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An Optimal Control Problem by a Hyperbolic System with Boundary Delay *

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Abstract. The paper deals with an optimal control problem by a system of semilinear hyperbolic equations with boundary differential conditions with delay. This problem is considered for smooth controls. Because this requirement it is impossible to prove optimality conditions of Pontryagin maximum principle type and classic optimality conditions of gradient type. Problems of this kind arise when modeling the dynamics of non-interacting age-structured populations. Independent variables in this case are the age of the individuals and the time during which the process is considered. The functions of the process state describe the age-related population density. The goal of the control problem may be to achieve the specified population densities at the end of the process. The problem of identifying the functional parameters of models can also be considered as the optimal control problem with a quadratic cost functional. For the problem we obtain a non-classic necessary optimality condition which is based on using a special control variation that provides smoothness of controls. An iterative method for improving admissible controls is developed. An illustrative example demonstrates the effectiveness of the proposed approach.

Keywords: hyperbolic system, boundary differential conditions with delay, necessary optimality condition, optimal control.

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1. Introduction

Optimal control problems with time delays play an important role in various fields of applications. The study of such problems began in the 60s of the XX century (see, for example, [6]) from ordinary differential equations.

Optimal control problems for specific classes of distributed parameter systems with delay were investigated in the following areas:
1) existence of optimal control [10; 11];
2) necessary optimality conditions [9];
3) computational methods based on ideas of dynamic programming, reductions to simpler problems, etc. [3; 5].

The study of optimal control problems for distributed parameter systems is closely related to the study of partial differential equations and/or integral equations, including generalized solutions of such systems. At the same time, it makes sense to consider a number of optimal control problems in a smooth control class. In this case solutions of the corresponding boundary value problems for differential equations could be considered in the classical or "almost" classical sense. In particular, the solution of inverse problems for partial differential equations can often be reduced to the corresponding problems of optimal control in the class of smooth control functions. In this paper we consider an optimal control problem by boundary differential conditions with delay for semi-linear hyperbolic equations. Such problems arise in modeling of age-structured biological populations. Boundary delay is connected with environmental factors, reproduction peculiarities, etc.

A principle difference from classical statements of optimal control problems is the investigation of problems in the class of smooth admissible controls. In this paper, we study the problem in the class of smooth controls that satisfy the pointwise constraints in each point. It is impossible to use the optimal control methods based on the Pontryagin maximum principle or the classic gradient optimality conditions. These methods are focused on the classes of discontinuous controls. We apply the idea of the general approach [2] based on using a special variation that provides smoothness of controls and satisfaction the constraint. This approach is based on the "inner" variation idea, originally applied by L. Zabello for optimal control problems in ordinary differential equations [12]. However L. Zabello used a combination of inner and needle variations. His approach, generally speaking, is effective for researching problems with lag, but in the class of discontinuous controls.

Another way of deriving constructive necessary optimality conditions was suggested in [4; 7; 8]. The new class of optimal control problems governed by the dissipative and discontinuous differential inclusion of the sweeping/Moreau process with controlled moving sets was investigated. Controls actions of moving sets must be continuous. Authors used the method of discrete approximations and combined it with advanced tools of first-order and second-order variational analysis and generalized differentiation. These techniques can be useful for the class of problems considered below.

2. Problem statement

Consider the system

$$x_t + A(s,t)x_s = f(x,s,t), \tag{2.1}$$

$$(s,t) \in \Pi,\ \Pi = S \times T,\ S = [s_0, s_1], T = [t_0, t_1],$$

Here $x = x(s,t)$ is a n-dimensional vector-function of state variables, $\mathbf{A} = \mathbf{A}(s,t)$ is a $(n \times n)$ - matrix. The system (2.1) is written in an invariant form, i.e., $\mathbf{A}$ is a diagonal matrix. In addition, we assume that the diagonal elements $a_i(s,t)$ of the matrix of coefficients possess constant signs in the rectangle Π:

$$a_i(s,t) > 0, \quad i = 1, 2, \ldots, m_1;$$
$$a_i(s,t) = 0, \quad i = m_1 + 1, m_1 + 2, \ldots, m_2;$$
$$a_i(s,t) < 0, \quad i = m_2 + 1, m_2 + 2, \ldots, n.$$

Respectively, the state vector $x = x(s,t)$ contains two subvectors

$$x^+ = (x_1, x_2, \ldots, x_{m_1}), \quad x^- = (x_{m_2+1}, x_{m_2+2}, \ldots, x_n),$$

which correspond to positive and negative diagonal elements of the matrix of coefficients.

Let the controlled initial-boundary conditions for the system (2.1) be given in the following form:

$$\frac{dx^+(s_0,t)}{dt} = g(x^+(s_0,t), x^+(s_0,t-h), u(t), t), \quad t \in T, \tag{2.2}$$

$$x(s,t_0) = x^0(s),\ s \in S,\ \ x^-(s_1,t) = \nu(t), t \in T,$$
$$x^+(s_0,t) = q(t), \ \ t \in [-h; t_0]; \ \ h > 0,$$

where h is a constant delay.

Control $u = u(t)$ is a smooth r-dimensional vector-function on segment T. It satisfies the constraint

$$u(t) \in U,\ t \in T, \tag{2.3}$$

where U is a compact set.

The problem is to minimize the functional

$$J(u) = \int_S \varphi(x(s,t_1), s)\, ds, \tag{2.4}$$

defined on the solutions of the problem (2.1), (2.2) for admissible control functions (2.3). Denote $y(t) = x^+(s_0, t-h)$.

The optimal control problem (2.1)-(2.4) is considered under the following suppositions:

1) the diagonal elements $a_i(s,t)$ of the matrix $\mathbf{A}$ are continuous and continuously differentiable in Π;
2) functions $x^0(s)$, $\nu(t)$, $q(t)$ are continuous with respect to their arguments on the sets S and T respectively and satisfy the conditions
$$\nu(t_0) = (x^0(s_1))^-, \quad q(t_0) = (x^0(s_0))^+;$$
3) functions $f(x,s,t)$ and $\varphi(x,s)$ are continuous with respect to their arguments, and they have continuous and bounded partial derivatives with respect to the state function x;
4) function $g(x^+, y, u, t)$ is continuous and continuously differentiable, and has bounded partial derivatives with respect to x^+, y and u.

Despite the smoothness of the controls, the solution of the initial-boundary problem for the hyperbolic system is understood in a generalized sense. It is suitable to use the definition of a generalized solution in terms of characteristics of the system. Let us consider characteristic curves determined by the ordinary differential equations

$$\frac{ds}{dt} = a_i(s,t), \quad i = 1, 2, \ldots, n. \tag{2.5}$$

Let $s_i = s_i(\xi, \tau; t)$ be a solution of (2.5), which passes through the point $(\xi, \tau) \in \Pi$. If there exists a classical solution of the system under consideration, then the given system is equivalent to the following one:

$$x_i(s,t) = x_i(\xi_i, \tau_i) + \int\limits_{\tau_i}^{t} f_i(x(\xi,\tau), \xi, \tau)|_{\xi = s_i(s,t;\tau)}\, d\tau, \tag{2.6}$$

$$(s,t) \in \Pi, \quad i = 1, 2, ..., n,$$

where (ξ_i, τ_i) is the initial point of i-th characteristic curve passing through (s,t).

By means of integral system (2.6) it is possible to prove the existence and uniqueness of a continuous in Π a generalized solution. Each component of this solution is continuously differentiable along the corresponding characteristic family [1]. So, instead of the left side of system (2.1) we consider the differential operator

$$\left(\frac{dx}{dt}\right)_A = \left(\left(\frac{dx_1}{dt}\right)_A, \left(\frac{dx_2}{dt}\right)_A, \ldots, \left(\frac{dx_n}{dt}\right)_A\right),$$

where $(dx_i/dt)_A$ is the derivative of i-th component of the state vector along the corresponding family of characteristic curves.

3. Increment formula

Consider two admissible processes, namely, the initial admissible process $\{u, x\}$ and the perturbated one $\{\widetilde{u} = u + \Delta u, \widetilde{x} = x + \Delta x\}$. Write the problem in the following form

$$\left(\frac{d\Delta x}{dt}\right)_A = \Delta f(x, s, t),$$

$$\Delta x^+(s_0, t) = 0,\ t \in [-h; t_0];\quad \Delta x(s, t_0) = 0,\quad s \in S;\ \Delta x^-(s_1, t) = 0,\quad t \in T;$$

$$\Delta x^+{}_t(s_0, t) = \Delta g(x^+(s_0, t), y(t), u(t), t), \tag{3.1}$$

where

$$\Delta g(x^+(s_0, t), y(t), u(t), t) = g(\widetilde{x}^+, \widetilde{y}, \widetilde{u}, t) - g(x^+, y, u, t) =$$

$$= \Delta_{\widetilde{u}} g(x^+, y, u, t) + \Delta_{\widetilde{x}^+} g(x^+, y, \widetilde{u}, t) + \Delta_{\widetilde{y}} g(\widetilde{x}^+, y, \widetilde{u}, t),$$

$$\Delta J(u) = \int\limits_S \Delta\varphi(x(s, t_1), s)\, ds.$$

Add the following terms to the increment formula for the cost functional

$$\iint\limits_{\Pi} \langle \psi(s, t), \left(\frac{d\Delta x}{dt}\right)_A - \Delta f(x, s, t)\rangle\, ds\, dt,$$

$$\int\limits_T \langle p(t), \Delta x^+{}_t(s_0, t) - \Delta g(x^+(s_0, t), y(t), u(t), t)\rangle\, dt,$$

where $\psi(s, t)$ and $p(t)$ are still undefined n-dimensional and m_1-dimensional vector-functions having the same smoothness properties as $x(s, t)$ and $x^+(t)$ respectively. Here $\langle ., . \rangle$ is a designation of a scalar product in Euclidean space of a corresponding dimension.

Applying integration by parts, we have

$$\Delta J(u) = \int\limits_S \Delta\varphi(x(s, t_1), s)\, ds + \int\limits_S [\langle \psi(s, t_1), \Delta x(s, t_1)\rangle -$$

$$- \langle \psi(s, t_0), \Delta x(s, t_0)\rangle]\, ds - \iint\limits_{\Pi} \langle \left(\frac{d\psi}{dt}\right)_A + A_s\psi, \Delta x(s, t)\rangle\, ds\, dt +$$

$$+ \int\limits_T [\langle \psi(s_1, t), A(s_1, t)\Delta x(s_1, t)\rangle - \langle \psi(s_0, t), A(s_0, t)\Delta x(s_0, t)\rangle]\, dt +$$

$$+\langle p(t_1), \Delta x^+(s_0,t_1)\rangle - \langle p(t_0), \Delta x^+(s_0,t_0)\rangle - \int_T \langle p_t, \Delta x^+(s_0,t)\rangle\, dt -$$

$$-\iint_\Pi \langle \psi(s,t), \Delta f(x,s,t)\rangle\, dsdt - \int_T \langle p(t), \Delta g(x^+,y,u,s,t)\rangle\, dt.$$

Introduce the following auxiliary functions

$$H(\psi(s,t), x(s,t), s,t) = \langle \psi(s,t), f(x,s,t)\rangle,$$

$$h(p(t), x^+(s_0,t), y(t), u(t), t) = \langle p(t), g(x^+(s_0,t), y(t), u(t), t)\rangle.$$

Then

$$\Delta h(p,x^+,y,u,t) = \Delta_{\widetilde{u}} h(p,x^+,y,u,t) +$$
$$+\Delta_{\widetilde{x}^+} h(p,x^+,y,\widetilde{u},t) + \Delta_{\widetilde{y}} h(p,\widetilde{x}^+,y,\widetilde{u},t),$$

where

$$\Delta_{\widetilde{u}} h(p,x^+,y,u,t) = h(p,x^+,y,\widetilde{u},t) - h(p,x^+,y,u,t),$$
$$\Delta_{\widetilde{x}^+} h(p,x^+,y,\widetilde{u},t) = h(p,\widetilde{x}^+,y,\widetilde{u},t) - h(p,x^+,y,\widetilde{u},t),$$
$$\Delta_{\widetilde{y}} h(p,\widetilde{x}^+,y,\widetilde{u},t) = h(p,\widetilde{x}^+,\widetilde{y},\widetilde{u},t) - h(p,\widetilde{x}^+,y,\widetilde{u},t).$$

Use the following expansions

$$\Delta\varphi(x(s,t_1),s) = \langle \frac{\partial\varphi(x(s,t_1),s)}{\partial x}, \Delta x(s,t_1)\rangle + o_\varphi(|\Delta x(s,t_1)|),$$

$$\Delta H(\psi,x,s,t) = \langle \frac{\partial H(\psi,x,s,t)}{\partial x}, \Delta x(s,t)\rangle + o_H(|\Delta x(s,t)|),$$

$$\Delta_{\widetilde{x}^+} h(p, x^+(s_0,t), y(t), \widetilde{u}, t) =$$
$$= \langle \frac{\partial h(p, x^+(s_0,t), y(t), \widetilde{u}, t)}{\partial x}, \Delta x^+(s_0,t)\rangle + o_h(|\Delta x^+(s_0,t)|),$$

$$\Delta_{\widetilde{y}} h(p,\widetilde{x}^+(s_0,t), y(t), \widetilde{u}, t) = \langle \frac{\partial h(p,\widetilde{x}^+(s_0,t), y(t), \widetilde{u}, t)}{\partial y}, \Delta y(t)\rangle + o_h(|\Delta y(t)|).$$

Transform the term

$$\frac{\partial h(p,\widetilde{x}^+(s_0,t), y(t), \widetilde{u}, t)}{\partial y} = \Delta_{\widetilde{u}} \frac{\partial h(p,\widetilde{x}^+(s_0,t), y(t), u, t)}{\partial y} +$$
$$+ \frac{\partial h(p,\widetilde{x}^+(s_0,t), y(t), u, t)}{\partial y},$$

where

$$\frac{\partial h(p,\widetilde{x}^+(s_0,t), y(t), u, t)}{\partial y} = \Delta_{\widetilde{x}^+} \frac{\partial h(p, x^+(s_0,t), y(t), u, t)}{\partial y} +$$

$$+\frac{\partial h(p,x^+(s_0,t),y(t),u,t)}{\partial y}.$$

Then, we obtain

$$\int\limits_T \langle \frac{\partial h(p,x^+(s_0,t),y(t),u,t)}{\partial y}, \Delta y\rangle\, dt =$$

$$=\int\limits_T \langle \frac{\partial h(p(t),x^+(s_0,t),x^+(s_0,t-h),u(t),t)}{\partial y}, \Delta x^+(s_0,t-h)\rangle\, dt =$$

$$=\int\limits_{t_0-h}^{t_0} \langle \frac{\partial h(p(\theta+h),x^+(s_0,\theta+h),x^+(s_0,\theta),u(\theta+h),\theta+h)}{\partial y}, \Delta x^+(s_0,\theta)\rangle\, d\theta+$$

$$+\int\limits_{t_0}^{t_1-h} \langle \frac{\partial h(p(\theta+h),x^+(s_0,\theta+h),x^+(s_0,\theta),u(\theta+h),\theta+h)}{\partial y}, \Delta x^+(s_0,\theta)\rangle\, d\theta.$$

Here we have used the following designation: $\theta = t-h, \ \ \theta\in[t_0-h,t_1-h]$. Then, reverting to a variable t, we get

$$\int\limits_{t_0}^{t_1-h} \langle \frac{\partial h(p(t+h),x^+(s_0,t+h),x^+(s_0,t),u(t+h),t+h)}{\partial y}, \Delta x^+(s_0,t)\rangle\, dt.$$

Let functions $\psi(s,t),\ p(t)$ be the solutions of the following adjoint problem

$$\left(\frac{d\psi}{dt}\right)_A + A_s\psi = -H_x(\psi,x,s,t), \quad \psi(s,t_1) = -\varphi_x(x(s,t_1),s), \tag{3.2}$$

$$\psi^+(s_1,t)=0; \ \ \psi^-(s_0,t)=0, \ \ t\in T;$$

$$p_t = \begin{cases} -h_x[t]-h_y[t+h]-A^+(s_0,t)\psi^+(s_0,t), & t\in[t_0;t_1-h],\\ -h_x[t]-A^+(s_0,t)\psi^+(s_0,t), & t\in[t_1-h;t_1].\end{cases}$$

$$p(t_1)=0; \ \ p(t)\equiv 0, \ t>t_1. \tag{3.3}$$

Here

$$h_x[t] = h_x(p(t),x^+(s_0,t),y(t),u(t),t),$$

$$h_y[t+h] = h_y(p(t+h),x^+(s_0,t+h),x^+(s_0,t),u(t+h),t+h).$$

Then, the increment formula for the functional takes the following form:

$$\Delta J(u) = -\int\limits_T \Delta_{\widetilde{u}} h(p(t),x^+(s_0,t),y(t),u(t),t)\, dt + \eta, \tag{3.4}$$

where

$$\eta = \int_S o_\varphi(|\Delta x(s,t_1)|)\, ds + \iint_\Pi (o_H(|\Delta x(s,t)|)\, dsdt +$$

$$+ \int_T [o_h(|\Delta x^+(s_0,t)|) + \langle \Delta_{\widetilde{u}} h_{x^+}(p(t), x^+(s_0,t), y(t), u(t), t), \Delta x^+(s_0,t)\rangle]\, dt +$$

$$+ \int_T [o_h(|\Delta y(t)|) + \langle \Delta_{\widetilde{u}} h_y(p(t), \widetilde{x}^+(s_0,t), y(t), u(t), t), \Delta y(t)\rangle]\, dt +$$

$$+ \int_T \langle \Delta_{\widetilde{x}^+} h_y(p(t), x^+(s_0,t), y(t), u(t), t), \Delta y(t)\rangle]\, dt.$$

Lemma 1. *Under the condition (3.1) the estimation of a state increment (analogously to [1]) takes the form*

$$\gamma(t) = \max_{(\xi,\tau)\in\Pi(t)} |\Delta x(\xi,\tau)| \le L_3 \int_{t_0}^{t} |\Delta u(\tau)|\, d\tau, \tag{3.5}$$

$$\Pi(t) = \{(\xi,\tau) \in \Pi : \tau \le t\}.$$

Proof. Consider the following system:

$$\Delta x_i(s,t) = \Delta x_i(\xi_i,\tau_i) + \int_{\tau_i}^{t} \Delta f_i(x(\xi,\tau),\xi,\tau)\, d\tau, \tag{3.6}$$

where (ξ_i,τ_i) is the initial point of i-th characteristic curve passing through (s,t). Denote

$$\gamma^+(t) = \max_{t_0\le\tau\le t} |\Delta x^+(s_0,\tau)|. \tag{3.7}$$

Under assumption of the problem

$$|\Delta f(x,s,t)| = |f(\widetilde{x},s,t) - f(x,s,t)| \le L|\widetilde{x} - x|,$$

$$|\Delta g(x,y,u,t)| = |g(\widetilde{x},\widetilde{y},\widetilde{u},t) - g(x,y,u,t)| \le L_1(|\widetilde{x}-x| + |\widetilde{y}-y| + |\widetilde{u}-u|),$$

where L is a Lipschitz constant for the function f, L_1 is a Lipschitz constant for the function g. Taking into account (3.6), (3.7) and conditions $\Delta x^-(s_1,t) = 0$, $\Delta x(s,t_0) = 0$, we get

$$|\Delta x_i(s,t)| \le \gamma^+(t) + L\int_{t_0}^{t} \gamma(\tau)\, d\tau, \quad i = 1,2,\ldots,m_1;$$

$$|\Delta x_i(s,t)| \leq L \int\limits_{t_0}^{t} \gamma(\tau)\,d\tau, \quad i = m_2+1, m_2+2, \ldots, n.$$

The right sides of the inequalities do not depend on i and s, then the following inequalities is valid

$$\gamma(t) \leq \sqrt{n}\gamma^+(t) + L\sqrt{n} \int\limits_{t_0}^{t} \gamma(\tau)\,d\tau. \tag{3.8}$$

Further, from (3.1),we get

$$|\Delta x^+(s_0,t)| \leq \int\limits_{t_0}^{t} |\Delta g(x^+,y,u,\tau)|\,d\tau \leq L_1 \int\limits_{t_0}^{t} (|\widetilde{x}^+(s_0,\tau) - x^+(s_0,\tau)| +$$

$$+ |\widetilde{y}(\tau) - y(\tau)| + |\widetilde{u}(\tau) - u(\tau)|)\,d\tau.$$

Consider

$$\int\limits_{t_0}^{t} |\Delta y(\tau)|\,d\tau = \int\limits_{t_0}^{t} |\widetilde{y}(\tau) - y(\tau)|\,d\tau =$$

$$= \int\limits_{t_0}^{t} |\widetilde{x}^+(s_0,\tau-h) - x^+(s_0,\tau-h)|\,d\tau =$$

$$= \int\limits_{t_0-h}^{t_0} |\widetilde{x}^+(s_0,\theta) - x^+(s_0,\theta)|\,d\theta + \int\limits_{t_0}^{t} |\widetilde{x}^+(s_0,\theta) - x^+(s_0,\theta)|\,d\theta =$$

$$= \int\limits_{t_0}^{t} |\widetilde{x}^+(s_0,\tau) - x^+(s_0,\tau)|\,d\tau.$$

Here we have used that

$$\tau - h = \theta, \quad \theta \in [t_0-h, t_1-h].$$

Then

$$|\Delta x^+(s_0,t)| \leq 2L_1 \int\limits_{t_0}^{t} |\Delta x^+(s_0,\tau)|\,d\tau + L_1 \int\limits_{t_0}^{t} |\Delta u(\tau)|\,d\tau.$$

From the inequality, we get

$$\gamma^+(t) \leq 2L_1 \int\limits_{t_0}^{t} \gamma^+(\tau)\,d\tau + L_1 \int\limits_{t_0}^{t} |\Delta u(\tau)|\,d\tau.$$

Using Gronwall-Bellman inequality, we obtain

$$\gamma^{+}(t) \leq L_2 \int_{t_0}^{t} |\Delta u(\tau)| \, d\tau, \quad L_2 = L_1 \cdot e^{2L_1(t_1 - t_0)}.$$

Take the inequality in (3.8) and reuse Gronwall-Bellman inequality. Then, we get estimation (3.5), where $L_3 = \sqrt{n} L_2 \cdot e^{\sqrt{n} L(t_1 - t_0)}$. □

4. Optimality condition

Apply the idea of the general approach [1] based on using a special variation that provides smoothness of control and satisfaction the constraint. The varied control takes the form

$$u_{\varepsilon,\delta}(t) = u(t + \varepsilon\delta(t)), \quad t \in T,$$

where $\varepsilon \in [0, 1]$ is a parameter of variation, $\delta(t)$ is a continuously differentiable function and satisfies the following conditions $t_0 \leq t + \delta(t) \leq t_1, \quad t \in T$. Since admissible controls belong to the class of smooth functions, the increment formula for the control function takes the form

$$\Delta u = \dot{u}(t)\varepsilon\delta(t) + o(\varepsilon).$$

Using (3.5), we get

$$\Delta J(u) = -\varepsilon \int_{T} \langle h_u, \dot{u} \rangle \delta(t) \, dt + o(\varepsilon). \tag{4.1}$$

Formulate a necessary optimality condition (analogously to results obtained in [1]).

Theorem 1. *If $\{u, x\}$ is the optimal process in the problem, there is valid the following condition*

$$\omega(t) = \langle h_u(p(t), x^{+}(s_0, t), y(t), u(t), t), \dot{u}(t) \rangle = 0, \quad t \in T,$$

where $p(t)$ is a solution to the adjoint problem (3.2), (3.3).

The convergence result is given in [1].

5. Iterative method

Describe the general scheme of the method (analogously to [1]).
1) Let $u^k(t)$ be an admissible control calculated on the k-th iteration.
2) For the control $u^k(t)$ solve problems (2.1), (3.2) and (3.3) to get functions $x^k(s,t)$ and $\psi^k(s,t)$, $p^k(t)$ respectively.
3) Calculate the value of the functional $J^k = J(u^k)$ and construct the function

$$\omega_k(t) = \langle h_u(p^k(t), x^k(s_0,t), u^k(t), t), \dot{u}^k(t)\rangle.$$

The function $\omega_k(t)$ consideres as a discrepancy of the fulfillment of the optimality condition. If $\omega_k(t) = 0$, then the control function u^k satisfies the optimality condition and the iteration process finishes.
4) If $\omega_k(t) \neq 0$, then consider a smooth variation of $u^k(t)$

$$u^k_{\varepsilon_k}(t) = u^k(t + \varepsilon_k \delta_k(t)),$$

$$\delta_k(t) = \frac{(t-t_0)(t_1-t)\omega_k(t)}{(t_1-t_0)\max\limits_{t\in T}|\omega_k(t)|},$$

where ε_k is a solution of the minimization problem

$$\varepsilon_k : \quad J(u^k_\varepsilon) \to \min, \quad \varepsilon \in [0,1].$$

5) The next approximation is given by the formula

$$u^{k+1}(t) = u^k_{\varepsilon_k}(t).$$

One of the following conditions can be considered as the stop criterion on some k-th iteration of the method.
a) The function $u^k(t)$ satisfies (with a given accuracy) the necessary optimality condition. For example, by the condition $\max\limits_{t\in T}|\omega_k(t)| \leq 10^{-5}$.
b) The value of the functional calculated on the previous iteration (that with the number $k-1$) is not improved, for example, $J^k - J^{k-1} > 10^{-6}$.
c) ε_k is closed to zero. This case is considered as no improvement of the functional on the method step.

6. Illustrative example

Consider the application of the described method to one test example. The algorithm was coded in Matlab 7.0. In the square $[0,4]\times[0,4]$ we consider the optimal control problem

$$x_{1t} + x_{1s} = x_1 + x_2 + f_1(s,t),$$

$$x_{2t} - x_{2s} = x_2 - f_2(s,t),$$

$$x_{1t}(0,t) = u \cdot x_1(0,t-0.5), \quad u(t) \in [0,3]; \quad x_1(0,t) = 0.3 \cdot t, \quad t \in [-0.5;0];$$
$$x_2(4,t) = 0.1 \cdot t, \quad x_1(s,0) = 0, \quad x_2(s,0) = s - 4.$$

The cost functional takes the form

$$J(u) = \frac{1}{2} \int\limits_S (x_1(s,4) - \overline{x}_1(s))^2 + (x_2(s,4) - \overline{x}_2(s))^2 \, ds \to min, \quad u \in U,$$

where $\overline{x}_1(s) = \overline{x}_1(s,4)$, $\overline{x}_2(s) = \overline{x}_2(s,4)$ are evaluated for the control

$$\overline{u}(t) = 2\cos\frac{t}{16} - \sin\frac{t}{2}.$$

The auxiliary functions and the adjoint problem:

$$H(\psi, x, s, t) = \psi_1 \cdot (x_1 + x_2 + f_1(s,t)) + \psi_2 \cdot (x_2 - f_2(s,t)),$$

$$\psi_{1t} + \psi_{1s} = -\psi_1, \quad \psi_{2t} - \psi_{2s} = -\psi_1 - \psi_2,$$
$$\psi_i(s,4) = \overline{x}_i(s) - x_i(s,4), \quad i = 1,2;$$
$$\psi_1(4,t) = 0; \quad \psi_2(0,t) = 0;$$
$$h(p,x,y,u,t) = p \cdot u \cdot y,$$

where $x_1(0,t-0.5) = y(t)$; $p(4) = 0$,

$$p_t = \begin{cases} -p \cdot u - \psi_1(0,t), & t \in [0;3.5), \\ -\psi_1(0,t), & t \in [3.5;4]. \end{cases}$$

We solve the problem under following functions: $f_1(s,t) = e^s \cos t$ and $f_2(s,t) = \sin t$. The initial control is

$$u^0(t) = 1 + \cos 1.2t.$$

The value of the functional is $J(u^0) = 7.61749$. We solved the problem by the described method. Results of calculation are presented at the Table 1.

We have obtained: $J(u^k) = 0.03854$, $\max\limits_{t \in T} |\omega_k(t)| = 0.02791$, the total number of iteration equals 48. The stop criterion is $\varepsilon_k \leq 10^{-5}$ (there is no improvement of the functional on the method step).

7. Conclusion

In this paper we considered the optimal control problem by hyperbolic system with a special type of boundary delay. The optimality condition in a class of smooth control is proved. We applied approach [1] which is based on using a special variation that provides smoothness of control and

Table 1

t	$\overline{u}(t)$	$u^0(t)$	$u^k(t)$
0	2	2	2
0.3	1.850	1.935	1.631
1.075	1.483	1.277	1.526
1.65	1.254	0.602	1.372
2.1	1.115	0.187	0.989
2.975	0.968	0.090	0.843
3.3	0.960	0.316	0.875
4	1.028	1.087	1.032

satisfaction the constraint in each point. We consider the application of the iterative method to solving the optimal control problem with boundary delay. Numerical experiment showed the efficiency and applicability of the method.

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Задача оптимального управления гиперболической системой с запаздыванием на границе

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Аннотация. В статье рассматривается задача оптимального управления системой полулинейных гиперболических уравнений, в которой граничные условия определяются из системы обыкновенных дифференциальных уравнений с запаздыванием. Задача рассматривается в классе гладких управляющих воздействий. В силу данного условия невозможно доказать условие оптимальности типа принципа максимума Л. С. Понтрягина и классические условия оптимальности градиентного типа. Задачи такого рода возникают при моделировании динамики невзаимодействующих между собой популяций с учетом возрастного распределения особей. Независимыми переменными в этом случае являются возраст особей и время, в течение которого рассматривается процесс. Функции состояния процесса описывают возрастные плотности популяций. Целью задачи управления может быть достижение заданных плотностей популяций в конечный момент времени. Проблема идентификация функциональных параметров моделей может также рассматриваться как задача оптимального управления с квадратичным целевым функционалом. Для указанной задачи получено неклассическое необходимое условие оптимальности, которое основано на применении специальной вариации управления, обеспечивающей гладкость управляющих функций. Предложен метод улучшения допустимых управлений. Эффективность предлагаемого подхода проиллюстрирована примером.

Ключевые слова: гиперболическая система, граничные дифференциальные условия с запаздыванием, необходимое условие оптимальности, оптимальное управление.

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