Серия «Математика»
2021. Т. 35. С. 60—72

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

УДК 519.6
MSC 35L60
DOI https://doi.org/10.26516/1997-7670.2021.35.60

Complete Riemann Solvers for the Hyperbolic GPR Model of Continuum Mechanics

U. Ariunaa[1], M. Dumbser[2], Ts. Sarantuya[1]

[1] *Mongolian University of Science and Technology, Ulaanbaatar, Mongolia*
[2] *University of Trento, Trento, Italy*

Abstract. In this paper, complete Riemann solver of Osher-Solomon and the HLLEM Riemann solver for unified first order hyperbolic formulation of continuum mechanics, which describes both of fluid and solid dynamics, are presented. This is the first time that these types of Riemann solvers are applied to such a complex system of governing equations as the GPR model of continuum mechanics. The first order hyperbolic formulation of continuum mechanics recently proposed by Godunov S. K., Peshkov I. M. and Romenski E. I., further denoted as GPR model includes a hyperbolic formulation of heat conduction and an overdetermined system of PDE. Path-conservative schemes are essential in order to give a sense to the non-conservative terms in the weak solution framework since governing PDE system contains non-conservative products, too. New Riemann solvers are implemented and tested successfully, which means it certainly acts better than standard local Lax-Friedrichs-type or Rusanov-type Riemann solvers. Two simple computational examples are presented, but the obtained computational results clearly show that the complete Riemann solvers are less dissipative than the simple Rusanov method that was employed in previous work on the GPR model.

Keywords: Riemann solvers, the hyperbolic GPR model, continuum mechanics, HLLEM Riemann solver.

1. Introduction

In this work, unified first order hyperbolic formulation of continuum mechanics, further denoted as GPR model proposed by Godunov, Peshkov and Romenski [3;9]. Especially, the Osher Riemann Solver [7] and the HLLEM

Riemann solver [2] are described. Simple and universal formulation of the HLLEM Riemann solver [2] is proposed that works for general conservative and non-conservative systems of hyperbolic equations. The computational results present for large set of different conservative and non-conservative hyperbolic systems, without stiff source term. Since the governing PDE system proposed in [3] contains non-conservative products, the use of so-called path-conservative schemes becomes necessary, in order to give a sense to the non-conservative terms in the framework of weak solutions. Riemann solvers are the key modern numerical schemes for hyperbolic systems and a large number of Riemann solvers are available in conservation form. In [4] a simple extension of the well-known Osher-Solomon solver to a rather general class of hyperbolic systems in non-conservative form was proposed. They have been applied the method to the shallow water equations with spatially variable and temporally fixed bottom as well as to the two-fluid debris flow model. Except the Osher-Solomon solver, Parés [8] and Castro et al. [1] concerned on path-conservative numerical schemes based on Godunov, Roe methods. It is important to mention, that high order fully-discrete one-step ADER-WENO finite volume schemes (see [5]) and ADER discontinuous Galerkin finite element schemes were used to solve the GPR model in the stiff relaxtion limit, but only a simple Rusanov flux was employed in [3]. Hence, the purpose of this paper is to extend complete Riemann solvers to instead of complex GPR model in order to improve the resolution of intermediate waves. Subsequently, other higher order nonlinear schemes for the GPR model have been proposed, such as the Split-WENO method [6] and ADER -WENO-ALE schemes [10]. The implementation and testing of the new Riemann solvers was successful and the complete Riemann solvers clearly behave better than standard local Lax-Friedrichs-type or Rusanov-type Riemann solvers.

2. Model

We consider the first order hyperbolic Godunov- Peshkov-Romenski (GPR) model [3; 9], which is the first successful attempt to build a unified and thermodynamically compatible formulation of continuum mechanics under a first order symmetric hyperbolic form that includes classical fluid mechanics and solid mechanics just as two special limiting cases of the same formulation. We refer to the recent work of Dumbser et al. [2], where a detailed introduction to this model is given and where the GPR model has been solved numerically for the first time using high order accurate Eulerian ADER-WENO and ADER-DG schemes on fixed grids, and where many numerical examples have been provided. The GPR model also includes a hyperbolic formulation of heat conduction and it can be written under the

form given in [2] as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0 \tag{2.1a}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial (\rho v_i v_k + p\delta_{ik} - \sigma_{ik})}{\partial x_k} = 0 \tag{2.1b}$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial A_{im} v_m}{\partial x_k} + v_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} \right) = -\frac{\psi_{ik}}{\theta_1(\tau_1)} \tag{2.1c}$$

$$\frac{\partial \rho J_i}{\partial t} + \frac{\partial (\rho J_i v_k + T\delta_{ik})}{\partial x_k} = -\frac{\rho H_i}{\theta_2(\tau_2)} \tag{2.1d}$$

$$\frac{\partial \rho s}{\partial t} + \frac{\partial (\rho s v_k + H_k)}{\partial x_k} = \frac{\rho}{\theta_1 T} \psi_{ik} \psi_{ik} + \frac{\rho}{\theta_2 T} H_i H_i \geq 0 \tag{2.1e}$$

The solutions of the above PDE system fulfill also the additional conservation of total energy

$$\frac{\partial \rho E}{\partial t} + \frac{\partial (v_k \rho E + v_i (p\delta_{ik} - \sigma_{ik}) + q_k)}{\partial x_k} = 0 \tag{2.2}$$

At this point we emphasize that the system above is an overdetermined system of PDE, hence in the numerical solution of the above model we solve the total energy conservation equation (2.2) and not the entropy equation (2.1e), in order to obtain the correct propagation of shock waves.
We use the following notations: ρ is the mass density, $[v_i] = \mathbf{v} = (u, v, w)$ is the velocity vector, $[A_{ik}] = \mathbf{A}$ is the distortion tensor, $[J_i] = \mathbf{J}$ is the thermal impulse vector, s is the entropy, $p = \rho^2 E_\rho$ is the pressure, $E = E(\rho, s, \mathbf{v}, \mathbf{A}, \mathbf{J})$ is the total energy potential, δ_{ik} is the Kronecker delta, $[\sigma_{ik}] = \boldsymbol{\sigma} = -[\rho A_{mi} E_{A_{mk}}]$ is the symmetric shear stress tensor, $T = E_s$ is the temperature, $[q_k] = \mathbf{q} = [E_s E_{J_k}]$ is the heat flux vector and $\theta_1 = \theta_1(\tau_1) > 0$ and $\theta_2 = \theta_2(\tau_2) > 0$ are positive scalar functions depending on the strain dissipation time $\tau_1 > 0$ and the thermal impulse relaxation time $\tau_2 > 0$, respectively.

The dissipative terms ψ_{ik} and H_i on the right hand side of the evolution equations for $\mathbf{A}, \mathbf{J}$ and s are defined as $[\psi_{ik}] = \boldsymbol{\psi} = [E_{A_{ik}}]$ and $[H_i] = \mathbf{H} = [E_{J_i}]$ respectively. Accordingly, the viscous stress tensor and the heat flux vector are directly related to the dissipative terms on the right hand side via $\boldsymbol{\sigma} = -\rho \mathbf{A}^\mathbf{T} \boldsymbol{\psi}$ and $\mathbf{q} = T\mathbf{H}$. Note that $E_\rho, E_s, E_{A_{ik}}$ and E_{J_i} denote the partial derivatives $\partial E/\partial \rho, \partial E/\partial s, \partial E/\partial A_{ik}$ and $\partial E/\partial J_i$, they are the energy gradients in the state space or the thermodynamic forces.

These equations express the mass conservation (2.1a), the momentum conservation (2.1b), the time evolution for the distortion tensor (2.1c), the time evolution for the thermal impulse (2.1d), the time evolution for the entropy (2.1e), and the total energy conservation (2.2). The PDE governing the time evolution of the thermal impulse (2.1d) looks similar

to the momentum equation (2.1b), where the temperature T takes the role of the pressure p. Therefore we refer to this equation as the thermal momentum equation. To close the above system, the total energy potential $E(\rho, s, \mathbf{v}, \mathbf{A}, \mathbf{J})$ must be specified. This potential definition will then generate all constitutive fluxes (i.e. non advective fluxes) and source terms by means of its partial derivatives with respect to the state variables. As s consequence the energy potential specification is fundamental for the model formulation. The total energy E is the sum of three terms (details can be found [3]), each of which represents the energy distributed in its corresponding scale. Thus, we assume that

$$E(\rho, s, \mathbf{v}, \mathbf{A}, \mathbf{J}) = E_1(\rho, s) + E_2(\mathbf{A}, \mathbf{J}) + E_3(\mathbf{v}) \tag{2.3}$$

E_1 is the ideal gas equation of state:

$$E_1(\rho, s) = \frac{c_0^2}{\gamma(\gamma - 1)},\ c_0^2 = \gamma \rho^{\gamma - 1} e^{s/c_V} \tag{2.4}$$

where c_0 has the meaning of the adiabatic sound speed, c_v and c_p are the specific heat capacities at constant volume and at constant pressure, which are related by the ratio of specific heats $\gamma = c_p/c_V$.
For the non-equilibrium, part of the total energy:

$$E_2(\mathbf{A}, \mathbf{J}) = \frac{c_s^2}{4} G_{ij}^{TF} G_{ij}^{TF} + \frac{\alpha^2}{2} J_i J_i \tag{2.5}$$

with

$$[G_{ij}^{TF}] = \mathrm{dev}(\mathbf{G}) = \mathbf{G} - \frac{1}{3}\mathrm{tr}(\mathbf{G})\mathbf{I}, \text{ and } \mathbf{G} = \mathbf{A}^T \mathbf{A} \tag{2.6}$$

Here, c_s is the characteristic velocity of propagation of transverse perturbations, α is the characteristic velocity of propagation of heat waves. The specific kinetic energy per unit mass:

$$E_3(\mathbf{v}) = \frac{1}{2} v_i v_i \tag{2.7}$$

For simplicity, in this paper we consider velocities c_s and α are assumed to be constant.

3. General Form of the Governing PDE

Here we want to show general form of nonlinear system of hyperbolic PDEs with non-conservative products and stiff source terms:

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{Q}}{\partial x} = \mathbf{S}(\mathbf{Q}), \quad x \in \Omega \subset \mathbb{R}, \quad t \in \mathbb{R}_0^+ \tag{3.1}$$

$\mathbf{Q}(x;t) \in \Omega_q \subset \mathbb{R}_q^n$ is the state vector, Ω_q is the space of admissible states (state space), $\mathbf{F}(\mathbf{Q})$ is a nonlinear flux that contains the conservative part, $\mathbf{B}(\mathbf{Q})$ is an $n \times n$ matrix that includes the purely non-conservative part of the system. $\mathbf{S}(\mathbf{Q})$ is a nonlinear algebraic source term.
When written in quasilinear form, the system (3.1) becomes

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{A}(\mathbf{Q}) \cdot \frac{\partial \mathbf{Q}}{\partial x} = \mathbf{S}(\mathbf{Q}) \tag{3.2}$$

where the matrix $\mathbf{A}(\mathbf{Q}) = \partial \mathbf{F}(\mathbf{Q})/\partial \mathbf{Q} + \mathbf{B}(\mathbf{Q})$ includes both the Jacobian of the conservative flux, as well as the non-conservative product.
The system is called hyperbolic if all the eigenvalues of the matrix $\mathbf{A}$ are real numbers and if there exists a complete set of n eigenvectors that are linearly independent. In the case $\mathbf{B}(\mathbf{Q}) = 0$, the PDE (3.1) reduces to a flux form and is hence called a system of conservation laws. In the following we will denote the eigenvalues of $\mathbf{A}(\mathbf{Q})$ with $\Lambda(\mathbf{Q}) = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, where the eigenvalues are ordered as $\lambda_1 \leq \lambda_2, \ldots, \leq \lambda_n$. The left eigenvectors of $\mathbf{A}(\mathbf{Q})$ associated with the eigenvalues will be denoted by $\mathbf{L}(\mathbf{Q}) = (\mathbf{l_1^T}, \mathbf{l_2^T}, ..., \mathbf{l_n^T})^\mathbf{T}$ and the matrix of right eigenvectors of $\mathbf{A}(\mathbf{Q})$ with $\mathbf{R}(\mathbf{Q}) = (\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_n})$. We furthermore assume that the left and right eigenvectors are orthonormal, i.e. $\mathbf{L} \cdot \mathbf{R} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
When the system (3.1) is hyperbolic, the matrix $\mathbf{A}(\mathbf{Q})$ can be diagonalized, hence $\mathbf{A}(\mathbf{Q}) = \mathbf{R}(\mathbf{Q})\mathbf{\Lambda}(\mathbf{Q})\mathbf{L}(\mathbf{Q})$.

4. Complete Riemann solvers

For solving the mathematical models presented in the previous section 3, three different numerical fluxes namely a standard Rusanov-type, an Osher–type and a HLLEM-type have been chosen and compared. The integration over the space–time control volume $[x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}] \times [t^n; t^{n+1}]$ and application of Gauss' theorem, the following integral formulation for the (3.1) is obtained:

$$\begin{aligned} \mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n &- \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) - \frac{\Delta t}{\Delta x}\left(\mathbf{D}^-_{i+\frac{1}{2}} + \mathbf{D}^+_{i-\frac{1}{2}}\right) \\ &- \Delta t \mathbf{B}(\mathbf{Q}_i^{n+\frac{1}{2}})\frac{\Delta \mathbf{Q}_i^n}{\Delta x} + \Delta t \mathbf{S}(\mathbf{Q}_i^{n+\frac{1}{2}}) \end{aligned} \tag{4.1}$$

where $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta t = t^{n+1} - t^n$ represent the mesh spacing and the time step, respectively. A new term $\mathbf{D}^{\pm}_{i+\frac{1}{2}}$ has been introduced in order to take into account the jumps of the solution $\mathbf{Q}\pm_{i+\frac{1}{2}}$ on the space-time element boundaries. This method is stable under conventional CFL condition:

$$\Delta t = CFL \frac{\Delta x}{\max\limits_{i=1} |\Lambda(Q_i^n)|}, \quad \text{with} \quad CFL \leq 1. \tag{4.2}$$

While eqiation (4.1) is exact relation, a numerical scheme is obtained by using a numerical flux, which is a function of two arguments.

4.1. Osher type Riemann solver

For the Osher-type scheme [7], the numerical flux is defined by

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}(\mathbf{Q}^{+}_{i+\frac{1}{2}}) + \mathbf{F}(\mathbf{Q}^{-}_{i+\frac{1}{2}})\right) - \frac{1}{2}\int_0^1 \left|\mathbf{A}\left(\Psi(s)\right)\right| \frac{\partial \Psi}{\partial s} ds \tag{4.3}$$

and the so-called jump terms by

$$\mathbf{D}^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\int_1^0 \mathbf{B}\left(\Psi(s)\right) \frac{\partial \Psi}{\partial s} ds, \tag{4.4}$$

with the segment path

$$\Psi(s) = \mathbf{Q}^{-}_{i+\frac{1}{2}} + s\left(\mathbf{Q}^{+}_{i+\frac{1}{2}} - \mathbf{Q}^{-}_{i+\frac{1}{2}}\right) \tag{4.5}$$

that connects the left state $\mathbf{Q}^{-}_{i+\frac{1}{2}}$ with the right state $\mathbf{Q}^{+}_{i+\frac{1}{2}}$.

4.2. HLLEM Riemann solver

Here we briefly summarize the HLLEM-type Riemann solver of Dumbser and Balsara [2], which allows to resolve intermediate fields with less numerical dissipation than the standard HLL method or local Lax-Friedrich-type schemes. In [2], the HLLEM-type Riemann solvers have been properly adapted to hyperbolic systems with non-conservative products using the family of path-conservative finite volume schemes of Parés [8] and Castro et al. [1].

In the following, we use the notation $\mathbf{Q}_L = \mathbf{Q}^{-}_{i+\frac{1}{2}}$ and $\mathbf{Q}_R = \mathbf{Q}^{+}_{i+\frac{1}{2}}$. In the HLLEM Riemann solver, the jump terms read

$$\mathbf{D}^{-}_{i+\frac{1}{2}} = -\frac{s_L}{s_R - s_L}\int_1^0 \mathbf{B}\left(\Psi(s)\right) \frac{\partial \Psi}{\partial s} ds, \tag{4.6}$$

$$\mathbf{D}^{+}_{i+\frac{1}{2}} = +\frac{s_R}{s_R - s_L}\int_1^0 \mathbf{B}\left(\Psi(s)\right) \frac{\partial \Psi}{\partial s} ds, \tag{4.7}$$

and the numerical fluxes are given by

$$\mathbf{F}_{\mathrm{HLLEM}} = \mathbf{F}_{\mathrm{HLL}} - \frac{\varphi s_R s_L}{s_R - s_L}\mathbf{R}_*(\bar{\mathbf{Q}})\,\boldsymbol{\delta}_*(\bar{\mathbf{Q}})\,\mathbf{L}_*(\bar{\mathbf{Q}})\,\left(\mathbf{Q}_R - \mathbf{Q}_L\right), \tag{4.8}$$

with the standard HLL flux

$$\mathbf{F}_{\mathrm{HLL}} = \frac{s_R\mathbf{F}_L - s_L\mathbf{F}_R}{s_R - s_L} + \frac{s_L s_R}{s_R - s_L}\left(\mathbf{Q}_R - \mathbf{Q}_L\right). \tag{4.9}$$

Here, $\bar{\mathbf{Q}} = \frac{1}{2}(\mathbf{Q}_L + \mathbf{Q}_R)$ is the arithmetic average of the left and the right state and φ in the range $0 \leq \varphi \leq 1$ is a flattener. To control the amount of anti-diffusion, the diagonal matrix $\boldsymbol{\delta}_*(\bar{\mathbf{Q}})$ is given by

$$\boldsymbol{\delta}_*(\bar{\mathbf{Q}}) = \mathbf{I} - \frac{\boldsymbol{\Lambda}_*^-}{s_L} - \frac{\boldsymbol{\Lambda}_*^+}{s_R}, \quad 0 < \boldsymbol{\delta}_*(\bar{\mathbf{Q}}) \leq 1 \tag{4.10}$$

with $\mathbf{I}$ the identity matrix. The computation in (4.10) is based on the diagonal elements $\boldsymbol{\Lambda}_* = \boldsymbol{\Lambda}_*(\bar{\mathbf{Q}})$. Suitable wave speed estimates for s_L and s_R are given by

$$s_L = \min(0, \boldsymbol{\Lambda}(\mathbf{Q}_L), \boldsymbol{\Lambda}(\bar{\mathbf{Q}})), \; s_R = \max(0, \boldsymbol{\Lambda}(\bar{\mathbf{Q}}), \boldsymbol{\Lambda}(\mathbf{Q}_R)). \tag{4.11}$$

Since equation (3.1) only gives an evolution equation for the cell averages $\mathbf{Q}_i^n$ but the interface flux $\mathbf{F}_{i+\frac{1}{2}}$ needs values at the element interface, a spatial reconstruction operator is needed that produces suitable interface values from the given cell averages.

4.3. Data reconstruction

Higher order spatial and temporal accuracy can be obtained by using a more sophisticated reconstruction operator. The computational domain Ω is discretized by a computational mesh, composed of conforming elements denoted by T_i, where the index i ranges from 1 to the total number of elements N_E. The discrete solution of PDE (3.1) is denoted by $u_h(x, t^n)$ and is represented by piecewise polynomials of maximum degree $N \geq 0$. Within each cell T_i we have

$$u_h(x, t^n) = \sum_{l=0}^{\mathcal{M}} \psi_l(x) \hat{u}_l^n = \psi_l(x) \hat{u}_l^n, \quad x \in T_i \tag{4.12}$$

where u_h is referred to as the discrete representation of the solution, while the coefficients $\hat{u}_l^n$ are usually called degree of freedom.
In order to derive the Path conservative WENO schemes, we first multiply the governing PDE system (3.1) with a test function ϕ_k defined by discrete solution space and integrate over the space-time control volume $\Omega \times [t^n, t^{n+1}]$.

$$\int_{t_n}^{t^{n+1}} \int_\Omega \phi_k \frac{\partial Q}{\partial t} dx dt + \int_{t_n}^{t^{n+1}} \int_\Omega \phi_k \left(\frac{\partial f}{\partial x} + B(Q) \frac{\partial f}{dx} \right) dx dt = \int_{t_n}^{t^{n+1}} \int_\Omega \phi_k S(Q) dx dt. \tag{4.13}$$

5. Result

In this section, we solve a test problem for continuum mechanics, used in [9]. The computational domain is given by $\Omega = [-1; 1]$. The final computational time is chosen as $t = 0.50$; the chosen physical parameters are $\gamma = 1.4, c_V = 1, \rho = 1, \alpha = 2$ and $c_s = 1$. The initial data for the Riemann problem in conservative variables are

$$\begin{aligned} \mathbf{Q}_L &= (1; 0; -0.1; 0; 1; 1; 0; 0; 0; 1; 0; 0; 0; 1; 0; 0; 0)\,; \\ \mathbf{Q}_R &= (1; 0; +0.1; 0; 1; 1; 0; 0; 0; 1; 0; 0; 0; 1; 0; 0; 0) \end{aligned} \tag{5.1}$$

which corresponds to an isolated shear layer. The numerical results obtained with the new Osher-type solver and of the HLLEM-type Riemann solver are compared with the simple Rusanov method. Furthermore, in the viscous fluid limit ($\tau_1 << 1, \tau_2 << 1$) we also show a comparison with the exact solution of the compressible Navier-Stokes equations. The obtained computational results are shown in Figs. 1, 2 and 3 for the Rusanov,

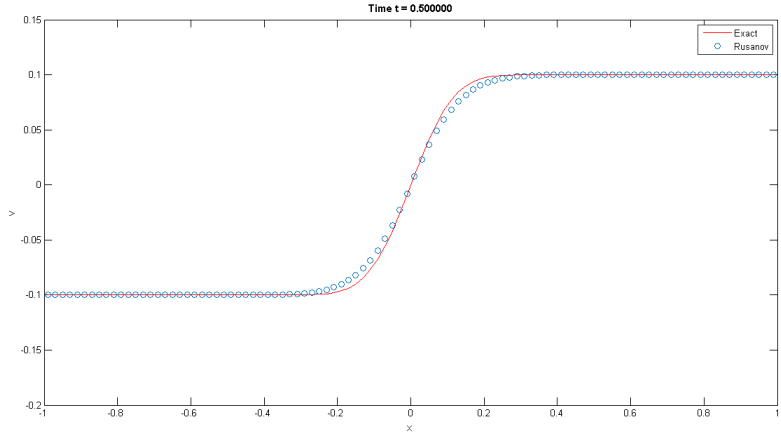

Figure 1. Rusanov Riemann solver in the viscous fluid limit. Comparison with the exact solution of the incompressible Navier-Stokes equations.

the Osher and the HLLEM type Riemann solver, respectively and are compared with the exact solution of the viscous fluid limit. For the relaxation times, in this case we have chosen $\tau_1 = 0.05$ and $\tau_2 = 0.05$.

Fig. 4 compares the numerical solution of the Osher and the Rusanov Riemann solvers and Fig. 5 compares the numerical solution of the HLLEM and the Rusanov Riemann solvers for the elastic solid limit ($\tau_1 = 10^{20}$ and $\tau_2 = 10^{20}$).

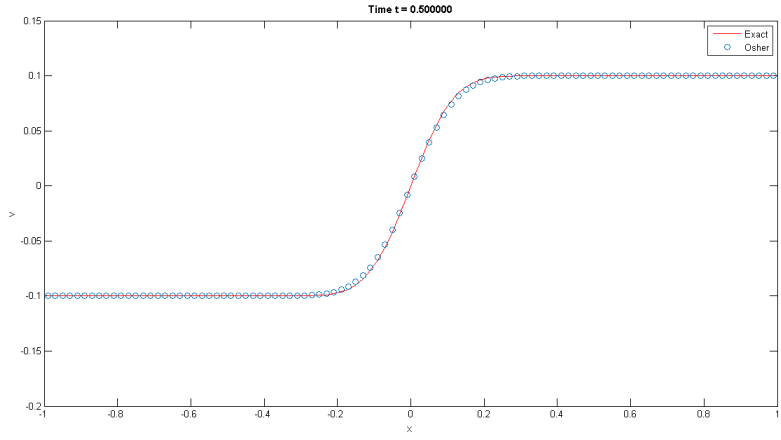

Figure 2. Osher Riemann solver with viscous fluid limit. Comparison with the exact solution of the incompressible Navier-Stokes equations.

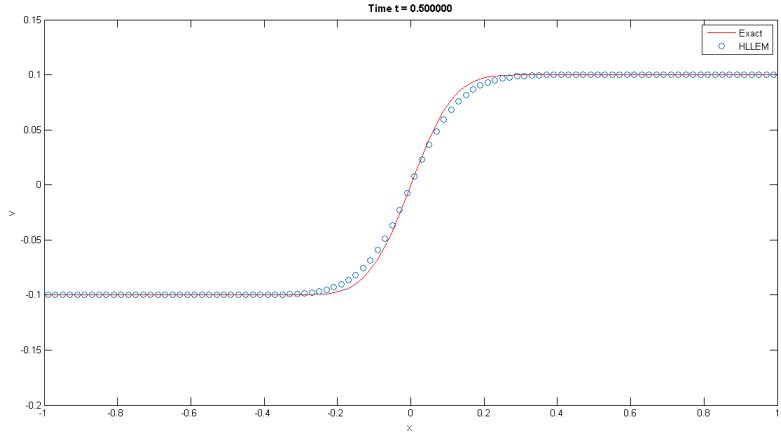

Figure 3. HLLEM Riemann solver with viscous fluid limit. Comparison with the exact solution of the incompressible Navier-Stokes equations.

6. Conclusions

We have presented and applied two complete Riemann solvers to the GPR model of continuum mechanics. To the best knowledge of the authors, this is the first time that these types of Riemann solvers are applied to such a complex system of governing equations as the GPR model of continuum mechanics. Since both Riemann solvers need the entire eigenstructure of the governing PDE system, which was discussed in [3], we rely on numerical linear algebra in order to compute the necessary eigenstructure

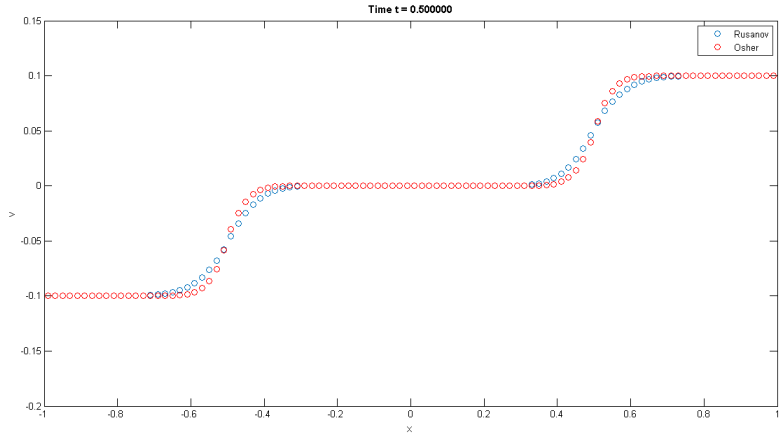

Figure 4. Comparison of the Osher-type solver with the simple Rusanov scheme in the case of an ideal elastic solid, i.e. in the limit $\tau_1 \to \infty$ and $\tau_2 \to \infty$.

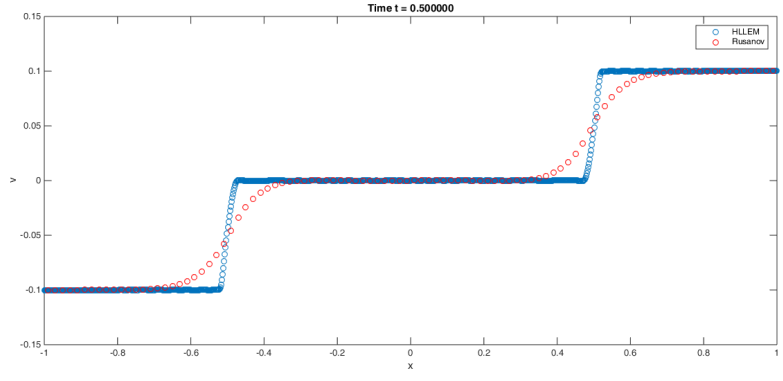

Figure 5. Comparison of the HLLEM solver with the simple Rusanov scheme in the case of an ideal elastic solid, i.e. in the limit $\tau_1 \to \infty$ and $\tau_2 \to \infty$.

numerically at the aid of the `eig` function of `MATLAB`. Two simple computational examples are presented, but the obtained computational results clearly show that the complete Riemann solvers are less dissipative than the simple Rusanov method that was employed in previous work on the GPR model, see [3]. Future work will concern the extension to multiple space dimensions and to moving meshes in the framework of high order accurate Arbitrary-Lagrangian-Eulerian WENO finite volume schemes [5].

References

1. Castro M., Gallardo J., Parés C. High-order finite volume schemes based on reconstruction of states for solving hyperbolic systems with nonconservative products. Applications to shallow-water systems. *Mathematics of Computations*, 2006, vol. 75, pp. 1103–1134. https://doi.org/10.1090/S0025-5718-06-01851-5
2. Dumbser M., Balsara D.S. A new effcient formulation of the HLLEM Riemann solver for general conservative and non-conservative hyperbolic systems. *J. Comp. Phys.*, 2016, vol. 304, pp. 275–319. https://doi.org/10.1016/j.jcp.2015.10.014
3. Dumbser M., Peshkov I., Romenski E., Zanotti O. High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: Viscous heat-conducting fluids and elastic solids. *J. Comp. Phys.*, 2016, vol. 314, pp. 824–862. https://doi.org/10.1016/j.jcp.2016.02.015
4. Dumbser M., Toro E.F. A simple extension of the Osher Riemann solver to non conservative hyperbolic systems. *J. Sci. Comput.*, 2011, vol. 48, pp. 70-88. https://doi.org/10.1007/s10915-010-9400-3
5. Dumbser M., Uuriintsetseg A., Zanotti O. On Arbitrary-Lagrangian-Eulerian one-step WENO schemes for stiff hyperbolic balance laws. *Communications in Computational Physics*, 2013, vol. 14, pp. 301–327. https://doi.org/10.4208/cicp.310112.120912a
6. Jackson H. A fast numerical scheme for the Godunov-Peshkov-Romenski model of continuum mechanics. *Journal of Computational Physics*, 2017, vol. 348, pp. 514-533. https://doi.org/10.1016/j.jcp.2017.07.055
7. Osher S., Solomon F. Upwind difference schemes for hyperbolic conservation laws. *Math. Comput.,* 1982, vol. 38, pp. 339–374. https://doi.org/10.1090/S0025-5718-1982-0645656-0
8. Parés C. Numerical methods for nonconservative hyperbolic systems: A theoretical framework. *SIAM Journal on Numerical Analysis*, 2006, vol. 44, pp. 300-321. https://doi.org/10.1137/050628052
9. Peshkov I., Romenski E. A hyperbolic model for viscous newtonian flows. *Contin. Mech. Thermodyn.*, 2016, vol. 28, pp. 85–104. https://doi.org/10.1007/s00161-014-0401-6
10. Walter B., Dumbser M., Loubére R. Cell centered direct Arbitrary-Lagrangian-Eulerian ADER-WENO finite volume schemes for nonlinear hyperelasticity. *Computers and Fluids*, 2016, vol. 134-135, pp. 111–129. https://doi.org/10.1016/j.jcp.2017.06.022

Ariunaa Uuriintsetseg, Master of Sciences (Engineer Mathematics), Senior Lecturer, Department of Mathematics, Mongolian University of Science and Technology, 34, Baga toiruu, Sukhbaatar district, Ulaanbaatar, 14191, Mongolia, tel.: +976 99737368, email: ariunaa@must.edu.mn

Michael Dumbser, Doctor of Sciences (Aerospace Engineering), Full Professor, Laboratory of Applied Mathematics, Faculty of Engineering, University of Trento, 77, Via Mesiano, Trento, 38050, Italy, tel.:+39 461 882646, email: michael.dumbser@unitn.it, ORCID iD https://orcid.org/0000-0002-8201-8372

Sarantuya Tsedendamba, Doctor of Sciences (Mathematics), Professor, Department of Mathematics, Mongolian University of Science and

Technology, 34, Baga toiruu, Sukhbaatar district, Ulaanbaatar, 14191, Mongolia, tel.: +976 99128063, email: saraa@must.edu.mn

Received 17.10.2020

Полные решатели Римана для гиперболической GPR модели механики сплошной среды

У. Ариунаа[1], М. Думбсер[2], Ц. Сарантуяа[1]

[1] *Монгольский университет науки и технологии, Улан-Батор, Монголия*
[2] *Университет Тренто, Тренто, Италия*

Аннотация. В статье рассмотрен полный решатель Римана Ошера-Соломона и решатель Римана HLLEM для унифицированной гиперболической формулировки механики сплошных сред первого порядка, которая описывает динамику как жидкости, так и твердого тела. Это первый случай, когда эти типы решателей Римана применяются к такой сложной системе определяющих уравнений, как модель механики сплошных сред с помощью георадаров. Гиперболическая формулировка механики сплошной среды первого порядка, недавно предложенная С. К. Годуновым, И. М. Пешковым и Е. И. Роменским, далее обозначаемая как модель GPR, включает гиперболическую формулировку теплопроводности и переопределенную систему ДУ с ЧП. Схемы с консервативным путём важны для того, чтобы дать смысл неконсервативным терминам в структуре слабого решения, поскольку управляющая система ДУ с ЧП также содержит неконсервативные продукты. Новые решатели Римана реализованы и успешно протестированы, что означает, что они определенно работают лучше, чем стандартные локальные решатели Римана типа Лакса – Фридрихса или Русанова. Представлены два простых вычислительных примера, но полученные результаты ясно показывают, что полные решатели Римана менее диссипативны, чем простой метод Русанова, который использовался в предыдущей работе с моделью GPR.

Ключевые слова: решатели Римана, гиперболическая модель GPR, механика сплошной среды, решатель Римана HLLEM.

Список литературы

1. Castro M., Gallardo J., Parés C. High-order finite volume schemes based on reconstruction of states for solving hyperbolic systems with nonconservative products. Applications to shallow-water systems // Mathematics of Computations. 2006. Vol. 75. P. 1103–1134. https://doi.org/10.1090/S0025-5718-06-01851-5
2. Dumbser M., Balsara D. S. A new effcient formulation of the HLLEM Riemann solver for general conservative and non-conservative hyperbolic systems // J. Comp. Phys. 2016. Vol. 304. P. 275–319. https://doi.org/10.1016/j.jcp.2015.10.014
3. High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: Viscous heat-conducting fluids and elastic solids // M. Dumbser, I. Peshkov, E. Romenski, O. Zanotti // J. Comp. Phys. 2016. Vol. 314. P. 824–862. https://doi.org/10.1016/j.jcp.2016.02.015
4. Dumbser M., Toro E. F. A simple extension of the Osher Riemann solver to non conservative hyperbolic systems // J. Sci. Comput. 2011. Vol. 48. P. 70–88. https://doi.org/10.1007/s10915-010-9400-3

5. Dumbser M., Uuriintsetseg A., Zanotti O. On Arbitrary-Lagrangian-Eulerian one-step WENO schemes for stiff hyperbolic balance laws // Communications in Computational Physics. 2013. Vol. 14. P. 301–327. https://doi.org/10.4208/cicp.310112.120912a
6. Jackson H. A fast numerical scheme for the Godunov-Peshkov-Romenski model of continuum mechanics // Journal of Computational Physics. 2017. Vol. 348. P. 514–533. https://doi.org/10.1016/j.jcp.2017.07.055
7. Osher S., Solomon F. Upwind difference schemes for hyperbolic conservation laws // Math. Comput. 1982. Vol. 38. P. 339–374. https://doi.org/10.1090/S0025-5718-1982-0645656-0
8. Parés C. Numerical methods for nonconservative hyperbolic systems: A theoretical framework // SIAM Journal on Numerical Analysis. 2006. Vol. 44. P. 300–321. https://doi.org/10.1137/050628052
9. Peshkov I., Romenski E. A hyperbolic model for viscous newtonian flows // Contin. Mech. Thermodyn. 2016. Vol. 28. P. 85–104. https://doi.org/10.1007/s00161-014-0401-6
10. Walter B., Dumbser M., Loubére R. Cell centered direct Arbitrary-Lagrangian-Eulerian ADER-WENO finite volume schemes for nonlinear hyperelasticity //Computers and Fluids. 2016. Vol. 134-135. P. 111–129. https://doi.org/10.1016/j.jcp.2017.06.022

Уурийнцэцэг Ариунаа, магистр наук (инженер математики), преподаватель, департамент математики, Монгольский университет науки и технологий, Монголия, 14191, г. Улан-Батор, 8-й хороо, Бага тоируу, 34, Сухэ-Баторский район, тел..: +976 99737368,
email: ariunaa@must.edu.mn

Михаэль Думбсер, доктор инженерных наук (аэрокосмическая техника), профессор, лаборатория прикладной математики, инженерный факультет, Университет Тренто, Италия, 38050, г. Тренто, Via Mesiano, 77, тел.:+39 461 882646, email: michael.dumbser@unitn.it,
ORCID iD https://orcid.org/0000-0002-8201-8372

Цэдэндамба Сарантуяа, доктор математических наук, профессор, департамент математики, Монгольский университет науки и технологий, Монголия, 14191, г. Улан-Батор, 8-й хороо, Бага тоируу 34, Сухэ-Баторский район, тел.: +976 99128063, email: saraa@must.edu.mn

Поступила в редакцию 17.10.2020