АЛГЕБРО-ЛОГИЧЕСКИЕ МЕТОДЫ В ИНФОРМАТИКЕ И ИСКУССТВЕННЫЙ ИНТЕЛЛЕКТ

ALGEBRAIC AND LOGICAL METHODS IN COMPUTER SCIENCE AND ARTIFICIAL INTELLIGENCE

Серия «Математика»
2020. Т. 34. С. 93—108

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

УДК 519.716
MSC 03B50, 08A99
DOI https://doi.org/10.26516/1997-7670.2020.34.93

Classification of Multioperations of Rank 2 by E-precomplete Sets

V. I. Panteleev, L. V. Riabets
Irkutsk State University, Irkutsk, Russian Federation

Abstract. In this paper multioperations defined on a two-element set and their closure operator based on composition operator and the equality predicate branching operator is considered. The composition operator is based on union of sets. The classification of multioperations based on their membership in precomplete sets has been obtained. It is shown that the number of equivalence classes is 129. All types of bases are described and it is proved that the maximum cardinality of a basis is 4.

Keywords: closure, equality predicate, multioperation, closed set, composition, precomlete set

1. Introduction

Along with classical functional systems over the set of k-valued functions ($k \geqslant 2$), the systems considering generalizations of functions of the k-valued logic defined on some finite set A and taking the set A subsets as their values have been studied for long time. These are partial functions, hyperfunctions, and multifunctions. The action of the composition operator for such systems was studied, for example, in [4–7; 18].

As a rule, the composition operator leads to a countable or continual classification. Since the mid-1970s, different authors offered closure operators that were significantly stronger than the composition operator and gave rise to finite classifications of functions. These operators include, in particular, the parametric and positive closure operators [8], E-closure operator [10]. Actions of the E-closure operator on the set of Boolean functions, partial Boolean functions and on the set of multi-valued logic functions were studied in [9–11]. In [12], all E-closed classes have been found for the set of partial Boolean functions. In [15; 17], the criterion of functional completeness has been formulated for the E-closure operator on the set of hyperfunctions on a two-element set and full structure of E-closed classes have been derived. The completeness criterion for the set of rank 2 multifunctions with respect to the E-closure operator has been obtained in [17].

The membership relation of functions to precomplete (or maximal) closed classes is a relation of equivalence and it allows one to construct classifications of the functions themselves and their complete sets — bases.

In [2; 19], it is shown that for the set of Boolean functions with a composition operator, the number of equivalence classes in terms of their membership in the maximal closed classes is 15, the maximum cardinality of the basis is 4. Equivalence classes and types of bases for various functions of k-valued logic and their generalizations can be found, for example, in [1; 3; 13; 14].

It is worth noting that along with the term "function", the term "operation" is also used, which we will further use.

This paper presents classification of multioperations of rank 2 in terms of their membership in E-precomplete closed classes. A set of multioperations can thus be divided into 129 equivalence classes. Among complete sets of multioperations, distinguished are minimal complete sets — bases. It is shown that there are no bases of cardinality greater than four. Using computer calculations, all types of bases were obtained.

2. Basic concepts and definitions

Let $E_2 = \{0, 1\}$. The set of all multioperations of rank 2 is denoted by M_2 and is defined as follows:

$$M_{2,n} = \left\{ f \mid f : E_2^n \to 2^{E_2} \right\}, \; M_2 = \bigcup_n M_{2,n},$$

In further representation, we will not distinguish between the one-element set and the element of that set. For the E_2 set, we will use the "$-$"

notation (dash) and the empty set will denote by $*$. Sometimes we use the term "operation" instead of "multioperation", if no ambiguity is caused.

The M_2 contains the set of hyperoperations (H_2), the sets of partial operations (O_2^*), and the set of Boolean operations (O_2) where

$$H_{2,n} = \left\{ f \mid f : E_2^n \to 2^{E_2} \setminus \{\varnothing\} \right\},\ H_2 = \bigcup_n H_{2,n},$$

$$\mathrm{O}_{2,n}^* = \{ f \mid f : E_2^n \to E_2 \cup \{\varnothing\} \},\ \mathrm{O}_2^* = \bigcup_n \mathrm{O}_{2,n}^*,$$

$$\mathrm{O}_{2,n} = \{ f \mid f : E_2^n \to E_2 \},\ \mathrm{O}_2 = \bigcup_n \mathrm{O}_{2,n}.$$

All binary sets from the E_2^n will be considered ordered according to their binary representation of natural numbers $0, \ldots, 2^n - 1$. The binary set corresponding to the natural number 0 is denoted as $\tilde{0}$, and to natural number $2^n - 1$ is denoted as $\tilde{1}$.

The multioperation f dependent on n variables will be written down as vector $(\tau_{\tilde{0}}, \ldots, \tau_{\tilde{1}})$ of length 2^n, where each element $\tau_{\tilde{\sigma}}$ is $f(\tilde{\sigma})$, $\tilde{\sigma} \in E_2^n$.

Let $f(x_1, \ldots, x_n)$, $f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)$ be multioperations. Composition of $f\big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\big)$ defines multioperation $g(x_1, \ldots, x_m)$ as follows: if the set $(\alpha_1, \ldots, \alpha_m) \in E_2^m$, then, by definition,

$$g(\alpha_1, \ldots, \alpha_m) = \bigcup_{\beta_i \in f_i(\alpha_1, \ldots, \alpha_m)} f(\beta_1, \ldots, \beta_n).$$

The composition operator defined in this way will be called composition by union or S_U-composition. It allows us to find the multioperation value on the sets composed of elements of the set 2^{E_2}, treating the element of the set as a constant operation.

We say that multioperation $g(x_1, \ldots, x_n)$ is obtained from operations $f_1(x_1, \ldots, x_n)$, $f_2(x_1, \ldots, x_n)$ using the equality predicate branching operator, if for some $i, j \in \{1, \ldots, n\}$ the following relation holds:

$$g(x_1, \ldots, x_n) = \begin{cases} f_1(x_1, \ldots, x_n), & \text{if } x_i = x_j, \\ f_2(x_1, \ldots, x_n), & \text{otherwise.} \end{cases}$$

In what follows, we will use the terminology of the ES_U-closure of the set of multioperations proposed in [16].

Let $A = (a_{ij})$ be a $m \times n$ matrix. Denote the columns of A by $A^1, ..., A^n$.

For multioperation $f(x_1, ..., x_s)$ we denote $f\begin{pmatrix} a_{11} & ... & a_{1n} \\ & \vdots & \\ a_{m1} & ... & a_{mn} \end{pmatrix}$ as a set of

$$\left\{ f(B^1, ..., B^s) \mid B^j \in \{A^1, ..., A^n\} \right\}.$$

Here, if $B^j = (b_1^j, ..., b_m^j)^T, j \in \{1, ..., s\}$, then

$$f(B^1, ..., B^s) \text{ is a } \begin{pmatrix} f(b_1^1, ..., b_1^s) \\ \vdots \\ f(b_m^1, ..., b_m^s) \end{pmatrix}.$$

The concept of preserving a predicate by an operation is standard. The set of multioperations preserving the predicate R is denoted by Pol R. In what follows, an m-ary predicate containing n sets will be defined by a $m \times n$ matrix, in which the columns are sets from the predicate.

The following sets of multioperations are introduced in [16]:
$K_1 = \{f \mid f(0, \dots, 0) \in \{0, -\}\}$; $K_2 = \{f \mid f(1, \dots, 1) \in \{1, -\}\}$;
$K_3 = \{f \mid f(0, \dots, 0) \in \{0, *\}\}$; $K_4 = \{f \mid f(1, \dots, 1) \in \{1, *\}\}$;
$K_5 = O_2^*$; $K_6 = H_2$; $K_7 = \{f \mid f(\tilde{\alpha}) \in \{*, 1, -\}, \tilde{\alpha} \in E_2^n\}$;
$K_8 = \{f \mid f(\tilde{\alpha}) \in \{*, 0, -\}, \tilde{\alpha} \in E_2^n\}$; $K_9 = \text{Pol } R_9$; $R_9 = \begin{pmatrix} 0 & 1 & * & - \\ 1 & 0 & * & - \end{pmatrix}$;
$K_{10} = \text{Pol } R_{10}$; $R_{10} = \begin{pmatrix} 0 & 1 & * & * & * & * & 0 & 1 & - \\ 1 & 0 & 0 & 1 & - & * & * & * & * \end{pmatrix}$;
$K_{11} = \text{Pol } R_{11}$; $R_{11} = \begin{pmatrix} 0 & * & * & * & * & 0 & 1 & - \\ 1 & 0 & 1 & - & * & * & * & * \end{pmatrix}$.

It is also shown [16] that these 11 sets are ES_U-precomplete closed classes on the set of all multioperations of rank 2 and there are no other ES_U-precomplete closed classes.

3. Classification of multioperations

For each multioperation, we explicitly define a vector of its membership in precomplete sets. The length of this vector is 11. We say that the i-th coordinate of the vector equals 1 if the multioperation belongs to K_i, and 0 otherwise.

On a set of all multioperations, we define the equivalence relation as follows: multioperations with the same vectors of membership in precomplete sets will be equivalent. Since the number of precomplete sets is 11, then the largest possible number of equivalence classes is 2^{11}.

Let $\overline{K}$ denote the set of operations that don't belong to the set K.

Lemma 1. *The number of equivalence classes of multioperations belonging to the sets K_1 and K_2, with respect to their membership in ES_U-precomplete sets, is 29.*

Proof. To enumerate all classes, we split the set of considered multioperations into 4 sets — $K_3 \cap K_4$, $K_3 \cap \overline{K}_4$, $\overline{K}_3 \cap K_4$, and $\overline{K}_3 \cap \overline{K}_4$.

Consider the *first* set. If multioperation f belongs to the set $K_3 \cap K_4$, then $f(0, \dots, 0) \in \{0\}$ and $f(1, \dots, 1) \in \{1\}$. Then the multioperation obviously belongs to the set K_{11}, and doesn't belong to the sets K_7 and K_8. Now we have to consider the membership in the sets K_5, K_6, K_9, K_{10}.

Let there be a set $\tilde{\alpha}$ such that $f(\tilde{\alpha}) \in \{*\}$. Then, f does not belong to class K_6. In this case, if the multioperation on either set does not return $-$, the membership in the set K_9 incurs the K_{10} membership. If f returns $-$ at least on one set, it cannot simultaneously belong to K_9 and to K_{10}.

Let multioperation f doesn't return $*$ on either of the sets. Hence, it belongs to class K_6. If the multioperation is a Boolean operation ($f \in O_2$), it is easy to notice that it belongs to the set K_9 if and only if it is in K_{10}. Otherwise, the multioperation does not belong to K_{10}.

Thus, the first set is split into 10 equivalence classes at most.

Now let us consider the *second* set. In this case, $f(0,\ldots,0) \in \{0\}$ and $f(1,\ldots,1) \in \{-\}$. Clearly, the multioperation doesn't belong to the sets K_5, K_7, K_9, K_{10}, K_{11}, and there remain no more than 4 variants of equivalence classes.

By analogy with the second set, we get that for the *third* set, the number of equivalence classes is at most 4.

Consider the last *fourth* set of multioperations $\overline{K}_3 \cap \overline{K}_4$. Then $f(\tilde{0}) \in \{-\}$ and $f(\tilde{1}) \in \{-\}$. In this case, the multioperation clearly doesn't belong to the sets K_5, K_{10}, K_{11}.

If multioperation f on some set returns 0 and nowhere returns 1, then it does not belong to K_7, K_9 and belongs to K_8 (2 equivalence classes). Similarly, if f returns 1 and does not return 0, then it does not belong to K_8, K_9 and belongs to K_7 (2 classes).

If there exist sets, on which f returns 1 and returns 0, then it does not belong to K_7, K_8 (4 equivalence classes).

But if the multioperation on either set does not return 1 and does not return 0, then it belongs to K_7, K_8 and there remain 3 equivalence classes at most.

Thus, it is shown that the fourth considered set can be divided into no more than 11 classes, and the total of obtained equivalence classes does not exceed 29.

Table 1 describes 29 multioperations of the set $K_1 \cap K_2$ appertaining to different equivalence classes.

The Lemma is proved. □

Lemma 2. *The number of equivalence classes of multioperations belonging to the sets K_1 and $\overline{K}_2$, with respect to their membership in ES_U-precomplete sets, is 28.*

Proof. As in the previous proof, we consider 4 cases: operations appertaining to $K_3 \cap K_4$, $K_3 \cap \overline{K}_4$, $\overline{K}_3 \cap K_4$, and $\overline{K}_3 \cap \overline{K}_4$.

In the *first* case, $f(0,\ldots,0) \in \{0\}$ and $f(1,\ldots,1) \in \{*\}$. Then f does not belong to classes K_6, K_7, K_9, it belongs to the set K_{11}. Thus, the

Table 1

Operations of the set $K_1 \cap K_2$, appertaining to different classes

No	operation	vector	No	operation	vector
1	(0011)	1 1 1 1 1 1 0 0 1 1 1	2	(0001)	1 1 1 1 1 1 0 0 0 0 1
3	$(0--1)$	1 1 1 1 0 1 0 0 1 0 1	4	$(00-1)$	1 1 1 1 0 1 0 0 0 0 1
5	$(0**1)$	1 1 1 1 1 0 0 0 1 1 1	6	$(0*01)$	1 1 1 1 1 0 0 0 0 1 1
7	$(0**00**1)$	1 1 1 1 1 0 0 0 0 0 1	8	$(0**--**1)$	1 1 1 1 0 0 0 0 1 0 1
9	$(0*-1)$	1 1 1 1 0 0 0 0 0 1 1	10	$(0**00*-1)$	1 1 1 1 0 0 0 0 0 0 1
11	$(000-)$	1 1 1 0 0 1 0 1 0 0 0	12	$(001-)$	1 1 1 0 0 1 0 0 0 0 0
13	$(0**-)$	1 1 1 0 0 0 0 1 0 0 0	14	$(0*1-)$	1 1 1 0 0 0 0 0 0 0 0
15	(-111)	1 1 0 1 0 1 1 0 0 0 0	16	(-001)	1 1 0 1 0 1 0 0 0 0 0
17	$(-**1)$	1 1 0 1 0 0 1 0 0 0 0	18	$(-*01)$	1 1 0 1 0 0 0 0 0 0 0
19	$(----)$	1 1 0 0 0 1 1 1 1 0 0	20	$(-11-)$	1 1 0 0 0 1 1 0 0 0 0
21	$(-00-)$	1 1 0 0 0 1 0 1 0 0 0	22	$(-01-)$	1 1 0 0 0 1 0 0 1 0 0
23	$(-000001-)$	1 1 0 0 0 1 0 0 0 0 0	24	$(-**-)$	1 1 0 0 0 0 1 1 1 0 0
25	$(-*--)$	1 1 0 0 0 0 1 1 0 0 0	26	$(-*1-)$	1 1 0 0 0 0 1 0 0 0 0
27	$(-*0-)$	1 1 0 0 0 0 0 1 0 0 0	28	$(-**01**-)$	1 1 0 0 0 0 0 0 1 0 0
29	$(-****01-)$	1 1 0 0 0 0 0 0 0 0 0			

number of equivalence classes in this case is no greater than 8. We get the same result in the *second* and *third* cases. In the *fourth* case, there will remain 4 equivalence classes at most.

Table 2 contains 28 multioperations that belong to different classes. The Lemma is proved. □

Lemma 3. *The number of equivalence classes of multioperations belonging to the sets* $\overline{K}_1$ *and* K_2*, with respect to their membership in* ES_U*-precomplete sets, is 28.*

Proof. The proof that there are no more than 28 equivalence classes is similar to the previous lemma. Multioperations and corresponding vectors are given in Table. 3. □

Lemma 4. *The number of equivalence classes of multioperations belonging to the sets* $\overline{K}_1$ *and* $\overline{K}_2$*, with respect to their membership in* ES_U*-precomplete sets, is 44.*

Proof. From the condition it follows that the considered multioperations satisfy the following conditions: $f(0,\ldots,0) \in \{1,*\}$, $f(1,\ldots,1) \in \{0,*\}$. We consider all variants.

1) $f(0,\ldots,0) \in \{1\}$, $f(1,\ldots,1) \in \{0\}$. Clearly, such operations are not in the sets K_3, K_4, K_7, K_8, K_{11}.

Table 2

Operations of the set $K_1 \cap \overline{K}_2$, appertaining to different classes

No	operation	vector	No	operation	vector
1	$(0***)$	1 0 1 1 1 0 0 1 0 1 1	2	$(000*)$	1 0 1 1 1 0 0 1 0 0 1
3	$(0*1*)$	1 0 1 1 1 0 0 0 0 1 1	4	$(011*)$	1 0 1 1 1 0 0 0 0 0 1
5	$(0*-*)$	1 0 1 1 0 0 0 1 0 1 1	6	$(00-*)$	1 0 1 1 0 0 0 1 0 0 1
7	$(0****1-*)$	1 0 1 1 0 0 0 0 0 1 1	8	$(01-*)$	1 0 1 1 0 0 0 0 0 0 1
9	(0000)	1 0 1 0 1 1 0 1 0 0 0	10	(0010)	1 0 1 0 1 1 0 0 0 0 0
11	$(00-0)$	1 0 1 0 0 1 0 1 0 0 0	12	$(01-0)$	1 0 1 0 0 1 0 0 0 0 0
13	$(0**0)$	1 0 1 0 1 0 0 1 0 0 0	14	$(0*10)$	1 0 1 0 1 0 0 0 0 0 0
15	$(0*-0)$	1 0 1 0 0 0 0 1 0 0 0	16	$(0****1-0)$	1 0 1 0 0 0 0 0 0 0 0
17	$(-***)$	1 0 0 1 0 0 1 1 0 1 1	18	$(---*)$	1 0 0 1 0 0 1 1 0 0 1
19	$(-*1*)$	1 0 0 1 0 0 1 0 0 1 1	20	$(-11*)$	1 0 0 1 0 0 1 0 0 0 1
21	$(-*0*)$	1 0 0 1 0 0 0 1 0 1 1	22	$(-00*)$	1 0 0 1 0 0 0 1 0 0 1
23	$(-01*)$	1 0 0 1 0 0 0 0 0 1 1	24	$(-**00*1*)$	1 0 0 1 0 0 0 0 0 0 1
25	(-000)	1 0 0 0 0 1 0 1 0 0 0	26	(-010)	1 0 0 0 0 1 0 0 0 0 0
27	$(-**0)$	1 0 0 0 0 0 0 1 0 0 0	28	$(-*10)$	1 0 0 0 0 0 0 0 0 0 0

Note that if the multioperation on either set does not return $*$ and does not return $-$, then it belongs to K_9 if and only if it belongs to K_{10}.

But if there is a set, on which the operation returns $*$, and there is no set, on which it returns $-$, then if it belongs to K_9, it also belongs to K_{10}.

If the multioperation on either set does not return $*$ and there is a set, on which it returns $-$, then such operation does not belong to K_{10}.

If there are binary sets, on which the multioperation return $*$ and $-$, then it cannot be in K_9 and in K_{10} at the same time.

Keeping these comments in mind, we get that the number of equivalence classes does not exceed 10.

2) $f(0,\ldots,0) \in \{1\}$, $f(1,\ldots,1) \in \{*\}$. The multioperations clearly do not belong to the sets K_3, K_6, K_8, K_9, and belong to K_4, K_{11}.

3) Similar to the previous variant.

4) $f(0,\ldots,0) \in \{*\}$, $f(1,\ldots,1) \in \{*\}$. In this case, the multioperations belong to the sets K_3, K_4, K_{11}, and don't belong to K_6. Now we have to consider their membership in the sets K_5, K_7, K_8, K_9, K_{10}.

If the operations return 0 and 1, then they are not in the sets K_7, K_8. The presence of $-$ among the operation values incurs fairness of the statement that if the operation is in K_9, then it does not belong to K_{10}. The absence of $-$ among the operation values makes fair the statement, that if the operation is in K_9, then it belongs to K_{10}. Thus, the number of equivalence classes does not exceed 6.

It is easy to notice that each of three other variants has no more than 4 classes of equivalence. And no more than 18 classes in total.

Table 3

Operations of the set $\overline{K}_1 \cap K_2$, appertaining to different classes

No	operation	vector	No	operation	vector
1	$(***1)$	0 1 1 1 1 0 1 0 0 1 1	2	$(*111)$	0 1 1 1 1 0 1 0 0 0 1
3	$(**01)$	0 1 1 1 1 0 0 0 0 1 1	4	$(*001)$	0 1 1 1 1 0 0 0 0 0 1
5	$(**-1)$	0 1 1 1 0 0 1 0 0 1 1	6	$(*1-1)$	0 1 1 1 0 0 1 0 0 0 1
7	$(*****0-1)$	0 1 1 1 0 0 0 0 0 1 1	8	$(*0-1)$	0 1 1 1 0 0 0 0 0 0 1
9	$(***-)$	0 1 1 0 0 0 1 1 0 1 1	10	$(*---)$	0 1 1 0 0 0 1 1 0 0 1
11	$(**1-)$	0 1 1 0 0 0 1 0 0 1 1	12	$(*11-)$	0 1 1 0 0 0 1 0 0 0 1
13	$(**0-)$	0 1 1 0 0 0 0 1 0 1 1	14	$(*00-)$	0 1 1 0 0 0 0 1 0 0 1
15	$(*01-)$	0 1 1 0 0 0 0 0 0 1 1	16	$(***00*1-)$	0 1 1 0 0 0 0 0 0 0 1
17	(1111)	0 1 0 1 1 1 1 0 0 0 0	18	(1001)	0 1 0 1 1 1 0 0 0 0 0
19	$(11-1)$	0 1 0 1 0 1 1 0 0 0 0	20	$(10-1)$	0 1 0 1 0 1 0 0 0 0 0
21	$(1**1)$	0 1 0 1 1 0 1 0 0 0 0	22	$(1*01)$	0 1 0 1 1 0 0 0 0 0 0
23	$(1*-1)$	0 1 0 1 0 0 1 0 0 0 0	24	$(1****0-1)$	0 1 0 1 0 0 0 0 0 0 0
25	$(111-)$	0 1 0 0 0 1 1 0 0 0 0	26	$(100-)$	0 1 0 0 0 1 0 0 0 0 0
27	$(1**-)$	0 1 0 0 0 0 1 0 0 0 0	28	$(1*0-)$	0 1 0 0 0 0 0 0 0 0 0

Table 4 contains 44 operations and corresponding classes. The Lemma is proved. □

Theorem 1. *The number of equivalence classes of multioperations with respect to their membership in the ES_U-precomplete sets is 129.*

Proof. The fairness of the statement follows from Lemmas 1–4. □

4. Basis Types

For the system of operations to be a basis, the conditions of completeness and minimality must be observed. According to the completeness criterion [16], the system of multioperations is complete if and only if none of the 11 precomplete sets $K_1 - K_{11}$ contains it as a whole. Minimality means that when either operation is removed from the system, the system becomes incomplete.

Operation No. 44 $(1**00*-0)$ in Table 4 does not belong to either of the precomplete sets, so the minimum basis cardinality is 1.

By f_K we denote a multioperation that does not belong to the set K.

Theorem 2. *The maximum basis cardinality of rank 2 multioperations with respect to the ES_U-closure operator is 4.*

Proof. Let the set B be a basis. Then it contains multioperation $f_{K_{11}}$. Consider all possible values of this operation on the binary sets $(0,\ldots,0)$ and $(1,\ldots,1)$. There are 8 possible variants: $(1-)$, (-0), (10), (00), (11), $(--)$, $(0-)$, (-1).

• In the first two variants, operation $f_{K_{11}}$ does not belong to another 7 precomplete sets, hence, the maximum basis cardinality does not exceed 4.

• Consider the third case: $f_{K_{11}}(0,\ldots,0) = 1$ and $f_{K_{11}}(1,\ldots,1) = 0$. In this case, $f_{K_{11}}$ is not in the sets K_1, K_2, K_3, K_4, K_7, K_8. In the set B, we select operation $f_{K_{10}}$, which can return (00), (11), (-0), $(0-)$, (-1), $(1-)$, $(--)$ on the sets $(0,\ldots,0)$ and $(1,\ldots,1)$. Therefore, $f_{K_{10}}$ does not belong to the set K_9 or K_6.

Table 4

Operations of the set $\overline{K}_1 \cap \overline{K}_2$, appertaining to different classes

No	operation	vector	No	operation	vector
1	$(****)$	0 0 1 1 1 0 1 1 1 1 1	2	$(**1*)$	0 0 1 1 1 0 1 0 0 1 1
3	$(*11*)$	0 0 1 1 1 0 1 0 0 0 1	4	$(**0*)$	0 0 1 1 1 0 0 1 0 1 1
5	$(*00*)$	0 0 1 1 1 0 0 1 0 0 1	6	$(*01*)$	0 0 1 1 1 0 0 0 1 1 1
7	$(*****01*)$	0 0 1 1 1 0 0 0 0 1 1	8	$(***00*1*)$	0 0 1 1 1 0 0 0 0 0 1
9	$(*--*)$	0 0 1 1 0 0 1 1 1 0 1	10	$(**-*)$	0 0 1 1 0 0 1 1 0 1 1
11	$(***--*-*)$	0 0 1 1 0 0 1 1 0 0 1	12	$(*****1-*)$	0 0 1 1 0 0 1 0 0 1 1
13	$(*1-*)$	0 0 1 1 0 0 1 0 0 0 1	14	$(*****0-*)$	0 0 1 1 0 0 0 1 0 1 1
15	$(*0-*)$	0 0 1 1 0 0 0 1 0 0 1	16	$(**0--1**)$	0 0 1 1 0 0 0 0 1 0 1
17	$(****01-*)$	0 0 1 1 0 0 0 0 0 1 1	18	$(***001-*)$	0 0 1 1 0 0 0 0 0 0 1
19	$(***0)$	0 0 1 0 1 0 0 1 0 1 1	20	$(*000)$	0 0 1 0 1 0 0 1 0 0 1
21	$(**10)$	0 0 1 0 1 0 0 0 0 1 1	22	$(*110)$	0 0 1 0 1 0 0 0 0 0 1
23	$(**-0)$	0 0 1 0 0 0 0 1 0 1 1	24	$(*0-0)$	0 0 1 0 0 0 0 1 0 0 1
25	$(*****1-0)$	0 0 1 0 0 0 0 0 0 1 1	26	$(*1-0)$	0 0 1 0 0 0 0 0 0 0 1
27	$(1***)$	0 0 0 1 1 0 1 0 0 1 1	28	$(111*)$	0 0 0 1 1 0 1 0 0 0 1
29	$(1*0*)$	0 0 0 1 1 0 0 0 0 1 1	30	$(100*)$	0 0 0 1 1 0 0 0 0 0 1
31	$(1*-*)$	0 0 0 1 0 0 1 0 0 1 1	32	$(11-*)$	0 0 0 1 0 0 1 0 0 0 1
33	$(1****0-*)$	0 0 0 1 0 0 0 0 0 1 1	34	$(10-*)$	0 0 0 1 0 0 0 0 0 0 1
35	(1010)	0 0 0 0 1 1 0 0 1 1 0	36	(1000)	0 0 0 0 1 1 0 0 0 0 0
37	$(1--0)$	0 0 0 0 0 1 0 0 1 0 0	38	$(10-0)$	0 0 0 0 0 1 0 0 0 0 0
39	$(1**0)$	0 0 0 0 1 0 0 0 1 1 0	40	$(1*00)$	0 0 0 0 1 0 0 0 0 1 0
41	$(1**00**0)$	0 0 0 0 1 0 0 0 0 0 0	42	$(1**--**0)$	0 0 0 0 0 0 0 0 1 0 0
43	$(1*-0)$	0 0 0 0 0 0 0 0 0 1 0	44	$(1**00*-0)$	0 0 0 0 0 0 0 0 0 0 0

• Let $f_{K_{11}}(0,\ldots,0) = 0$ and $f_{K_{11}}(1,\ldots,1) = 0$. Then $f_{K_{11}}$ does not belong to the sets K_2, K_4, K_7, K_9, K_{10}.

Consider $f_{K_1}(0,\ldots,0) \in \{1,*\}$. If $f_{K_1}(0,\ldots,0) = 1$, then f_{K_1} does not belong to K_3, K_8. Then it turns out that the set B contains two multioperations not appertaining to 9 classes.

If $f_{K_1}(0,\dots,0) = *$, then $f_{K_1} \notin K_6$. Consider values $f_{K_1}(1,\dots,1) \in \{*, 0, 1, -\}$. If $f_{K_1}(1,\dots,1) = 1$ or $f_{K_1}(1,\dots,1) = -$, then f_{K_1} is not in K_8 or K_5 respectively.

We can suppose that the set B either contains $f_{K_{11}} = (00), f_{K_1} = (*0)$, or $f_{K_{11}} = (00), f_{K_1} = (**)$. Consider $f_{K_3}(0,\dots,0) \in \{1, -\}$. In this case, f_{K_3} does not belong to K_8 or K_5 respectively. We get that B contains 3 multioperations not appertaining to 10 precomplete classes.

• Variant $f_{K_{11}}(0,\dots,0) = 1$ and $f_{K_{11}}(1,\dots,1) = 1$ is dual to the previous one.

• Let $f_{K_{11}}(0,\dots,0) = -$ and $f_{K_{11}}(1,\dots,1) = -$. In this case, $f_{K_{11}}$ does not belong to the sets K_3, K_4, K_5, K_{10}. In the set B, we select operation f_{K_1} and consider its possible values on the binary set $(0,\dots,0)$ specifically.

•△ Let $f_{K_1}(0,\dots,0) = 1$. So $f_{K_1}(1,\dots,1)$ can be $0, *, 1, -$. If $f_{K_1}(\tilde{1}) = 0$, the operation is not in the sets K_7, K_8, K_2. And if $f_{K_1}(1,\dots,1) = *$, the operation does not belong to the sets K_6, K_8, K_9. In both variants, the basis contains 4 multioperations at most.

In the rest two variants, the operation does not belong to the sets K_8, K_9, and hence, the considered set still belongs to the sets K_2, K_6, K_7. Consider operation f_{K_2}. If $f_{K_2}(1,\dots,1) = 0$, then the operation is not in the set K_7, and if $f_{K_2}(1,\dots,1) = *$, then the operation does not belong to the set K_6.

•△ Let us get to the case when $f_{K_1}(0,\dots,0) = *$. If $f_{K_1}(1,\dots,1) = 0$, the operation does not belong to the sets K_2, K_6, K_7, K_9. If $f_{K_1}(1,\dots,1) = 1$, then the operation is not in the sets K_6, K_8, K_9. The case when $f_{K_1}(\tilde{1}) = -$, excludes its membership in the sets K_6, K_9, and the case when $f_{K_1}(\tilde{1}) = *$ excludes the membership in the sets K_2, K_6.

Thus, it can be assumed that the set B either contains multioperations $f_{K_{11}} = (--)$ and $f_{K_1} = (*-)$, or $f_{K_{11}} = (--)$ and $f_{K_1} = (**)$.

Let $\{(--), (**)\} \subseteq B$. Consider f_{K_7}, which returns 0 on some set. We write out all possible values of this operation on the sets $(0,\dots,0)$ and $(1,\dots,1)$ and the sets that it does not belong to

$f_{K_7}(0,\dots,0)$	0	0	0	0	1	1	1	1	$-$	$-$	$-$	$-$	$*$	$*$	$*$	$*$
$f_{K_7}(1,\dots,1)$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$
	K_9	K_8	K_9	K_9	K_8	K_8	K_8	K_8	K_9	K_9	K_9		K_9	K_9		K_9

Clearly, one can remove multioperation $f_{K_{11}}$ or f_{K_1} from the set B, thus the set B will contain no more than 4 multioperations.

Now let $\{(--), (*-)\} \subseteq B$. We draw up a table for f_{K_7} similar to that discussed above:

$f_{K_7}(0,\dots,0)$	0	0	0	0	1	1	1	1	$-$	$-$	$-$	$-$	$*$	$*$	$*$	$*$
$f_{K_7}(1,\dots,1)$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$
	K_2	K_8	K_2		K_8	K_8	K_8	K_8	K_2	K_8	K_2		K_2	K_8	K_2	

and notice that the set B either contains operations $(--), (*-), (0-)$, or one can remove operation $f_{K_{11}}$ or f_{K_1} from it.

So, we assume that the set B contains operations $(--), (*-), (0-)$ that belong to the sets K_2, K_8. Consider operation f_{K_8}, which returns 1 on a certain set. We draw up a table for it, similar to those considered above:

$f_{K_8}(0,\dots,0)$	0	0	0	0	1	1	1	1	$-$	$-$	$-$	$-$	$*$	$*$	$*$	$*$
$f_{K_8}(1,\dots,1)$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$
	K_2		K_2		K_2		K_2		K_2		K_2		K_2		K_2	

Three cases from this table must be discussed: $(01), (-1), (*1)$. In other cases, we either consider cases where the set B contains operation $(11), (1-)$, or we can remove operation f_{K_7} or $f_{K_{11}}$ or f_{K_1} from it.

In case (01), operation $f_{K_7} = (0-)$ can be removed from the set B; in case (-1), operation $f_{K_{11}} = (--)$ can be removed from the set B; in case $(*1)$, operation $f_{K_1} = (*-)$ can be removed from the set B.

• Let $f_{K_{11}}(0,\dots,0) = 0$ and $f_{K_{11}}(1,\dots,1) = -$. In this case, $f_{K_{11}}$ does not belong to the sets K_4, K_5, K_7, K_9, K_{10}.

We select multioperation f_{K_1} in the set B. If $f_{K_1}(0,\dots,0) = 1$, then f_{K_1} is not in the sets K_3, K_8.

Let $f_{K_1}(0,\dots,0) = *$. Then $f_{K_1}(1,\dots,1) \in \{0,*\}$ excludes the sets K_2, K_6. The value $f_{K_1}(1,\dots,1) = 1$ excludes the sets K_6, K_8. And now, if $f_{K_1}(1,\dots,1) = -$, then we exclude the set K_6.

Thus, it can be assumed that the set B contains operations $f_{K_{11}} = (0-)$ and $f_{K_1} = (*-)$. Consider f_{K_8} that returns 1 on a certain set. We write out all possible values of this multioperation on the sets $(0,\dots,0)$ and $(1,\dots,1)$ and the sets that it does not belong to

$f_{K_8}(0,\dots,0)$	0	0	0	0	1	1	1	1	$-$	$-$	$-$	$-$	$*$	$*$	$*$	$*$
$f_{K_8}(1,\dots,1)$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$	0	1	$*$	$-$.
	K_2		K_2		K_2	K_3	K_2	K_3	K_2	K_3	K_2	K_3	K_2		K_2	

Thus, the set B either contains operations $(0-), (*-), (01)$, or $(0-)$, $(*-)$, $(*1)$, or one can remove operation $f_{K_{11}}$ or f_{K_1} from it.

•△ Consider $\{(0-), (*-), (01)\} \subseteq B$. All operations of the set belong to the sets of K_2, K_3. We consider operation f_{K_2} and present its values on $(0,\dots,0)$ and $(1,\dots,1)$ and the sets, to which it does not belong in the following table:

$f_{K_2}(0,\dots,0)$	0	1	$-$	$*$	0	1	$-$	$*$
$f_{K_2}(1,\dots,1)$	0	0	0	0	$*$	$*$	$*$	$*$.
		K_3	K_3			K_3	K_3	

Four cases from this table must be discussed: (00), $(*0)$, $(0*)$, and $(**)$. In case (00), one can remove operation $f_{K_{11}} = (0-)$ from the set B, in cases $(*0)$ and $(**)$ one can remove operation $f_{K_1} = (*-)$ from the set B.

If $f_{K_2} = (0*)$, we will add operation f_{K_3} to the set B and find out, which of the previously added operations can be removed from the set B. To do this, we will make a table of membership of multioperations in precomplete classes:

Table 5
Membership in precomplete classes
of the set B operations and possible variants of f_{K_3}

I	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	II
$(0*)$	+	−	+	+	+	−	−	+	−	+	+	
$(0-)$	+	+	+	−	−	+	−	+	−	−	−	
(01)	+	+	+	+	+	+	−	−	+	+	+	
$(*-)$	−	+	+	−	−	−	+	+	−	+	+	
$(1*)$	−	−	−	+	+	−	+	−	−	+	+	$(0*)$
(10)	−	−	−	−	+	+	−	−	+	+	−	(01)
(11)	−	+	−	+	+	+	+	−	−	−	−	(01)
$(1-)$	−	+	−	−	−	+	+	−	−	−	−	(01)
(-0)	+	−	−	−	−	+	−	+	−	−	−	$(0-)$
(-1)	+	+	−	+	−	+	+	−	−	−	−	$(0-)$
$(-*)$	+	−	−	+	−	−	+	+	−	+	+	$(-*)$
$(--)$	+	+	−	−	−	+	+	+	+	−	−	$(0-)$

Column I in table 5 contains operations from B and all possible multioperations of type f_{K_3}. The following 11 columns indicate membership of multioperations in precomplete sets. Column II for multioperations f_{K_3} specifies which operation from B they can replace.

Thus, the set B contains 4 multioperations.

•△ Let $\{(0-), (*-), (*1)\} \subseteq B$. This case is also reduced to sequential analysis of the possible values of f_{K_2} and f_{K_3}, except that in a table similar to Table 5, operation (01) from B must be replaced with $(*1)$.

• Let $f_{K_{11}}(0, \dots, 0) = -$ and $f_{K_{11}}(1, \dots, 1) = 1$. Then $f_{K_{11}}$ does not belong to the sets K_3, K_5, K_8, K_9, K_{10}. This variant is dual to the previous one. □

We say that two bases B_1 and B_2 of the same cardinality are of different types if they don't have the same sets of vectors of membership in the precomplete sets.

Resulting from computer-aided enumeration, it has been found that there is 1 type of basis of the cardinality 1, 1525 types of basis of the cardinality 2, 29623 types of basis of the cardinality 3, and 18744 types of basis of the cardinality 4.

5. Conclusion

The paper considers classification of multioperations, defined on the two-element set, in terms of their membership in ES_U-precomplete classes. The next step is to obtain all E-closed classes for multioperations given on a two-element set and to construct a completeness criterion for multioperations defined on a three-element set.

References

1. Kazimirov A.S., Panteleev V.I., Tokareva L.V. Classification and Enumeration of Bases in Clone of All Hyperfunctions on Two-Elements Set. *The Bulletin of Irkutsk State University. Series Mathematics*, 2014, vol. 10, pp. 61-78. (in Russian)
2. Krnić L. Types of bases in the algebra of logic. *Glasnik matematicko-fizicki i astronomski. Ser. 2*, 1965, vol. 20, p. 23-32.
3. Lau D., Miyakawa M. Classification and enumeration of bases in $P_k(2)$. *Asian-European Journal of Mathematics*, 2008, vol. 01, no 02, pp. 255-282.
4. Lo Czu Kai. Maximal closed classes on the Set of Partial Many-valued Logic Functions. *Kiberneticheskiy Sbornik*, Moscow, Mir Publ., 1988, vol. 25, pp. 131-141. (in Russian)
5. Lo Czu Kai. Completeness theory on Partial Many-valued Logic Functions. *Kiberneticheskiy Sbornik*, Moscow, Mir Publ., 1988, vol. 25, pp. 142-157. (in Russian)
6. Machida H. Hyperclones on a Two-Element Set. *Multiple-Valued Logic. An International Journal*, 2002, no. 8(4), pp. 495-501.
7. Machida H., Pantovic J. On Maximal Hyperclones on $\{0,1\}$ — a new approach. *Proceedings of 38th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2008)*, 2008, pp. 32-37.
8. Marchenkov S.S. On the Expressibility of Functions of Many-Valued Logic in Some Logical-Functional Classes. *Discrete Math. Appl.*, 1999, no. 4, pp. 563-581. https://doi.org/10.4213/dm400
9. Marchenkov S.S. Closure Operators with Predicate Branching. *Bulletin of Moscow State University. Series 1. Mathematics and Mechanics*, 2003, no. 6, pp. 37–39. (in Russian)
10. Marchenkov S.S. The Closure Operator with the Equality Predicate Branching on the Set of Partial Boolean Functions. *Discrete Math. Appl.*, 2008, no. 3, pp. 381-389. https://doi.org/10.4213/dm1015
11. Marchenkov S.S. The E-closure Operator on the Set of Partial Many-valued Logic Functions. *Mathematical problems in cybernetics*, Moscow, Fizmatlit Publ., 2013, vol. 19, pp. 227–238. (in Russian)
12. Matveev S.S. Construction of All E-closed Classes of Partial Boolean Functions. *Mathematical problems in cybernetics*, Moscow, Fizmatlit Publ., 2013, vol. 18, pp. 239-244. (in Russian)
13. Miyakawa M., Stojmenović I., Lau D., Rosenberg I. Classification and basis enumerations in many-valued logics. *Proc. 17th International Symposium on Multi-Valued logic.* Boston, 1987, pp. 151-160.
14. Miyakawa M., Stojmenović I., Lau D., Rosenberg I. Classification and basis enumerations of the algebras for partial functions. *Proc. 19th International Symposium on Multi-Valued logic.* Rostock, 1989, pp. 8-13.

15. Panteleyev V.I., Riabets L.V. The Closure Operator with the Equality Predicate Branching on the Set of Hyperfunctions on Two-Element Set. *The Bulletin of Irkutsk State University. Series Mathematics*, 2014, vol. 10, pp. 93-105. (in Russian)
16. Panteleev V.I., Riabets L.V. The Completeness Criterion for Closure Operator with the Equality Predicate Branching on the Set of Multioperations on Two-Element Set. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 29, pp. 68–85. https://doi.org/10.26516/1997-7670.2019.29.68
17. Panteleyev V.I., Riabets L.V. E-closed Sets of Hyperfunctions on Two-Element Set. *J. Sib. Fed. Univ. Math. Phys.*, 2020, vol. 13, no. 2, pp. 231-241. https://doi.org/10.17516/1997-1397-2020-13-2-231-241
18. Romov B.A. Hyperclones on a Finite Set. *Multiple-Valued Logic. An International Journal*, 1998, vol. 3(2), pp. 285-300.
19. Yablonskij S.V. On the Superpositions of Logic Functions. *Mat. Sbornik*, 1952, vol. 30, no. 2(72), pp. 329-348. (in Russian)

Vladimir Panteleev, Doctor of Sciences (Physics and Mathematics), Institute of Mathematics and Information Technologies, Irkutsk State University, Russian Federation, 664000, Irkutsk, K. Marx st., 1, tel.:+7(3952)242214 e-mail: vl.panteleyev@gmail.com, ORCID iD https://orcid.org/0000-0003-4766-486X.

Leonid Riabets, Candidate of Sciences (Physics and Mathematics), Associate Professor, Institute of Mathematics and Information Technologies, Irkutsk State University, Russian Federation, 664000, Irkutsk, K. Marx st., 1, tel.:+7(3952)242214 e-mail: l.riabets@gmail.com, ORCID iD https://orcid.org/0000-0003-4047-9573.

Received 24.10.2020

Классификация мультиопераций ранга 2 относительно E-предполных множеств

В. И. Пантелеев, Л. В. Рябец

Иркутский государственный университет, Иркутск, Российская Федерация

Аннотация. В работе рассматриваются мультиоперации, задаваемые на двухэлементном множестве, и замыкание, использующее операторы суперпозиции и разветвления по предикату равенства. Оператор суперпозиции основан на объединения множеств. Приводится классификация мультиопераций относительно принадлежности предполным множествам. Показывается, что число классов эквивалентности равно 129. Описываются все типы базисов и доказывается, что максимальная мощность базиса равна 4.

Ключевые слова: замыкание, предикат равенства, мультиоперация, замкнутое множество, суперпозиция, предполные множества.

Известия Иркутского государственного университета.
2020. Т. 34. Серия «Математика». С. 93–108

Список литературы

1. Казимиров А. С., Пантелеев В. И., Токарева Л. В. Классификация и перечисление базисов клона всех гиперфункций ранга 2 // Известия Иркутского государственного университета. Серия Математика. 2014. Т. 7. С. 61–78.
2. Krnić L. Types of bases in the algebra of logic // Glasnik matematicko-fizicki i astronomski. Ser. 2. 1965. Vol. 20. P. 23–32.
3. Lau D., Miyakawa M. Classification and enumerations of bases in $P_k(2)$ // Asian-European Journal of Mathematics. 2008. Vol. 01, № 02. P. 255–282.
4. Ло Джукай. Максимальные замкнутые классы в множестве частичных функций многозначной логики // Кибернетический сборник. Новая серия. М. : Мир, 1988. Вып. 25. С. 131–141.
5. Ло Джукай. Теория полноты для частичных функций многозначной логики // Кибернетический сборник. Новая серия. М. : Мир, 1988. Вып. 25. С. 142–157.
6. Machida H. Hyperclones on a Two-Element Set // Multiple-Valued Logic. An International Journal. 2002. N 8(4). P. 495–501.
7. Machida H., Pantovic J. On Maximal Hyperclones on $\{0,1\}$ — a new approach // Proceedings of 38th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2008). 2008. P. 32–37.
8. Марченков С. С. О выразимости функций многозначной логики в некоторых логико-функциональных языках // Дискретная математика. 1999. Вып. 4. С. 110–126. https://doi.org/10.4213/dm400
9. Марченков С. С. Оператор замыкания с разветвлением по предикату равенства на множестве частичных булевых функций // Дискретная математика. 2008. Вып. 3. С. 80–88. https://doi.org/10.4213/dm1015
10. Марченков С. С. Операторы замыкания с разветвлением по предикату // Вестник МГУ. Серия 1, Математика и механика. 2003. № 6. С. 37–39.
11. Марченков С. С. Оператор *E*-замыкания на множестве частичных функций многозначной логики // Математические вопросы кибернетики. М. : Физматлит, 2013. Вып. 19. С. 227–238.
12. Матвеев С. А. Построение всех *E*-замкнутых классов частичных булевых функций // Математические вопросы кибернетики. М. : Физматлит, 2013. Вып. 18. С. 239–244.
13. Classification and basis enumerations in many-valued logics / M. Miyakawa, I. Stojmenović, D. Lau, I. Rosenberg // Proc. 17th International Symposium on Multi-Valued logic. Boston, 1987. P. 151–160.
14. Classification and basis enumerations of the algebras for partial functions / M. Miyakawa, I. Stojmenović, D. Lau, I. Rosenberg // Proc. 19th International Symposium on Multi-Valued logic. Rostock, 1989. P. 8–13.
15. Пантелеев В. И., Рябец Л. В. Оператор замыкания с разветвлением по предикату равенства на множестве гиперфункций ранга 2 // Известия Иркутского государственного университета. Серия Математика. 2014. Т. 10. С. 93–105.
16. Panteleev V. I., Riabets L. V. The Completeness Criterion for Closure Operator with the Equality Predicate Branching on the Set of Multioperations on Two-Element Set // Известия Иркутского государственного университета. Серия Математика. 2019. Т. 29. С. 68–85. https://doi.org/10.26516/1997-7670.2019.29.68
17. Panteleyev V. I., Riabets L. V. E-closed Sets of Hyperfunctions on Two-Element Set // Журнал СФУ. Серия: Математика и физика. 2020. Т. 13, № 2. С. 231–241. https://doi.org/10.17516/1997-1397-2020-13-2-231-241
18. Romov B. A. Hyperclones on a Finite Set // Multiple-Valued Logic. An International Journal. 1998. Vol. 3(2). P. 285–300.

19. Яблонский С. В. О суперпозициях функций алгебры логики // Математический сборник. 1952 Т. 30, № 2(72). С. 329–348.

Владимир Иннокентьевич Пантелеев, доктор физико-математических наук, доцент, Институт математики и информационных технологий, Иркутский государственный университет, Российская Федерация, 664003, г. Иркутск, ул. К. Маркса, 1 тел.: +7(3952)242214
e-mail: vl.panteleyev@gmail.com, ORCID iD
https://orcid.org/0000-0003-4766-486X.

Леонид Владимирович Рябец, кандидат физико-математических наук, доцент, Институт математики и информационных технологий, Иркутский государственный университет, Российская Федерация, 664003, г. Иркутск, ул. К. Маркса, 1 тел.: +7(3952)242214
e-mail: l.riabets@gmail.com,
ORCID iD https://orcid.org/0000-0003-4047-9573.

Поступила в редакцию 24.10.2020