof. By Lemma 6 and the Aubin-Lions lemma (cf. [7]), it is easy to show existence of the subsequence with the required condition. $\square$

The following lemma is concerned with approximation of the test function η.

Lemma 8. *For $\eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V)$ with $\eta(T) = 0$, there exists $\{\eta_n\} \subset W^{1,2}(0, T; V)$ such that*

$$\eta_n \in L^2(0, T; V_n), \eta_n(T) = 0, \eta_n(0) \to \eta(0) \text{ in } H \text{ as } n \to \infty,$$
$$\eta_n \to \eta \text{ in } L^2(0, T; V) \text{ and } \eta_{nt} \to \eta_t \text{ in } L^2(0, T; H) \text{ as } n \to \infty.$$

Proof of the existence. Put $u_k = u_{n_k}$ and $\eta_k = \eta_{n_k}$ for $k \in \mathbb{Z}_{>0}$. Since u_k is the solution of P_{n_k}, by Lemma 4 we obtain

$$\rho \int_{Q(T)} u_{ktt} \cdot \eta_k dx dt + \gamma \int_{Q(T)} u_{kxx} \cdot \eta_{kxx} dx dt + \int_{Q(T)} f(\varepsilon_{n_k}) u_{kx} \cdot \eta_{kx} dx dt = 0,$$

and

$$-\rho \int_{Q(T)} u_{kt} \cdot \eta_k dx dt + \gamma \int_{Q(T)} u_{kxx} \cdot \eta_{kxx} dx dt + \int_{Q(T)} f(\varepsilon_{n_k}) u_{kx} \cdot \eta_{kx} dx dt$$
$$= -\int_0^1 v_{0n_k} \eta_k(0) dx \text{ for } k \in \mathbb{Z}_{>0}.$$

By letting $k \to \infty$ in this equation, Lemmas 7 and 8 guarantee that u satisfies the condition in Definition 1. Hence, the existence of the solution to P has been proved. □

5. Conclusion

In this paper we have established existence and uniqueness of a weak solution to the initial boundary value problem for the beam equation accompanying with the nonlinear stress and strain functions. We note that we consider the stress function as a continuous function f on $\mathbb{R}$ having no singularity and the uniqueness is proved thanks to application of the approximate dual problem. In near future we will investigate the similar problem in case the stress function has the singularity such that $f(r) \to \infty$ as $r \downarrow -1$.

References

1. Aiki T. Weak solutions for Falk's Model of Shape Memory Alloys. *Math. Methods Appl. Sci.*, 2000, vol. 23, pp. 299-319.
2. Aiki T., Kosugi C. Numerical schemes for ordinary differential equations describing shrinking and stretching motion of elastic materials. *Adv. Math. Sci. Appl.*, 2020, vol. 29, pp. 459-494.
3. Brocate M., Sprekels J. *Hysteresis and phase transitions*. Springer, Appl. Math. Sci., 1996, vol. 121.
4. Furihata D., Matsuo T. *Discrete variational derivative method: A structure preserving numerical method for partial differential equations*. Chapman & Hall CRC Publ., 2010.
5. Holzapfel Gerhard A. *Nonlinear solid mechanics: a continuum approach for engineering*. John Wiley & Sons Publ., 2000.
6. Ladyzenskaja O.A., Solonnikov V.A., Ural'ceva N.N. *Linear and Quasi-Linear Equations of Parabolic Type*. Transl. Math. Monograph, Vol. 23, Amer. Math. Soc., Providence, R. I. 1968.
7. Lions J.L. *Quelques méthodes de résolution des problèmes aux limites non linéairs*. Paris, Dunod, Gauthier - Villars Publ., 1969.
8. Niezgódka M., Pawlow I. A generalized Stefan problem in several space variables. *Appl. Math. Opt.*, 1983, vol. 9, pp. 193-224.
9. Ogden Raymond William. Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubber like solids. *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.*, 1972, vol. 328, pp. 567-583.
10. J. C. Simo, C. Miehe, Associative coupled thermoplasticity at finite strains: Formulation, numerical analysis and implementation. *Comput. Methods in App. Mech. Engi.*, 1992, vol. 98, pp. 41-104.
11. Yoshikawa S. Weak solutions for the Falk model system of shape memory alloys in energy class. *Math. Methods Appl. Sci.*, 2005, vol. 28, pp. 1423-1443.

12. Yoshikawa S. An error estimate for structure — preserving finite difference scheme for the Falk model system of shape memory alloys. *IMA J. Numer. Anal.*, 2017, vol. 37, pp. 477-504.

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Существование и единственность слабых решений для модели, представляющей движения кривых из эластичных материалов

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Аннотация. Рассмотрена начально-краевая задача для уравнения балки с нелинейной деформацией. В нашей предыдущей работе эта задача была рассмотрена в виде математической модели для растягивающих и сжимающих движений кривой из эластичного материала на плоскости. Цель статьи — установить единственность и существование слабых решений. В частности, единственность доказывается применением приближенного метода двойственных уравнений.

Ключевые слова: задача со свободной границей, периодические решения.

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