Серия «Математика»
2020. Т. 34. С. 3—17

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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УДК 517.977
MSC 49J15, 49M25
DOI https://doi.org/10.26516/1997-7670.2020.34.3

On Resolution of an Extremum Norm Problem for the Terminal State of a Linear System

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Abstract. We study extremum norm problems for the terminal state of a linear dynamical system using methods of parameterization of admissible controls.

Piecewise continuous controls are approximated in the class of piecewise linear functions on a uniform grid of nodes of the time interval by linear combinations of special support functions. In this case, the restriction of a control of the original problem to the interval induces the same restrictions for the variables of the finite-dimensional problems.

The finite-dimensional version of a minimum norm problem can effectively be resolved with the help of modern convex optimization programs. In the case of two variables, we propose an analytical method of resolution that uses a one-dimensional minimization problem for a parabola over a segment.

For a non-convex norm maximization problem, the finite-dimensional version is resolved globally by exhaustive search over the vertices of a hypercube. The proposed approach provides further insights into global resolution of non-convex optimal control problems and is exemplified by some illustrative problems.

Keywords: linear control system, extremum norm problems for the terminal state, piecewise linear approximation, finite-dimensional problems.

1. Introduction

Extremum norm problems for the terminal state of a linear control system have a long history. On the other hand, they are still actively investigated in view of their scientific relevance and interest.

There are several effective methods of resolution of a minimum norm problem which are due to its convex structure. These methods are based on nonlocal improvements using the matrix conjugate system [1; 5; 10].

A maximum norm problem for the terminal state is multi-extremal, which renders its global resolution essentially more difficult. The existing methods are focused on improving extreme controls and are based on global optimality conditions [2; 7; 8]. Since numerical verification of global optimality conditions is still problematic, these methods provide usually weakened and incomplete versions of optimality criteria. In this sense, the global resolution of the problem is, in general, not fully methodological and as such contains heuristic elements. In other words, a maximum norm problem is still open for further research.

In the present paper, extremum norm problems for the terminal state of a linear system are considered in the framework of the control function parameterization technique [4; 6; 11]. In this case, the approximation is done in the class of piecewise linear functions on a given grid of nods of the time interval, so that the optimal control problem is transformed to a finite-dimensional version of extremum problem for a quadratic function over a hypercube.

The resulting problem can be solved in a finite number of iterations [9]. In order to simplify the resolution procedure, we propose an iteration-free algorithm for the two-dimensional problem (minimum of a paraboloid over a square) that uses a geometric interpretation.

The global resolution of the finite-dimensional maximum problem is elementary implemented through a complete or specialized exhaustive search over the given set of hypercube vertices. Therefore, in the framework of the presented approach, the original nonconvex problem is approximated by a similar finite-dimensional problem of the given dimension. The guaranteed global resolution of the later is done by an acceptable finite exhaustive search.

Our resolution technique is tested on two illustrative problems.

2. Statement of the problem. Parametrization of the control

Introduce the following variables: $t \in [t_0, T]$, the time, $u(t) \in R$, the control, and $x(t) \in R^n$, the state, satisfying the linear system

$$\dot{x} = A(t)x + b(t)u, \quad x(t_0) = x^0, \tag{2.1}$$

where $A(t) \in R^{n\times n}$, $b(t) \in R^n$ are continuous functions on $[t_0, T]$.

The set V of admissible controls contains piecewise continuous functions $u(t)$ with the standard constraint

$$u(t) \in [u_-, u_+], \quad t \in [t_0, T]. \tag{2.2}$$

We define on the set V the terminal functional (the norm of the terminal state)

$$\Phi(u) = \frac{1}{2}\langle x(T), x(T)\rangle \tag{2.3}$$

and consider the corresponding extremum problems (maximum and minimum).

Transform these variational problems into finite-dimensional problems using simple parametrizations of admissible controls.

We present the first parametrization scheme in the class of piecewise constant functions. Introduce on the interval $[t_0, T]$ a uniform network Δ_1 of nods $t_i = t_0 + ih$, $i = \overline{0, m}$ with the mesh $h = \frac{T-t_0}{m}$.

Set $T_j = (t_{j-1}, t_j]$, $j = \overline{1, m}$ and define the characteristic functions

$$\chi_j(t) = \begin{cases} 1, & t \in T_j, \\ 0, & t \notin T_j. \end{cases}$$

Let $y = (y_1, ..., y_m)$ be a collection of parameters. Construct the controls

$$u(t, y) = \sum_{j=1}^{m} y_j \chi_j(t), \quad t \in [t_0, T]$$

and introduce the subset of admissible controls

$$V_1 = \{u(\cdot, y) : y_j \in [u_-, u_+], \ \ j = \overline{1, m}\}.$$

Let $x(t, y)$, $t \in [t_0, T]$ be a solution of the phase system (2.1) corresponding to the control $u(t, y)$. Then,

$$x(t, y) = x(t, 0) + \sum_{j=1}^{m} y_j x^j(t), \quad t \in [t_0, T], \tag{2.4}$$

where $x^j(t)$ is a solution of the phase Cauchy problem

$$\dot{x} = A(t)x + b(t)\chi_j(t), \quad x(t_0) = 0.$$

Consider the functional $\Phi(u)$ on the set of controls V_1. Using the representation (2.4) we obtain the formula

$$\Phi_1(y) = \Phi(0) + \langle d, y\rangle + \frac{1}{2}\langle Xy, Xy\rangle,$$

where

d is the m-vector with entries $\langle x(T,0), x^j(T)\rangle, \;\; j=\overline{1,m},$

X is the $(n\times m)$ - matrix with columns $x^j(T), \;\; j=\overline{1,m}.$

Now, we describe the second parametrization scheme in the class of piecewise linear functions. Introduce on $[t_0,T]$ a uniform network Δ_2 of nods $t_i =$
$= t_0 + ih, \; i=\overline{0,m+1}$ with the mesh $h=\frac{T-t_0}{m+1}$. Denote the intervals

$$T_0=[t_0,t_1], \quad T_j=[t_{j-1},t_{j+1}], \; j=\overline{1,m}, \quad T_{m+1}=[t_m,t_{m+1}]$$

and define the following support functions on $[t_0,T]$

$$\varphi_0(t)=\begin{cases} \frac{1}{h}(t_1-t), & t\in T_0,\\ 0, & t\notin T_0,\end{cases} \qquad \varphi_{m+1}(t)=\begin{cases} \frac{1}{h}(t-t_m), & t\in T_{m+1},\\ 0, & t\notin T_{m+1},\end{cases}$$

$$\varphi_j(t)=\begin{cases} \frac{1}{h}(t-t_{j-1}), & t\in[t_{j-1},t_j],\\ \frac{1}{h}(t_{j+1}-t), & t\in[t_j,t_{j+1}],\\ 0, & t\notin T_j,\end{cases} \qquad j=\overline{1,m}.$$

The values at the nods are given by

$$\varphi_j(t_i)=\begin{cases} 1, & j=i,\\ 0, & j\neq i,\end{cases} \qquad i,j=\overline{0,m+1}.$$

Let $z=(z_0,...,z_{m+1})$ be a collection of parameters. Construct the controls

$$u(t,z)=\sum_{j=0}^{m+1} z_j\varphi_j(t), \quad t\in[t_0,T].$$

They are continuous, piecewise linear functions with the nodes t_i and the values $u(t_i,z)=z_i, \; i=\overline{0,m+1}$. In other words, the control $u(t,z)$ is the linear spline on the network Δ_2 corresponding to the values $\{z_0,...,z_{m+1}\}$.

Introduce the subset of admissible controls

$$V_2=\{u(\cdot,z): z_j\in[u_-,u_+], \;\; j=\overline{0,m+1}\}.$$

Let $x(t,z), \; t\in[t_0,T]$ be a phase trajectory corresponding to the control $u(t,z)$. We have

$$x(t,z)=x(t,0)+\sum_{j=0}^{m+1} z_j\hat{x}^j(t), \quad t\in[t_0,T], \tag{2.5}$$

where $\hat{x}^j(t)$ is a solution of the Cauchy problem

$$\dot{x}=A(t)x+b(t)\varphi_j(t), \quad x(t_0)=0.$$

Consider the functional $\Phi(u)$ on the set of controls V_2. Using the representation (2.5), we obtain the formula

$$\Phi_2(z) = \Phi(0) + \langle \hat{d}, z\rangle + \frac{1}{2}\langle \hat{X}z, \hat{X}z\rangle,$$

where
$\hat{d} \in R^{m+2}$ is the vector with entries $\langle x(T,0), \hat{x}^j(T)\rangle$, $j = \overline{0, m+1}$,
$\hat{X} \in R^{n\times(m+2)}$ is the matrix with columns $\hat{x}^j(T)$, $j = \overline{0, m+1}$.

We note that the quadratic functions $\Phi_1(y)$, $\Phi_2(z)$ are convex. This allows us to qualify the corresponding extreme problems on simple sets: $z, y \in [u_-, u_+]$.

3. A minimum norm problem for the terminal state

Consider the problem

$$\Phi(u) = \frac{1}{2}\langle x(T,u), x(T,u)\rangle \to min, \quad u \in V. \tag{3.1}$$

After the first parameterization, we obtain the special convex programming problem with respect to the vector $y = (y_1, ..., y_m)$

$$\Phi_1(y) = \Phi(0) + \langle d, y\rangle +$$
$$+\frac{1}{2}\langle Xy, Xy\rangle \to min, \quad y \in [u_-, u_+]. \tag{3.2}$$

Such a quadratic problem can be solved in a finite number of iterations [9].

Consider some special cases of the problem (3.2) with respect to the dimension m, which are solved without an iterative process using explicit formulas.

Let $m = 1$, i.e. the problem (3.1) is solved on the set of const-controls $u(t) = y_1$, $t \in [t_0, T]$. The corresponding one-dimensional problem (3.2) has the form (minimum of a convex parabola over an interval)

$$\Phi_1(y_1) = \Phi(0) + y_1\langle x(T,0), x^1(T)\rangle +$$
$$+\frac{1}{2}y_1^2\langle x^1(T), x^1(T)\rangle \to min, \quad y_1 \in [u_-, u_+].$$

Assuming that $x^1(T) \neq 0$ define the stationary point

$$y_1^* = -\frac{\langle x(T,0), x^1(T)\rangle}{\langle x^1(T), x^1(T)\rangle}.$$

Then the solution of the one-dimensional problem is obvious:

$$y_1^{min} = \begin{cases} y_1^*, & y_1^* \in [u_-, u_+], \\ u_-, & y_1^* < u_-, \\ u_+, & y_1^* > u_+. \end{cases}$$

Next, let $m = 2$, i.e. the problem (3.1) is solved on the set of controls

$$u(t) = \begin{cases} y_1, & t \in [t_0, t_1], \\ y_2, & t \in (t_1, T], \end{cases}$$

with the fixed switching point $t_1 = \frac{t_0+T}{2}$.

The corresponding two-dimensional problem (3.2) is:

to minimize the function of two variables $\Phi_1(y_1, y_2)$ over the square

$$Y = \{(y_1, y_2) : y_i \in [u_-, u_+], \quad i = 1, 2\}.$$

Suppose that the matrix $X \in R^{n\times 2}$ is full rank, i.e. the Gram matrix $X^T X$ is positive definite. Then, the unique stationary point $y^* = (y_1^*, y_2^*)$ of the function $\Phi_1(y)$ is determined from the linear system

$$\bigtriangledown \Phi_1(y) = d + X^T X y = 0.$$

This is a minimum point of the function $\Phi_1(y)$ on R^2. In this case, the level curves of the function $\Phi_1(y)$ are ellipses centered at the point y^*.

Next, we solve analytically the minimization problem for the paraboloid $\Phi_1(y)$ over the square Y, using the information on the location of the point y^* and the geometric interpretation of the problem based on the level curves of the objective function.

If $y^* \in Y$, then we obtain the solution of the two-dimensional minimization problem: $y^{min} = y^*$.

We consider the general case $y^* \notin Y$.

Possible situations when the constraint is violated for one variable:

1) $y_1^* < u_-, \ \ y_2^* \in [u_-, u_+] \ \ \Rightarrow$ solve the one-dimensional problem

$$\Phi_1(u_-, y_2) \to min, \quad y_2 \in [u_-, u_+] \ \ \Rightarrow \ \ y_2^-,$$

set $y_1^{min} = u_-, \quad y_2^{min} = y_2^-$;

2) $y_1^* > u_+, \ \ y_2^* \in [u_-, u_+] \ \ \Rightarrow$ solve the one-dimensional problem

$$\Phi_1(u_+, y_2) \to min, \quad y_2 \in [u_-, u_+] \ \ \Rightarrow \ \ y_2^+,$$

set $y_1^{min} = u_+ \quad y_2^{min} = y_2^+$;

3) $y_1^* \in [u_-, u_+], \ \ y_2^* < u_- \ \ \Rightarrow$ solve the one-dimensional problem

$$\Phi_1(y_1, u_-) \to min, \quad y_1 \in [u_-, u_+] \ \ \Rightarrow \ \ y_1^-,$$

set $y_1^{min} = y_1^-, \quad y_2^{min} = u_-$;

4) $y_1^* \in [u_-, u_+], \ \ y_2^* > u_+ \ \ \Rightarrow$ solve the one-dimensional problem

$$\Phi_1(y_1, u_+) \to min, \quad y_1 \in [u_-, u_+] \ \ \Rightarrow \ \ y_1^+,$$

set $y_1^{min} = y_1^+, \quad y_2^{min} = u_+$.

Possible situations when the constraint is violated for two variables $y_1^*, \ y_2^*$:

5) $y_1^* < u_-, \;\; y_2^* < u_- \;\; \Rightarrow$ solve two one-dimensional problems

$$\Phi_1(u_-, y_2) \to min, \quad y_2 \in [u_-, u_+] \;\; \Rightarrow \;\; y_2^-,$$

$$\Phi_1(y_1, u_-) \to min, \quad y_1 \in [u_-, u_+] \;\; \Rightarrow \;\; y_1^-,$$

set

$$(y_1^{min}, y_2^{min}) = \begin{cases} (u_-, y_2^-), & \Phi_1(u_-, y_2^-) \le \Phi_1(y_1^-, u_-), \\ (y_1^-, u_-), & \Phi_1(u_-, y_2^-) \ge \Phi_1(y_1^-, u_-), \end{cases}$$

6) $y_1^* > u_+, \;\; y_2^* > u_+ \;\; \Rightarrow$ solve two one-dimensional problems

$$\Phi_1(u_+, y_2) \to min, \quad y_2 \in [u_-, u_+] \;\; \Rightarrow \;\; y_2^+,$$

$$\Phi_1(y_1, u_+) \to min, \quad y_1 \in [u_-, u_+] \;\; \Rightarrow \;\; y_1^+,$$

set

$$(y_1^{min}, y_2^{min}) = \begin{cases} (u_+, y_2^+), & \Phi_1(u_+, y_2^+) \le \Phi_1(y_1^+, u_+), \\ (y_1^+, u_+), & \Phi_1(u_+, y_2^+) \ge \Phi_1(y_1^+, u_+), \end{cases}$$

7) $y_1^* < u_-, \;\; y_2^* > u_+ \;\; \Rightarrow$ solve two one-dimensional problems

$$\Phi_1(u_-, y_2) \to min, \quad y_2 \in [u_-, u_+] \;\; \Rightarrow \;\; y_2^-,$$

$$\Phi_1(y_1, u_+) \to min, \quad y_1 \in [u_-, u_+] \;\; \Rightarrow \;\; y_1^+,$$

set

$$(y_1^{min}, y_2^{min}) = \begin{cases} (u_-, y_2^-), & \Phi_1(u_-, y_2^-) \le \Phi_1(y_1^+, u_+), \\ (y_1^+, u_+), & \Phi_1(u_-, y_2^-) \ge \Phi_1(y_1^+, u_+), \end{cases}$$

8) $y_1^* > u_+, \;\; y_2^* < u_- \;\; \Rightarrow$ solve two one-dimensional problems

$$\Phi_1(u_+, y_2) \to min, \quad y_2 \in [u_-, u_+] \;\; \Rightarrow \;\; y_2^+,$$

$$\Phi_1(y_1, u_-) \to min, \quad y_1 \in [u_-, u_+] \;\; \Rightarrow \;\; y_1^-,$$

set

$$(y_1^{min}, y_2^{min}) = \begin{cases} (u_+, y_2^+), & \Phi_1(u_+, y_2^+) \le \Phi_1(y_1^-, u_-), \\ (y_1^-, u_-), & \Phi_1(u_+, y_2^+) \ge \Phi_1(y_1^-, u_-), \end{cases}$$

This procedure provides an exact solution of a two-dimensional minimum problem avoiding the use of iterative algorithms.

Remark 1. The finite-dimensional problem after the second parameterization is solved similarly. In this case, the two-dimensional problem corresponds to the case when $m = 0$ with the variables $z_0, \; z_1$.

4. A maximum norm problem for the terminal state

Consider a norm maximization problem typical in nonconvex optimization. After the first parameterization, the problem can be stated as follows:

$$\varphi(y) \to max, \quad y \in Y. \tag{4.1}$$

Here,

$$\varphi(y) = \langle d, y\rangle + \frac{1}{2}\langle Xy, Xy\rangle,$$

$$Y = \{y = (y_1, ..., y_m) : \ y_i \in [u_-, u_+], \ \ i = \overline{1, m}\}.$$

Suppose that the matrix $X \in R^{n\times m}$ has maximal rank, i.e. $\varphi(y)$ is a strictly convex function.

Consider the set of corner points of the hypercube Y (2^m vertices)

$$Y_* = \{y \in Y : \ y_i = u_- \vee u_+, \ \ i = \overline{1, m}\}.$$

We will use the following known result: if y^0 is a global solution of the problem(4.1), then $y^0 \in Y_*$.

State the problem on the set of corner points

$$\varphi(y) \to max, \quad y \in Y_*. \tag{4.2}$$

According to the previous statement, the problems (4.1) and (4.2) are equaivalent.

The resolution of the problem (4.2) in the simplest case is implemented by the complete exhaustive search of 2^m corner points and corresponding values of the function φ. If this procedure is not admissible in view of computational costs, then one can apply an exhaustive search of corner points with a monotone increase of the values of $\varphi(y)$. The conditional gradient method on the set Y_* is particulary suitable in this case. We describe now an iteration of the method.

Let $y^k \in Y_*$. Then,

$$y^{k+1} = \underset{y\in Y_*}{argmax}\langle \nabla\varphi(y^k), y\rangle \quad \Leftrightarrow$$

$$y_i^{k+1} = \underset{y_i=u_-\vee u_+}{argmax}\, \langle \nabla_i\varphi(y^k), y_i\rangle, \quad i = \overline{1, m}.$$

Hence, for $i = \overline{1, m}$

$$y_i^{k+1} = \begin{cases} u_-, & \nabla_i\varphi(y^k) < 0, \\ u_+, & \nabla_i\varphi(y^k) > 0, \\ u_- \vee u_+, & \nabla_i\varphi(y^k) = 0. \end{cases}$$

It is clear that

$$y^{k+1} \in Y_*, \quad \langle \nabla\varphi(y^k), y^{k+1} - y^k\rangle \geq 0.$$

If $y^{k+1} \neq y^k$, then the function φ is strictly increasing:

$$\varphi(y^{k+1}) - \varphi(y^k) > \langle \nabla\varphi(y^k), y^{k+1} - y^k \rangle.$$

Let $\nabla_i\varphi(y^k) = 0$ for some index $i \in \{1, ..., m\}$. Then, $y_i^{k+1} = u_- \vee u_+$, i.e. one can always guarantee the condition $y_i^{k+1} \neq y_i^k$, which implies the increasing property: $\varphi(y^{k+1}) > \varphi(y^k)$.

Therefore, the stopping condition of the iterative search is the equality $y^{k+1} = y^k$, which implies that $\nabla_i\varphi(y^k) \neq 0,\ i = \overline{1, m}$.

We indicate one version for testing a point y^k for optimality in the case when the improvement method stops: $y^{k+1} = y^k$. Check if the values of the function φ at the corner points $y^{k,j},\ j = \overline{1, m}$ adjacent to y^k have increased as compared to $\varphi(y^k)$.

If at some point the function φ has increased, then the corresponding vertex $y^{k,j}$ is selected as the initial approximation for the next iteration cycle of the conditional gradient method. Otherwise, the vertex y^k ia a local maximum of the problem (4.2).

We note that the adjacent corner point $y^{k,j}$ is obtained from y^k by switching j-th coordinate y_j^k between two values $u_-,\ u_+$: if $y_j^k = u_-(u_+)$, then $y_j^{k,j} = u_+(u_-)$.

Remark 2. The maximum norm problem in the framework of second parametrization with the vector of variables $z =$
$= (z_0, ..., z_{m+1})$ and the objective function

$$\Phi_2(z) = \Phi(0) + \langle \hat{d}, z \rangle + \frac{1}{2}\langle \hat{X}z, \hat{X}z \rangle$$

is solved similarly.

5. Illustrative problems

Example 1. Optimal energy control of the harmonic oscillator [3]:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = -1, \ \ x_2(0) = 1;$$

$$|u(t)| \leq 1, \quad t \in [0, \pi];$$

$$\Phi(u) = \frac{1}{2}(x_1^2(\pi) + x_2^2(\pi)).$$

We give formulas for solving the phase system

$$x_1(t, u) = sint - cost + \int_0^t sin(t - \tau)u(\tau)d\tau,$$

$$x_2(t,u) = sint + cost + \int_0^t cos(t-\tau)u(\tau)d\tau.$$

Implement the first parameterization for $m = 2,\ y = (y_1, y_2),\ t_1 = \frac{\pi}{2}$. After simple calculations we obtain

$$\Phi(0) = 1, \quad X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

$$\Phi_1(y) = 1 + 2y_1 + y_1^2 + y_2^2.$$

The variables $y_1,\ y_2$ in the expression for $\Phi_1(y)$ are separated, i.e. the problems $\Phi_1(y) \to x,\ |y| \le 1$ are solved elementary.

The first problem $\Phi_1(y) \to min,\ |y| \le 1$ has a unique solution $y_1^{min} = -1,\ y_2^{min} = 0$, and $\Phi_1(y^{min}) = 0$. Hence, the control

$$u_{min}(t) = \begin{cases} -1, & t \in [0, \frac{\pi}{2}], \\ 0, & t \in (\frac{\pi}{2}, \pi]. \end{cases}$$

is optimal for the original problem

$$\Phi(u) \to min, \quad u \in V.$$

The second problem $\Phi_1(y) \to max,\ |y| \le 1$ has two solutions $y_1^{max} = = 1,\ y_2^{max} = \pm 1$, and $\Phi_1(y^{max}) = 5$. The corresponding controls are

$$u_{max}(t) = 1, \quad t \in [0, \pi],$$

$$u_{max}(t) = \begin{cases} 1, & t \in [0, \frac{\pi}{2}], \\ -1, & t \in (\frac{\pi}{2}, \pi]. \end{cases}$$

It is worth noting that the optimal control in the problem $\Phi(u) \to max, \quad u \in V$

$$u_{opt}(t) = \begin{cases} 1, & t \in [0, \frac{3\pi}{4}], \\ -1, & t \in (\frac{3\pi}{4}, \pi], \end{cases}$$

with the values $\Phi(u_{opt}) = 3 + 2\sqrt{2} \approx 5,82$ is known.

This control is derived in [2; 8] as a result of a non-trivial numerical implementation based on the global maximum condition. The controls $u_{max}(t)$ constructed analytically have simple structure and provide a good approximation to the optimal value of the functional (with the deviation 14%). In this case, the exact optimal control is obtained from the parameterization procedure for $m = 4$ by the exhaustive search of 2^4 corner points.

Example 2. The extremum norm problem for the terminal state for the two-stage system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x_1(0) = x_1^0, \ \ x_2(0) = x_2^0;$$

$$|u(t)| \le 1, \quad t \in [0, T];$$

$$\Phi(u) = \frac{1}{2}(x_1^2(T) + x_2^2(T)).$$

The terminal state of a phase trajectory $x(t, u)$ is given by the formulas

$$x_1(T, u) = x_1^0 + x_2^0 T + \int_0^T (T - t)u(t)dt,$$

$$x_2(T, u) = x_2^0 + \int_0^T u(t)dt.$$

For approximate resolution of the problem $\Phi(u) \to min$ with the numerical data $T = 2,\ x_1^0 = 2,\ x_2^0 = -1$ we apply both parameterization procedures.

Implement the first parametrization for $m = 2$. In this case,

$$\Phi(0) = \frac{1}{2}, \quad X = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$\Phi_1(y) = \Phi(0) + \langle d, y\rangle + \frac{1}{2}\langle Xy, Xy\rangle.$$

The stationary point of the paraboloid $\Phi_1(y)$ is defined from the linear system

$$d + X^T X y = 0,$$

which takes the form

$$13y_1 + 7y_2 = 4, \quad 7y_1 + 5y_2 = 4$$

with the solution $y_1^* = -\frac{1}{2},\ y_2^* = \frac{3}{2}$.

The point $y_2^* = \frac{3}{2}$ violates the constraint $|y_2| \le 1$. Hence, in accordance with the resolution scheme from Section 3 we assume that $y_2^+ = 1$ and solve the first equation

$$13y_1 + 7y_2^+ = 4 \quad \Rightarrow \quad y_1^+ = -\frac{3}{13}.$$

Consequently, the minimum point is

$$y_1^{min} = -\frac{3}{13}, \quad y_2^{min} = 1,$$

with the corresponding control

$$u(t, y^{min}) = \begin{cases} -\frac{3}{13}, & t \in [0, 1], \\ 1, & t \in (1, 2], \end{cases}$$

and the value of the functional $\Phi_1(y^{min}) = \frac{1}{26}$.

Next, consider the second parametrization in the class of linear controls when $m = 0$.

In this case,

$$z = (z_0, z_1), \quad u(t, z) = z_0\varphi_0(t) + z_1\varphi_1(t), \quad t \in [0, 2],$$

$$\varphi_0(t) = \frac{1}{2}(2 - t), \quad \varphi_1(t) = \frac{1}{2}t.$$

In the corresponding formula for the functional

$$\Phi_2(z) = \Phi(0) + \langle \hat{d}, z\rangle + \frac{1}{2}\langle \hat{X}z, \hat{X}z\rangle$$

we have

$$\hat{d} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ 1 & 1 \end{pmatrix}.$$

Therefore, we arrive at the linear system for determining the stationary point

$$25z_0 + 17z_1 = 9, \quad 17z_0 + 13z_1 = 9.$$

The solution: $z_0^* = -1, \ z_1^* = 2$. The point $z_1^* = 2$ violates the constraint $|z_1| \leq 1$. Hence, we assume that $z_1^+ = 1$ and solve the first equation:

$$25z_0 + 17z_1^+ = 9 \quad \Rightarrow \quad z_0^+ = -\frac{8}{25}.$$

Finally, we obtain the minimum point

$$z_0^{min} = -\frac{8}{25}, \quad z_1^{min} = 1,$$

with the corresponding control

$$u(t, z^{min}) = \frac{33}{50}t - \frac{8}{25}, \quad t \in [0, 2]$$

and the value of the functional $\Phi_2(z^{min}) = \frac{2}{25}$.

Comparing two parametrizations by the values of the functional, we conclude that the first parametrization is better than the second one:

$$\Phi_1(y^{min}) = \frac{1}{26} < \Phi_2(z^{min}) = \frac{2}{25}.$$

For the maximum norm problem ($\Phi(u) \to max, \ u \in V$) the optimal control is known [8]:

$$u_{opt}(t) = -1, \quad t \in [0, 2].$$

In the framework of the considered parameterizations, it is obtained by the exhaustive search over four vertices of the square which leads to the optimal result:

$$y_1^{max} = y_2^{max} = -1, \quad z_0^{max} = z_1^{max} = -1.$$

6. Conclusion

Extremum norm problems for the terminal state of a linear system are considered over the set of piecewise constant or piecewise linear controls on a given grid of approximation nodes. The corresponding finite-dimensional minimum problem is convex and is efficiently resolved by standard quadratic programming algorithms. The specific character of the nonconvex maximum problem allows to obtain a global solution by means of finite exhaustive search over the collection of corner points of an admissible set.

References

1. Arguchintsev A.V., Dykhta V. A., Srochko V. A. Optimal control: nonlocal conditions, computational methods, and the variational principle of maximum. *Russian Mathematics*, 2009, vol. 53, no. 1, pp. 1–35.
2. Antonik V. G., Srochko V. A. Method for nonlocal improvement of extreme controls in the maximization of the terminal state norm. *Computational Mathematics and Mathematical Physics*, 2009, vol. 49, no. 5, pp. 762–775.
3. Galyaev A. A., Lysenko P. V. Energy-optimal control of harmonic oscillator. *Automation and Remote Control*, 2019, vol. 80, pp. 16–29. htts://doi.org/10.1134/S0005231019010021
4. Gorbunov V. K. The reduction of optimal control problems for finite-dimensional. *Computational Mathematics and Mathematical Physics*, 1978, vol. 18, no. 5, pp. 1083–1095.
5. Srochko V.A. *Iterative methods for solving optimal control problems.* Moscow, Fizmatlit Publ., 2000. 160 p. (in Russian)
6. Srochko V.A., Aksenyushkina E.V. Parameterization of Some Control Problems by Linear Systems. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 30, pp. 83–98. htts://doi.org/10.26516/1997-7670.2019.30.83 (in Russian)
7. Strekalovsky A. S. *Elements of nonconvex optimization.* Novosibirsk, Nauka Publ., 2003. 356 p. (in Russian)
8. Strekalovsky A. S., Sharankhaeva E. V. Global search in a nonconvex optimal control problem. *Computational Mathematics and Mathematical Physics*, 2005, vol. 45, no. 10, pp. 1719–1734.
9. Sukharev A. G., Timokhov A. V., Fedorov V. V *Course of optimization methods.* Moscow, Nauka Publ., 1986. 328 p. (in Russian)
10. Tyatyushkin A. I. *Multi-method technology for optimization of control systems.* Novosibirsk, Nauka Publ., 2006. 343 p. (in Russian)
11. Chernov A.V. On application of Gaussian functions for numerical solution of optimal control problems. *Automation and Remote Control*, 2019, vol. 80, pp. 1026–1040. htts://doi.org/10.1134/S0005231019060035

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Received 26.10.20

К решению задач на экстремум нормы конечного состояния линейной системы

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Аннотация. Задачи на экстремум нормы конечного состояния линейной динамической системы изучаются с позиций методов параметризации допустимых управлений. Аппроксимация кусочно-непрерывных управлений проводится в классе кусочно-линейных функций на равномерной сетке узлов отрезка времени и оформляется как линейная комбинация специального набора опорных функций. При этом интервальное ограничение на управление в исходной задаче переходит в аналогичные ограничения на переменные конечномерных задач.

Конечномерный вариант задачи на минимум нормы допускает эффективное решение с помощью современных программ выпуклой оптимизации. Для случая двух переменных предлагается аналитический метод решения, использующий одномерную задачу минимизации параболы на отрезке.

Для невыпуклой задачи максимизации нормы конечномерная версия решается в глобальном смысле на основе перебора вершин гиперкуба. Предлагаемый подход открывает дополнительные возможности глобального решения невыпуклых задач оптимального управления.

Проводится апробация представленной технологии решения на иллюстративных задачах.

Ключевые слова: линейная система управления, задачи на экстремум нормы конечного состояния, кусочно-линейная аппроксимация, конечномерные задачи.

Список литературы

1. Аргучинцев А. В., Дыхта В. А., Срочко В. А. Оптимальное управление: нелокальные условия, вычислительные методы и вариационный принцип максимума // Известия вузов. Математика. 2009. № 1. С. 3–43.
2. Антоник В. Г., Срочко В. А. Методы нелокального улучшения экстремальных управлений в задаче на максимум нормы конечного состояния // Журнал вычислительной математики и математической физики. 2009. Т. 49, № 5. С. 791–804.
3. Галяев А. А., Лысенко П. В. Оптимальное по энергии управление гармоническим осциллятором // Автоматика и телемеханика. 2019. № 1. С. 21–37. https://doi.org/10.1134/S0005231019010021

4. Горбунов В. К. О сведении задач оптимального управления к конечномерным // Журнал вычислительной математики и математической физики. 1978. Т. 18, № 5. С. 1083–1095.
5. Срочко В. А. Итерационные методы решения задач оптимального управления. М. : Физматлит, 2000. 160 с.
6. Срочко В. А., Аксенюшкина Е. В. Параметризация некоторых задач управления линейными системами // Известия Иркутского государственного университета. Серия Математика. 2019. Т. 30. С. 83–98. https://doi.org/10.26516/1997-7670.2019.30.83
7. Стрекаловский А. С. Элементы невыпуклой оптимизации. Новосибирск : Наука, 2003. 356 с.
8. Стрекаловский А. С., Шаранхаева Е. В. Глобальный поиск в невыпуклой задаче оптимального управления // Журнал вычислительной математики и математической физики. 2005. Т. 45, № 10. С. 1785–1800.
9. Сухарев А. Г., Тимохов А. В., Федоров В. В. Курс методов оптимизации. М. : Наука, 1986. 328 с.
10. Тятюшкин А. И. Многометодная технология оптимизации управляемых систем. Новосибирск : Наука, 2006. 343 с.
11. Чернов А. В. О применении функций Гаусса для численного решения задач оптимального управления // Автоматика и телемеханика. 2019. № 6. С. 51–69. https://doi.org/10.1134/S0005231019060035

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Поступила в редакцию 26.10.2020