Серия «Математика»
2020. Т. 33. С. 80—95

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

УДК 510.67
MSC 03C30, 03C15, 03C50
DOI https://doi.org/10.26516/1997-7670.2020.33.80

Hierarchy of Families of Theories and Their Rank Characteristics *

S. V. Sudoplatov[123]

[1] *Sobolev Institute of Mathematics, Novosibirsk, Russian Federation*
[2] *Novosibirsk State Technical University, Novosibirsk, Russian Federation*
[3] *Novosibirsk State University, Novosibirsk, Russian Federation*

Abstract. Studying families of elementary theories produces an information on behavior and interactions of theories inside families, possibilities of generations and their complexity. The complexity is expressed by rank characteristics both for families and their elements inside families.

We introduce and describe a hierarchy of families of theories and their rank characteristics including dynamics of ranks. We consider regular families which based on a family of urelements — theories in a given language, and on a step-by-step process producing the required hierarchy. An ordinal-valued set-theoretic rank is used to reflect steps of this process. We introduce the rank RS and related ranks for regular families, with respect to sentence-definable subfamilies and generalizing the known RS-rank for families of urelements, as well as their degrees. Links and dynamics for these ranks and degrees are described on a base of separability of sets of urelements. Graphs and families of neighbourhoods witnessing ranks are introduced and characterized. It is shown that decompositions of families of neighbourhoods and their rank links, for discrete partitions, produce the additivity and the possibility to reduce complexity measures for families into simpler subfamilies.

Keywords: family of theories, closure, urelement, hierarchy, rank, decomposition.

* The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project No. 0314-2019-0002), and Committee of Science in Education and Science Ministry of the Republic of Kazakhstan (Grant No. AP05132546).

1. Introduction

We continue to study families of theories [10–14] connected with their ranks [6–9; 15] describing a hierarchy of families of theories and their rank characteristics.

We introduce and describe a hierarchy of families of theories and their rank characteristics including dynamics of ranks. Preliminary notions and notations are represented in Section 2. In Section 3, we consider regular families based on a family of urelements — theories in a given language, and on a step-by-step process producing the required hierarchy. An ordinal set-theoretic rank is used to reflect steps of this process. We introduce ranks $\mathrm{RS}^{\forall}$, $\mathrm{RS}^{\exists}$, RS with respect to sentence-definable subfamilies and generalizing the known RS-rank [15] for families of urelements, as well as their degrees. Links and dynamics for these ranks and degrees are described on a base of separability of sets of urelements. In Section 4, graphs and families of neighbourhoods witnessing ranks are introduced and characterized. It is shown that decompositions of families of neighbourhoods and their rank links, for discrete partitions, produce the additivity and the possibility to reduce complexity measures for families into simpler subfamilies.

Throughout the paper we consider complete first-order theories T in relational languages $\Sigma(T)$ and use the terminology in [11–15].

2. Preliminaries

Definition [12]. Let $\overline{\mathcal{T}}_{\Sigma}$ be the set of all complete elementary theories of a relational language Σ. For a set $\mathcal{T}\subset\overline{\mathcal{T}}_{\Sigma}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}'\equiv\mathcal{A}_E$, $\mathcal{A}_E=\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i)\in\mathcal{T}$. As usual, if $\mathcal{T}=\mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T}\subset\overline{\mathcal{T}}$, where $\overline{\mathcal{T}}$ is the union of all $\overline{\mathcal{T}}_{\Sigma}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0\subseteq\mathcal{T}$, where new language symbols with respect to the theories in $\mathcal{T}_0$ are empty.

For a set $\mathcal{T}\subset\overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi)\subseteq\Sigma$ we denote by $\mathcal{T}_{\varphi}$ the set $\{T\in\mathcal{T}\mid\varphi\in T\}$. The set $\mathcal{T}_{\varphi}$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_{\varphi}$ is also called (*formula-* or *sentence-*)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence-*)*$\mathcal{T}$-definable*, or simply *s-definable*.

Proposition 2.1 [12]. *If $\mathcal{T}\subset\overline{\mathcal{T}}$ is an infinite set and $T\in\overline{\mathcal{T}}\setminus\mathcal{T}$ then $T\in\mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any sentence $\varphi\in T$ the set $\mathcal{T}_{\varphi}$ is infinite.*

If T is an accumulation point for $\mathcal{T}$ then we also say that T is an *accumulation point* for $\mathrm{Cl}_E(\mathcal{T})$.

Definition [6]. Let $\mathcal{T}$ be a family of first-order complete theories in a language Σ. For a set Φ of Σ-sentences we put $\mathcal{T}_\Phi = \{T \in \mathcal{T} \mid \Phi \subseteq T\}$. A family of the form $\mathcal{T}_\Phi$ is called *d-definable* (in $\mathcal{T}$). If Φ is a singleton $\{\varphi\}$ then $\mathcal{T}_\varphi = \mathcal{T}_\Phi$ is called *s-definable* as above.

Theorem 2.2 [6]. *A subfamily $\mathcal{T}' \subseteq \mathcal{T}$ is d-definable in $\mathcal{T}$ if and only if $\mathcal{T}'$ is E-closed in $\mathcal{T}$, i.e., $\mathcal{T}' = \mathrm{Cl}_E(\mathcal{T}') \cap \mathcal{T}$.*

Definition [6]. A d-definable set $\mathcal{T}_\Phi$ is called $\mathcal{T}$*-consistent* if $\mathcal{T}_\Phi \neq \emptyset$, and $\mathcal{T}_\Phi$ is called *locally* $\mathcal{T}$*-consistent* if for any finite $\Phi_0 \subseteq \Phi$, $\mathcal{T}_{\Phi_0}$ is $\mathcal{T}$-consistent.

Theorem 2.3 (Compactness) [6]. *For any E-closed family $\mathcal{T}$, every locally $\mathcal{T}$-consistent d-definable set $\mathcal{T}_\Phi$ is $\mathcal{T}$-consistent.*

3. Hierarchy of families and their ranks

Let Σ be a language and $\mathcal{T}_\Sigma$ be the family of all complete theories in the language Σ. We consider both an approach for the construction of hereditarily finite sets [1] with urelements in $\mathcal{T}_\Sigma$ and, more generally, of sets in $V_{\alpha,\Sigma}$, where for ordinals α the sets $V_{\alpha,\Sigma}$ are defined by the following regular process (cf. [3, Section 2.6]):

a) $V_{0,\Sigma} = \mathcal{T}_\Sigma$;

b) $V_{\alpha,\Sigma} = \mathcal{P}\left(\bigcup\limits_{\gamma\leq\beta} V_{\gamma,\Sigma}\right)$, if $\alpha = \beta + 1$;

c) $V_{\alpha,\Sigma} = \bigcup\limits_{\beta<\alpha} V_{\beta,\Sigma}$, if α is a limit ordinal.

A set $\mathcal{T}$ with $\mathcal{T} \in V_{\alpha,\Sigma} \setminus \mathcal{T}_\Sigma$ for some ordinal α is called *regular*. Each regular set $\mathcal{T}$ has an ordinal $\rho(\mathcal{T})$ which is called the *rank* of $\mathcal{T}$ and is defined as the least ordinal with $\mathcal{T} \in V_{\rho(\mathcal{T}),\Sigma}$.

Clearly, $\rho(\mathcal{T}) \geq 1$ for any regular $\mathcal{T}$, and if $\mathcal{T} \in \mathcal{T}'$, for regular $\mathcal{T}'$, then $\rho(\mathcal{T}) < \rho(\mathcal{T}')$. Besides, if all elements in a set $\mathcal{T}'$ are regular then $\mathcal{T}'$ is regular, too, with $\rho(\mathcal{T}') = \bigcup\{\rho(\mathcal{T}) \mid \mathcal{T} \in \mathcal{T}'\}$ or $\rho(\mathcal{T}') = (\bigcup\{\rho(\mathcal{T}) \mid \mathcal{T} \in \mathcal{T}'\}) + 1$ depending on limit values for $\rho(\mathcal{T})$ with $\mathcal{T} \in \mathcal{T}'$.

For any regular family $\mathcal{T}$ we denote by $\mathrm{ur}(\mathcal{T})$ the set of all urelements in $\mathcal{T}_\Sigma$ which used for the construction of $\mathcal{T}$:

1) if $\rho(\mathcal{T}) = 1$ then $\mathcal{T} \subset \mathcal{T}_\Sigma$ and $\mathrm{ur}(\mathcal{T}) = \mathcal{T}$;

2) if $\rho(\mathcal{T}) = \alpha > 1$ then $\mathrm{ur}(\mathcal{T}) = \bigcup\limits_{\mathcal{T}'\in\mathcal{T}} \mathrm{ur}(\mathcal{T}')$.

Replacing $\mathrm{ur}(\mathcal{T})$ by a regular family $\mathcal{T}'$ we can define the following process for $V_{\alpha,\mathcal{T}'}$ instead of $V_{\alpha,\Sigma}$ in the following way:

a) $V_{0,\mathcal{T}'} = \mathcal{T}'$;

b) $V_{\alpha,\mathcal{T}'} = \mathcal{P}\left(\bigcup\limits_{\gamma\leq\beta} V_{\gamma,\mathcal{T}'}\right)$, if $\alpha = \beta + 1$;

c) $V_{\alpha,\mathcal{T}'} = \bigcup\limits_{\beta<\alpha} V_{\beta,\mathcal{T}'}$, if α is a limit ordinal.

A set $\mathcal{T}$ with $\mathcal{T} \in V_{\alpha,\mathcal{T}'} \setminus \mathcal{T}'$ for some ordinal α is called *regular* with respect to $\mathcal{T}'$. Clearly, each regular $\mathcal{T}$ with respect to $\mathcal{T}'$ is regular in the previous sense.

Each regular set $\mathcal{T}$, with respect to $\mathcal{T}'$, has an ordinal $\rho_{\mathcal{T}'}(\mathcal{T})$ which is called the *rank* of $\mathcal{T}$ with respect to $\mathcal{T}'$ and it is defined as the least ordinal with $\mathcal{T} \in V_{\rho_{\mathcal{T}'}(\mathcal{T}),\mathcal{T}'}$ or $\mathcal{T} = V_{\rho_{\mathcal{T}'}(\mathcal{T}),\mathcal{T}'}$.

Clearly, $\rho(\mathcal{T}) \geq 1$ for any regular $\mathcal{T}$ with respect to $\mathcal{T}'$, and if $\mathcal{T}_1 \in \mathcal{T}_2$, for regular sets $\mathcal{T}_1$, $\mathcal{T}_2$ with respect to $\mathcal{T}'$, then $\rho_{\mathcal{T}'}(\mathcal{T}_1) < \rho_{\mathcal{T}'}(\mathcal{T}_2)$. Besides, if all elements in $\mathcal{T}''$ are regular with respect to $\mathcal{T}'$, then $\mathcal{T}''$ is regular too, with $\rho_{\mathcal{T}'}(\mathcal{T}'') = \bigcup\{\rho_{\mathcal{T}'}(\mathcal{T}) \mid \mathcal{T} \in \mathcal{T}''\}$ or $\rho_{\mathcal{T}'}(\mathcal{T}'') = (\bigcup\{\rho_{\mathcal{T}'}(\mathcal{T}) \mid \mathcal{T} \in \mathcal{T}''\}) + 1$ depending on limit values for $\rho_{\mathcal{T}'}(\mathcal{T})$ with $\mathcal{T} \in \mathcal{T}''$.

Example 3.1. Recall [5] that a pair (X, Y) is called a *hypergraph* if $Y \subseteq \mathcal{P}(X)$. If X is a set of complete theories then Y consists of subsets of X implying $\rho(y) = 1$ for $y \in Y$. Thus, $\rho(Y) \leq 2$, $\rho_X(Y) \leq 2$, and $\rho_X(Y) = 2$ if and only if $Y \neq \emptyset$.

More generally, if $X = \mathcal{T}$ is a regular family, (X, Y) is a hypergraph, then $\rho(Y) \leq \rho(X) + 2$, with the equality $\rho(Y) = \rho(X) + 2$ if Y contains an element y with $\rho(y) = \rho(X) + 1$.

Below for simplicity we consider constructions based on $\mathrm{ur}(\mathcal{T})$ and $\rho(\mathcal{T})$ although these constructions can be generalized for regular families $\mathcal{T}'$ and ranks $\rho_{\mathcal{T}'}(\mathcal{T})$.

For a regular family $\mathcal{T}$ with a set $\mathrm{ur}(\mathcal{T})$ and a permutation f on $\mathcal{T}_\Sigma$ we denote by $f(\mathcal{T})$ the result of simultaneous replacements of $T \in \mathrm{ur}(\mathcal{T})$ by $f(T)$. The family $f(\mathcal{T})$ is called the *f-copy* or simply the *copy* of $\mathcal{T}$.

Clearly, $\rho(\mathcal{T}) = \rho(f(\mathcal{T}))$ for any permutation f on $\mathcal{T}_\Sigma$. Besides, $f(\mathcal{T}) = \mathcal{T}$ if and only if the restriction of f till $\mathrm{ur}(\mathcal{T})$ is a bijection permutating elements in $\mathcal{T}$.

Let $\mathcal{T}$ be a regular family, φ be a Σ-sentence. We denote by $\mathcal{T}^{\forall}_{\varphi}$ the *s-definable* subfamily of $\mathcal{T}$ consisting of all $\mathcal{T}' \in \mathcal{T}$ whose all urelements contain φ:

$$\mathcal{T}^{\forall}_{\varphi} = \{\mathcal{T}' \in \mathcal{T} \mid \varphi \in T \text{ for each } T \in \mathrm{ur}(\{\mathcal{T}'\})\}.$$

Similarly we denote by $\mathcal{T}^{\exists}_{\varphi}$ the *s-definable* subfamily of $\mathcal{T}$ consisting of all $\mathcal{T}' \in \mathcal{T}$ whose some urelements contain φ:

$$\mathcal{T}^{\exists}_{\varphi} = \{\mathcal{T}' \in \mathcal{T} \mid \varphi \in T \text{ for some } T \in \mathrm{ur}(\{\mathcal{T}'\})\}.$$

Similarly sets of urelements, the sets $\mathcal{T}^{\forall}_{\varphi}$ and $\mathcal{T}^{\exists}_{\varphi}$ are called (φ-)*neighbourhoods.*

Clearly, if $\mathcal{T}$ consists of urelements then for any sentence φ,

$$\mathcal{T}_\varphi^\forall = \mathcal{T}_\varphi^\exists = \mathcal{T}_\varphi$$

whereas in general case the following inclusion holds:

$$\mathcal{T}_\varphi^\forall \subseteq \mathcal{T}_\varphi^\exists, \tag{3.1}$$

possibly strict, if some but not all urelements in $\mathrm{ur}(\mathcal{T})$ contain the sentence φ.

If

$$\mathcal{T}_\varphi^\forall = \mathcal{T}_\varphi^\exists \tag{3.2}$$

the neighbourhoods $\mathcal{T}_\varphi^\forall$ and $\mathcal{T}_\varphi^\exists$ are denoted by $\mathcal{T}_\varphi$ as well.

Remark 3.2. By the definition, equality (3.2) holds, i.e., $\mathcal{T}_\varphi$ exists, if and only if for any $\mathcal{T}' \in \mathcal{T}$, $(\mathrm{ur}(\mathcal{T}'))_\varphi = \emptyset$ or $(\mathrm{ur}(\mathcal{T}'))_\varphi = \mathrm{ur}(\mathcal{T}')$. In particular, as above, $\mathcal{T}_\varphi$ exists if $\mathcal{T}$ consists of urelements.

Proposition 3.3. *The equality $\mathcal{T}_\varphi^\forall = \mathcal{T}_\varphi^\exists$ holds for a regular family $\mathcal{T}$ and any sentence $\varphi \in F(\Sigma)$ if and only if $|\mathrm{ur}(\mathcal{T}')| \leq 1$ for any $\mathcal{T}' \in \mathcal{T}$.*

Proof. Let the equality (3.2) holds for the family $\mathcal{T}$ and any sentence $\varphi \in F(\Sigma)$. Suppose that $\mathrm{ur}(\mathcal{T}')$ contains two distinct theories T_1, T_2 for some $\mathcal{T}' \in \mathcal{T}$. Taking a sentence φ with $\varphi \in T_1$ and $\neg\varphi \in T_2$ we obtain $\mathcal{T}' \in \mathcal{T}_\varphi^\exists$, witnessed by T_1, and $\mathcal{T}' \notin \mathcal{T}_\varphi^\forall$, witnessed by T_2. Therefore $\mathcal{T}_\varphi^\forall \subsetneqq \mathcal{T}_\varphi^\exists$ contradicting the equality (3.2).

Conversely, if $\mathrm{ur}(\mathcal{T}')$ are at most singletons for any $\mathcal{T}' \in \mathcal{T}$ then for any sentence $\varphi \in F(\Sigma)$ we have $\varphi \in T$ for some $T \in \mathrm{ur}(\mathcal{T}')$ if and only if $\varphi \in T$ for all $T \in \mathrm{ur}(\mathcal{T}')$, implying $\mathcal{T}_\varphi^\forall = \mathcal{T}_\varphi^\exists$. □

Definition. A regular family $\mathcal{T}$ is called *normal* if $\mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}')) = \mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}''))$ for any $\mathcal{T}', \mathcal{T}'' \in \mathcal{T}$ with $\mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}')) \cap \mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}'')) \neq \emptyset$, i.e., $\mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}'))$ and $\mathrm{Cl}_E(\mathrm{ur}(\mathcal{T}''))$ are disjoint or equal.

By the definition any regular family $\mathcal{T}$ consisting of copies of a family $\mathcal{T}'$ is normal. Besides, copies of families with disjoint E-closures form normal families, too.

Since finite sets of urelements are E-closed, Remark 3.2 and Proposition 3.3 immediately imply:

Corollary 3.4. *If a regular family $\mathcal{T}$ consists of urelements and / or elements without urelements, in particular, if $\rho(\mathcal{T}) = 1$, then $\mathcal{T}$ is normal and satisfies the equality* (3.2) *for any sentence $\varphi \in F(\Sigma)$.*

Let $\mathcal{T}$ be a regular family, $* \in \{\forall, \exists\}$. Similarly to the RS-rank [15] we define the RS^*-ranks for $\mathcal{T}$ as follows.

For the family $\mathcal{T}$ with $\mathrm{ur}(\mathcal{T}) = \emptyset$ we put the ranks $\mathrm{RS}^*(\mathcal{T}) = -1$, for families $\mathcal{T}$ with $\mathrm{ur}(\mathcal{T}) \neq \emptyset$ we put $\mathrm{RS}^*(\mathcal{T}) \geq 0$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}^*(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent Σ-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}^*(\mathcal{T}^*_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}^*(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}^*(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}^*(\mathcal{T}) = \alpha$ if $\mathrm{RS}^*(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}^*(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}^*(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}^*(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e^*-totally transcendental* if $\mathrm{RS}^*(\mathcal{T})$ is an ordinal, where $* \in \{\forall, \exists\}$.

The RS^*-ranks produce measures of complexity for families $\mathcal{T}$ with respect to their partitions into s-definable subfamilies. Here, the set $\mathrm{ur}(\mathcal{T})$ can be complicated enough, with large $\mathrm{RS}(\mathrm{ur}(\mathcal{T}))$, although $\mathrm{RS}^*(\mathcal{T})$ can be small including the value 0, if $\mathcal{T}$ has few disjoint s-definable parts.

For instance, any family $\mathcal{T}$ with $\mathcal{T} = \{\mathrm{ur}(\mathcal{T})\}$ and $\mathrm{RS}(\mathrm{ur}(\mathcal{T})) \geq \alpha$, for an ordinal α, satisfies $\mathrm{RS}^*(\mathcal{T}) = 0$.

Remark 3.5. The inclusion (3.1) implies

$$\mathrm{RS}^{\forall}(\mathcal{T}) \leq \mathrm{RS}^{\exists}(\mathcal{T}) \tag{3.3}$$

for any family $\mathcal{T}$ preserving disjointness of pass from families $\mathcal{T}^{\forall}_{\varphi_n}$ to families $\mathcal{T}^{\exists}_{\varphi_n}$, where families $\mathcal{T}^{\forall}_{\varphi_n}$ witness the value of $\mathrm{RS}^{\forall}(\mathcal{T})$. Thus, in such a case if $\mathcal{T}$ is $e^{\exists}$-totally transcendental then $\mathcal{T}$ is $e^{\forall}$-totally transcendental.

At the same time the inequality (3.3) can fail if the families $\mathcal{T}^{\exists}_{\varphi_n}$ are not disjoint. Indeed, we can extend an arbitrary family $\mathcal{T}'$ by two-element sets $\{T^{m,n}_0, T^{m,n}_1\}$ such that for an extended family $\mathcal{T} \supset \mathcal{T}'$, the disjoint families $(\mathcal{T}')^{\forall}_{\varphi_m}$ and $(\mathcal{T}')^{\forall}_{\varphi_n}$ are preserved: $\mathcal{T}^{\forall}_{\varphi_m} = (\mathcal{T}')^{\forall}_{\varphi_m}$, $\mathcal{T}^{\forall}_{\varphi_n} = (\mathcal{T}')^{\forall}_{\varphi_n}$, and $(\mathcal{T}')^{\forall}_{\varphi_m}$, $(\mathcal{T}')^{\forall}_{\varphi_n}$ are properly extended by $\{T^{m,n}_0, T^{m,n}_1\}$ and $\{T^{n,m}_0, T^{n,m}_1\}$ till $\mathcal{T}^{\exists}_{\varphi_m}$ and $\mathcal{T}^{\exists}_{\varphi_n}$, respectively, with $\varphi_n \in T^{m,n}_0 \setminus T^{m,n}_1$ and $\varphi_m \in T^{n,m}_0 \setminus T^{n,m}_1$.

These extensions produce nonempty intersections for s-definable subfamilies $\mathcal{T}^{\exists}_{\varphi_n}$ implying $\mathrm{RS}^{\exists}(\mathcal{T}) = 0$.

If $\mathcal{T}$ is e^*-totally transcendental, with $\mathrm{RS}^*(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}^*(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}^*(\mathcal{T}^*_{\varphi_i}) = \alpha$, $* \in \{\forall, \exists\}$.

By inclusion (3.1), for a family $\mathcal{T}$ if $\mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T})$ then $\mathrm{ds}^{\forall}(\mathcal{T}) \leq \mathrm{ds}^{\exists}(\mathcal{T})$ again for any family $\mathcal{T}$ preserving disjointness of pass from families $\mathcal{T}^{\forall}_{\varphi_n}$ to families $\mathcal{T}^{\exists}_{\varphi_n}$. And by the arguments above the value $\mathrm{ds}^{\exists}(\mathcal{T})$ can decrease with respect to $\mathrm{ds}^{\forall}(\mathcal{T})$ till even $\mathrm{ds}^{\exists}(\mathcal{T}) = 1$.

If in the definition of RS^* and ds^* the families $\mathcal{T}^*_{\varphi_n}$ are replaced by $\mathcal{T}_{\varphi_n}$ we obtain the values $\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}(\mathcal{T})$ of the rank RS and the degree ds, respectively, as well as the notion of e-totally transcendence.

Clearly, $\mathrm{RS}(\mathcal{T}) \leq \mathrm{RS}^{\forall}(\mathcal{T})$ for any family $\mathcal{T}$, and if $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T})$ then $\mathrm{ds}(\mathcal{T}) \leq \mathrm{ds}^{\forall}(\mathcal{T})$. In particular, if $\mathcal{T}$ is $e^{\forall}$-totally transcendental then $\mathcal{T}$ is e-totally transcendental.

By the definition if the equality (3.2) holds for the family $\mathcal{T}$ and any sentence φ then $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T})$, and if $\mathcal{T}$ is e- or e^*-totally transcendental then $\mathcal{T}$ is e-, $e^{\forall}$-, and $e^{\exists}$-totally transcendental with $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}^{\forall}(\mathcal{T}) = \mathrm{ds}^{\exists}(\mathcal{T})$. In particular, in view of Remark 3.2, these equalities are satisfied for families $\mathcal{T}$ consisting of urelements.

Thus, we have the following:

Proposition 3.6. *If $\mathcal{T}$ is a regular family with $\rho(\mathcal{T}) = 1$ then $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T})$, and if $\mathcal{T}$ is e- or e^*-totally transcendental for some $* \in \{\forall, \exists\}$ and with $\mathrm{ur}(\mathcal{T}) \neq \emptyset$ then $\mathcal{T}$ is e-, $e^{\forall}$-, and $e^{\exists}$-totally transcendental with $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}^{\forall}(\mathcal{T}) = \mathrm{ds}^{\exists}(\mathcal{T})$.*

Having the inequalities $\mathrm{RS}(\mathcal{T}) \leq \mathrm{RS}^{\forall}(\mathcal{T}) \leq \mathrm{RS}^{\exists}(\mathcal{T})$, $\mathrm{ds}^{\forall}(\mathcal{T}) \leq \mathrm{ds}^{\exists}(\mathcal{T})$ for $\mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T})$, and $\mathrm{ds}(\mathcal{T}) \leq \mathrm{ds}^{\forall}(\mathcal{T})$ for $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T})$, if disjointness of s-definable subfamilies is preserved, we will show that the difference can be arbitrarily large.

Theorem 3.7. 1. *For any $\alpha, \beta, \gamma \in \mathrm{Ord} \cup \{\infty\}$ with $\alpha \leq \beta \leq \gamma$ there is a regular family $\mathcal{T}$ such that $\rho(\mathcal{T}) = 2$, $\mathrm{RS}(\mathcal{T}) = \alpha$, $\mathrm{RS}^{\forall}(\mathcal{T}) = \beta$, $\mathrm{RS}^{\exists}(\mathcal{T}) = \gamma$.*

2. *For any ordinal α and natural k, m, n with $0 < k \leq m \leq n$ there is a regular family $\mathcal{T}$ such that $\rho(\mathcal{T}) = 2$, $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T}) = \alpha$, $\mathrm{ds}(\mathcal{T}) = k$, $\mathrm{ds}^{\forall}(\mathcal{T}) = m$, $\mathrm{ds}^{\exists}(\mathcal{T}) = n$.*

Proof. 1. For the realization $\mathrm{RS}(\mathcal{T}) = \alpha$ we just use the arguments for the proof of [15, Proposition 3.11] forming theories in a family $\mathcal{T}_0$ by 0-ary predicates witnessing the required rank. Now we replace urelements by singletons obtaining a family $\mathcal{T}'_0$. In such a case we have $\rho(\mathcal{T}'_0) = 2$ and $(\mathcal{T}'_0)^{\forall}_{\varphi} = (\mathcal{T}'_0)^{\exists}_{\varphi} = (\mathcal{T}'_0)_{\varphi}$ for any sentence φ.

For the realization $\mathrm{RS}^{\forall}(\mathcal{T}) = \beta$ we extend the family $\mathcal{T}'_0$ by two-element families $\{T_0, T'_0\}$ of new theories in a language of 0-ary predicates such that similarly to $\mathcal{T}_0$ both or non of T and T' contain predicates witnessing the required rank $\mathrm{RS}^{\forall}(\mathcal{T}) \geq \mathrm{RS}(\mathcal{T})$. Now, in order to separate subfamilies $\mathcal{T}^{\forall}_{\varphi}$ and $\mathcal{T}^{\exists}_{\varphi}$, for sentences φ, witnessing the difference between $\mathrm{RS}^{\forall}(\mathcal{T})$ and $\mathrm{RS}(\mathcal{T})$, we extend the family of two-element sets $\{T_0, T'_0\}$ by two-element sets $\{T_1, T'_1\}$ such that $\varphi \in T_1$ and $\neg\varphi \in T'_1$. We denote the obtained family of singletons and two-element sets by $\mathcal{T}'_1$.

Finally, for the realization $\mathrm{RS}^{\exists}(\mathcal{T}) = \gamma$ we extend the family $\mathcal{T}'_1$ by two-element sets $\{T, T'\}$ of new theories in a language of 0-ary predicates such that T contains predicates witnessing the required rank $\mathrm{RS}^{\exists}(\mathcal{T}) \geq \mathrm{RS}^{\forall}(\mathcal{T})$ and T' does not contain these predicates. Thus, the difference between $\mathrm{RS}^{\exists}(\mathcal{T})$ and $\mathrm{RS}^{\forall}(\mathcal{T})$ is witnessed by sentences contained in some but not all theories in $\{T, T'\}$.

2. For the realization $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}^{\forall}(\mathcal{T}) = \mathrm{RS}^{\exists}(\mathcal{T}) = \alpha$, $\mathrm{ds}(\mathcal{T}) = k \leq \mathrm{ds}^{\forall}(\mathcal{T}) = m \leq \mathrm{ds}^{\exists}(\mathcal{T}) = n$ we repeat the process in the item 1, using the arguments for the proof of [15, Proposition 3.11], such that $\mathcal{T}'_0$ has k

s-definable subfamilies witnessing $\mathrm{ds}(\mathcal{T}'_0) = k$, $\mathcal{T}'_1$ has m s-definable subfamilies witnessing $\mathrm{ds}(\mathcal{T}'_1) = m$, and the required family $\mathcal{T}$ has n s-definable subfamilies witnessing $\mathrm{ds}(\mathcal{T}) = n$.

The required family $\mathcal{T}$, both in Items 1 and 2, consists of singletons and two-element sets implying $\rho(\mathcal{T}) = 2$. □

Arguments above show that the triplet $(\mathrm{RS}(\mathcal{T}), \mathrm{RS}^{\forall}(\mathcal{T}), \mathrm{RS}^{\exists}(\mathcal{T}))$ can be varied arbitrarily enough as well as $(\mathrm{ds}(\mathcal{T}), \mathrm{ds}^{\forall}(\mathcal{T}), \mathrm{ds}^{\exists}(\mathcal{T}))$. Moreover, by the definition, using Morleyzation, these variations can be modelled by families $\mathcal{T}$ of theories in languages of 0-ary predicates and with $\rho(\mathcal{T}) = 2$.

Now we study connections between the pairs $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ and

$$(\mathrm{RS}(\mathrm{ur}(\mathcal{T})), \mathrm{ds}(\mathrm{ur}(\mathcal{T})))$$

for families $\mathcal{T}$ of rank $\rho(\mathcal{T})$.

Since sentences separating families $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{T}$ separate urelements $T_1 \in \mathrm{ur}(\mathcal{T}_1)$ and $T_2 \in \mathrm{ur}(\mathcal{T}_2)$, we have the following inequalities: $\mathrm{RS}(\mathcal{T}) \leq \mathrm{RS}(\mathrm{ur}(\mathcal{T}))$, and if $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(\mathrm{ur}(\mathcal{T}))$ then $\mathrm{ds}(\mathcal{T}) \leq \mathrm{ds}(\mathrm{ur}(\mathcal{T}))$.

At the same time $\mathcal{T}$ can have more accumulation points than $\mathrm{ur}(\mathcal{T})$. Indeed, if $\mathrm{ur}(\mathcal{T})$ is e-minimal, with unique accumulation point T, then an appropriate $\mathcal{T}$ can have accumulation points T, $\{T\}$, $\{T, \{T\}\}$, $\{T, \{T\}, \{T, \{T\}\}\}$ etc. Since there are unboundedly many these accumulation points we have the following:

Proposition 3.8. *For any infinite regular family $\mathcal{T}_0$ of urelements in a given language and any cardinality λ there is a family $\mathcal{T}$ with $\mathrm{ur}(\mathcal{T}) = \mathcal{T}_0$ and with λ accumulation points.*

Remark 3.9. Be the definition if $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$ then $\mathrm{ds}(\mathcal{T}) \in \omega \setminus \{0\}$. Besides, for a permutation $f \in S(\mathcal{T}_\Sigma)$, $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(f(\mathcal{T}))$, and $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}(f(\mathcal{T}))$, where $\mathrm{RS}(\mathcal{T})$ is an ordinal, if and only if f can be extended till a map f' on the set of Σ-sentences preserving RS- and ds-values, via sentences $f'(\varphi)$, for images of s-definable subfamilies of $f(\mathcal{T})$. In particular, RS- and ds-values for $\mathcal{T}$ and $f(\mathcal{T})$ coincide if f preserves s-definable subfamilies in the definition of RS.

Remark 3.10. If $\mathcal{T}$ is a family consisting of some copies of a family $\mathcal{T}'$ then $\mathcal{T}$ can have distinct properties with respect to rank and degree of a family of theories. Indeed, if $\mathcal{T} = \{\{T_1, \{T_2\}\}, \{T_2, \{T_1\}\}\}$ for some distinct theories T_1, T_2 then $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = 1$ since $\{T_1, \{T_2\}\}$ and $\{T_2, \{T_1\}\}$ cannot be separated by sentences, whereas $\mathrm{RS}(\{T_1, T_2\}) = 0$ and $\mathrm{ds}(\{T_1, T_2\}) = 2$. Similarly, one can not separate copies with common urelements. Moreover, it is easy to construct step-by-step a family $\mathcal{T}$ with $|\mathrm{ur}(\mathcal{T})| = n$ such that $\mathrm{RS}(\mathcal{T}) = 0$, $\mathrm{ds}(\mathcal{T}) = 1$, $\mathrm{RS}(\mathrm{ur}(\mathcal{T})) = 0$, $\mathrm{ds}(\mathrm{ur}(\mathcal{T})) = n$.

Definition. Families $\mathcal{T}_1$ and $\mathcal{T}_2$ in a language Σ are called *disjoint* if they do not have common urelements: $\mathrm{ur}(\mathcal{T}_1) \cap \mathrm{ur}(\mathcal{T}_2) = \emptyset$.

Notice that the effect described in Remark 3.10 does not occur for disjoint families:

Proposition 3.11. *For any pairwise disjoint copies $\mathcal{T}_i$ of a nonempty regular family $\mathcal{T}$ with finitely many urelements, $i < n$, $n \in \omega \setminus \{0\}$, the degree equals the cardinality of the set of these copies:* $\mathrm{ds}(\{\mathcal{T}_i \mid i < n\}) = n$.

Proof. Since $\mathcal{T}_i$ are disjoint they can be separated by sentences φ_i being disjunctions of sentences separating urelements of the copies. Indeed, since there are finitely many urelements in $\mathcal{T}' = \bigcup\limits_{i<n} \mathrm{ur}(\mathcal{T}_i)$, we can find sentences ψ_T isolating each element T in $\mathcal{T}'$. Taking disjunctions φ_i of sentences ψ_T for $T \in \mathrm{ur}(\mathcal{T}_i)$ we isolate $\mathcal{T}_i$. Having n isolating sentences we obtain $\mathrm{ds}(\{\mathcal{T}_i \mid i < n\}) = n$. □

Theorem 3.12. *For any two disjoint subfamilies $\mathcal{T}_1$ and $\mathcal{T}_2$ of an E-closed family $\mathcal{T}$ of urelements the following conditions are equivalent:*

(1) *$\mathcal{T}_1$ and $\mathcal{T}_2$ are separated by some sentence φ: $\mathcal{T}_1 \subseteq \mathcal{T}_\varphi$ and $\mathcal{T}_2 \subseteq \mathcal{T}_{\neg\varphi}$;*

(2) *E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint in $\mathcal{T}$: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) \cap \mathcal{T} = \emptyset$;*

(3) *E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) = \emptyset$.*

Proof. (1) ⇒ (3). Assuming that $\mathcal{T}_1 \subseteq \mathcal{T}_\varphi$ and $\mathcal{T}_2 \subseteq \mathcal{T}_{\neg\varphi}$ we obtain $\mathrm{Cl}_E(\mathcal{T}_1) \subseteq \mathcal{T}_\varphi$ and $\mathrm{Cl}_E(\mathcal{T}_2) \subseteq \mathcal{T}_{\neg\varphi}$ since these E-closures preserve φ and $\neg\varphi$, respectively. As $\mathcal{T}_\varphi \cap \mathcal{T}_{\neg\varphi} = \emptyset$ we have $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) = \emptyset$.

(3) ⇒ (2) is obvious.

(2) ⇒ (1). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be non-separated by sentences. Then for any sentence φ with $\mathcal{T}_1 \subseteq \mathcal{T}_\varphi$ some theory $T \in \mathcal{T}_2$ contains φ. Moreover, the families $\mathcal{T}_\varphi \cap \mathcal{T}_2$ are infinite. Since $\mathrm{Cl}_E(\mathcal{T}_1)$ is E-closed it is d-definable in $\mathcal{T}$ by Theorem 2.2, with $\mathrm{Cl}_E(\mathcal{T}_1) = \mathcal{T}_\Phi$ for some set Φ of sentences. Similarly, $\mathrm{Cl}_E(\mathcal{T}_2) = \mathcal{T}_\Psi$ for some set Ψ of sentences. Now the E-closed family $\mathcal{T}_\Phi \cap \mathcal{T}_\Psi = \mathcal{T}_{\Phi\cup\Psi}$ is locally consistent by the conjecture. Using Compactness we obtain that $\mathcal{T}_{\Phi\cup\Psi}$ is consistent contradicting $\mathcal{T}_\Phi \cap \mathcal{T}_\Psi = \mathcal{T}_\Phi \cap \mathcal{T}_\Psi \cap \mathcal{T} = \emptyset$. □

Notice that E-closeness of $\mathcal{T}$ is necessary for Theorem 3.12 since otherwise, for instance, taking disjoint $\mathcal{T}_1$ and $\mathcal{T}_2$ with common accumulation points outside $\mathcal{T}$ we can not separate $\mathcal{T}_1$ and $\mathcal{T}_2$ by a neighbourhood $\mathcal{T}_\varphi$.

In fact, Theorem 3.12 is connected with a general theorem that any compact Hausdorff space is normal [2, Theorem 3.1.9], i.e., any disjoint closed sets X, Y in a compact Hausdorff space are separated by disjoint open sets U, V with $X \subseteq U$ and $Y \subseteq V$. Here we consider disjoint clopen sets separating disjoint closed sets.

Now we generalize Proposition 3.11 for families with infinitely many urelements.

Proposition 3.13. *For any pairwise disjoint nonempty families $\mathcal{T}_i$ composed by E-closed sets of urelements, $i < n$, $n \in \omega \setminus \{0\}$, the degree equals the number of these families:* $\mathrm{ds}(\{\mathcal{T}_i \mid i < n\}) = n$.

Proof. By Theorem 3.12, $\{\mathcal{T}_i \mid i < n\}$ consists of n isolated points implying $\mathrm{ds}(\{\mathcal{T}_i \mid i < n\}) = n$. $\square$

Clearly, Proposition 3.13 can fail if $\mathcal{T}_i$ are composed by sets of urelements which are not E-closed. Indeed, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint families with $\rho(\mathcal{T}_1) = \rho(\mathcal{T}_2) = 1$ and a common accumulation point for $\mathrm{ur}(\mathcal{T}_1)$ and for $\mathrm{ur}(\mathcal{T}_2)$, then we can not separate $\mathcal{T}_1$ and $\mathcal{T}_2$ by a sentence φ producing $\mathrm{ds}(\{\mathcal{T}_1, \mathcal{T}_2\}) = 1$.

The following example shows that there are infinite disjoint families $\mathcal{T}_i$, $i \in \omega$, composed of urelements such that $\mathcal{T} = \{\mathcal{T}_i \mid i \in \omega\}$ has minimal rank and degree, i.e., satisfies $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = 1$.

Example 3.14. We consider a family $\mathcal{T}'$ of theories T_{ij}, $i, j \in \omega$, of a language $\Sigma = \{Q_i^{(0)} \mid i \in \omega\} \cup \{R_{ij}^{(0)} \mid i, j \in \omega\}$ in the following way: $Q_i, R_{ij} \in T_{ij}$, $\neg Q_k, \neg R_{rs} \in T_{ij}$ for $k \neq i$, $\langle r, s\rangle \neq \langle i, j\rangle$, $i, j \in \omega$. Clearly, Q_i, R_{ij} witness $\mathrm{RS}(\mathcal{T}') = 2$, $\mathrm{ds}(\mathcal{T}') = 1$ with accumulation points $T_{i,\infty}$ and T_∞ satisfying $Q_i, \neg R_{ij} \in T_{i,\infty}$, $\neg Q_i, \neg R_{ij} \in T_\infty$, $i, j \in \omega$. At the same time, the family $\mathcal{T}$, consisting of families $\mathcal{T}_j = \{T_{ij} \mid i \in \omega\}$, $j \in \omega$, has $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = 1$ since $\mathcal{T}_j$ are not separated by sentences.

Similarly Example 3.14, considering infinite families $\mathcal{T}'$ of theories, with $(\mathrm{RS}(\mathcal{T}'), \mathrm{ds}(\mathcal{T}')) = (\alpha, n)$, for given ordinal α and natural n, one can reduce the pair (α, n) till arbitrary (β, m) with $\beta \leq \alpha$, where $m \leq n$ for $\beta = \alpha$:

Theorem 3.15. *For any ordinals $\alpha \geq \beta$ and natural $m, n > 0$, with $m \leq n$ if $\alpha = \beta$, and for any family $\mathcal{T}$ of theories such that $\rho(\mathcal{T}) = 1$, $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T})) = (\alpha, n)$ there is a family $\mathcal{T}'$ with $\mathrm{ur}(\mathcal{T}') = \mathcal{T}$ and $(\mathrm{RS}(\mathcal{T}'), \mathrm{ds}(\mathcal{T}')) = (\beta, m)$.*

Proof. If $\alpha = \beta = 0$ then $|\mathcal{T}| = n$. Now taking an arbitrary partition $\mathcal{T}'$ of $\mathcal{T}$ into m nonempty sets we obtain $\mathrm{ur}(\mathcal{T}') = \mathcal{T}$ and $(\mathrm{RS}(\mathcal{T}'), \mathrm{ds}(\mathcal{T}')) = (0, m)$, where elements of $\mathcal{T}'$ are separated by disjunctions of sentences separating elements of $\mathcal{T}$.

If $\alpha = \beta > 0$ and $m < n$ we take n copies of families $\mathcal{T}_i$ of theories such that $\mathcal{T} = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n$, appropriate sentences φ_i witness $\mathrm{RS}(\mathcal{T}_i) = \alpha$, $\mathrm{ds}(\mathcal{T}_i) = 1$, $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T})) = (\beta, n)$. Taking $\mathcal{T}' = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_m \cup \{\mathcal{T}_{m+1} \cup \ldots \cup \mathcal{T}_n\}$ we obtain $\mathrm{ur}(\mathcal{T}') = \mathcal{T}$, $(\mathrm{RS}(\mathcal{T}'), \mathrm{ds}(\mathcal{T}')) = (\alpha, m)$, which is witnessed by sentences φ_i, $i \leq m$.

If $\alpha > \beta$ we fix m disjoint neighbourhoods $\mathcal{T}_{\varphi_i}$, witnessing $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \beta$, $\mathrm{ds}(\mathcal{T}_{\varphi_i}) = 1$, $\mathrm{ds}(\mathcal{T}_{\varphi_1} \cup \ldots \cup \mathcal{T}_{\varphi_m}) = m$, and we set $\mathcal{T}'_0 = \mathcal{T} \setminus (\mathcal{T}_{\varphi_1} \cup \ldots \cup \mathcal{T}_{\varphi_m})$. Now the family $\mathcal{T}' = \mathcal{T}_{\varphi_1} \cup \ldots \cup \mathcal{T}_{\varphi_m} \cup \{\mathcal{T}'_0\}$ has $\mathrm{ur}(\mathcal{T}') = \mathcal{T}$, $(\mathrm{RS}(\mathcal{T}'), \mathrm{ds}(\mathcal{T}')) = (\beta, m)$, which is witnessed by sentences for $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \beta$, $\mathrm{ds}(\mathcal{T}_{\varphi_i}) = 1$, $1 \leq i \leq m$. $\square$

Remark 3.16. Families $\mathcal{T}'$ in the proof of Theorem 3.15 are composed by d-definable subfamilies of $\mathcal{T}$.

4. Graphs and families of neighbourhoods witnessing ranks

In this section we introduce and study structures witnessing ranks of given families.

It is known (cf. [4, p. 335]) that formulas φ_i used in the definition of the rank $\mathrm{RS}(\cdot)$ form a tree with the root $\forall x(x \approx x)$, where any vertex φ for a neighbourhood $\mathcal{T}_\varphi$, in a step $\geq \alpha$ for $\mathrm{RS}(\mathcal{T})$, is connected by arcs $u = (\varphi, \varphi_i)$ with infinitely many pairwise inconsistent vertices φ_i for neighbourhoods $\mathcal{T}_{\varphi_i} \subset \mathcal{T}_\varphi$. That graph Γ, consisting of the arcs (φ, φ_i), is called the *graph witnessing the rank* $\mathrm{RS}(\mathcal{T})$ and denoted by $\Gamma_0(\mathcal{T})$.

Clearly, the system of vertices φ of the graph $\Gamma_0(\mathcal{T})$ defines the family $N_0(\mathcal{T})$ of the neighbourhoods $\mathcal{T}_\varphi$ with the relation $\subseteq$, which is denoted by $\mathcal{N}_0(\mathcal{T}) = \langle N_0(\mathcal{T}); \subseteq\rangle$, and vice versa.

The structures $\Gamma_0(\mathcal{T})$ and $\mathcal{N}_0(\mathcal{T})$ can recognize if $\mathcal{T}$ is e-totally transcendental or not. Thus, $\Gamma_0(\mathcal{T})$ and $\mathcal{N}_0(\mathcal{T})$ will be accordingly called *e-totally transcendental* or not.

Thus, we have the following:

Theorem 4.1. *For any nonempty regular family $\mathcal{T}$ the following conditions are equivalent:*
(1) $\mathcal{T}$ *is e-totally transcendental;*
(2) $\Gamma_0(\mathcal{T})$ *is e-totally transcendental;*
(3) $\mathcal{N}_0(\mathcal{T})$ *is e-totally transcendental.*

If $\mathrm{RS}(\mathcal{T})$ is an ordinal α, we mark the vertices φ in $\Gamma(\mathcal{T})$ by ordinals $l(\varphi) = \beta \leq \alpha$ starting with atoms φ by labels $l(\varphi) = 0$, continuing with $l(\varphi) = \beta + 1$ for arcs $u = (\varphi, \varphi_i)$ with $l(\varphi_i) = \beta$, and with $l(\varphi) = \beta$ for limit ordinals β and labels $l(\varphi_i) = \gamma < \beta$ with $\mathcal{T}_{\varphi_i} \subset \mathcal{T}_\varphi$, and finalize with $\forall x(x \approx x)$ by the label α.

In such a case the root $\forall x(x \approx x)$ is unique vertex with the label α if $\mathrm{ds}(\mathcal{T}) = 1$, or it has $n = \mathrm{ds}(\mathcal{T}) > 1$ pairwise inconsistent successors $\varphi_1, \ldots, \varphi_n$ with $\mathrm{RS}(\mathcal{T}_{\varphi_j}) = \alpha$ witnessing $\mathrm{ds}(\mathcal{T}) = n$.

The graph $\Gamma_0(\mathcal{T})$ expanded by the labels above is called the *graph witnessing the rank* $\mathrm{RS}(\mathcal{T}) = \alpha$ and denoted by $\Gamma(\mathcal{T})$.

Elements $\mathcal{T}_\varphi$ of $\mathcal{N}_0(\mathcal{T})$ can be also marked by ordinals which labels φ and witness the rank $\mathrm{RS}(\mathcal{T}) = \alpha$. Therefore we expand $\mathcal{N}_0(\mathcal{T})$ by these witnessing labels and obtain the expanded structure denoted by $\mathcal{N}(\mathcal{T})$.

Clearly, the universe $N(\mathcal{T})$ of $\mathcal{N}(\mathcal{T})$ is a family which control $\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}(\mathcal{T})$. Thus both $\Gamma(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$ code the steps for the values $\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}(\mathcal{T})$ and the values of supremum for labels of $\Gamma(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$ as well as the numbers of elements with maximal values define the ranks and degrees for $\Gamma(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$ denoted by $\mathrm{RS}(\Gamma(\mathcal{T}))$ and $\mathrm{ds}(\Gamma(\mathcal{T}))$ for $\Gamma(\mathcal{T})$, and $\mathrm{RS}(\mathcal{N}(\mathcal{T}))$ and $\mathrm{ds}(\mathcal{N}(\mathcal{T}))$ for $\mathcal{N}(\mathcal{T})$.

By the definition the values $\mathrm{RS}(\mathcal{T})$, $\mathrm{RS}(\Gamma(\mathcal{T}))$, $\mathrm{RS}(\mathcal{N}(\mathcal{T}))$ are equal each other, as well as $\mathrm{ds}(\mathcal{T})$, $\mathrm{ds}(\Gamma(\mathcal{T}))$, $\mathrm{ds}(\mathcal{N}(\mathcal{T}))$. Thus studying the rank $\mathrm{RS}(\cdot)$ we can replace $\mathcal{T}$ by $\Gamma(\mathcal{T})$ or $\mathcal{N}(\mathcal{T})$.

Definition. A family $\mathcal{T}'$ is called RS-ranking if $\mathcal{T}'$ consists of s-definable families $\mathcal{T}_\varphi$ forming $\mathcal{N}(\mathcal{T})$ for some family $\mathcal{T}$. In such a case we say that $\mathcal{T}'$ is the RS-ranking family for $\mathcal{T}$.

By the definition any family $\mathcal{T}$ has a RS-ranking family $\mathcal{T}'$ which is denoted by $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$. We have $\rho(\mathcal{F}_{\mathrm{RS}}(\mathcal{T})) = \rho(\mathcal{T}) + 1$.

Proposition 4.2. *For any nonempty family $\mathcal{T}$ the* RS*-ranking family $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ is uniquely defined if and only if* $\mathrm{RS}(\mathcal{T}) = 0$.

Proof. If $\mathrm{RS}(\mathcal{T}) = 0$, with $\mathrm{ds}(\mathcal{T}) = n$, then $\mathcal{T}$ is uniquely divided into n disjoint s-definable parts producing unique $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$.

If $\mathrm{RS}(\mathcal{T}) > 0$ then by the definition of RS we can remove infinitely many s-definable parts P from $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ witnessing the value $\mathrm{RS}(\mathcal{T})$ such that the reduced proper subfamily of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ witnesses again the value $\mathrm{RS}(\mathcal{T})$. It means that $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ is not uniquely defined. □

Remark 4.3. Each element $\mathcal{T}'$ of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ can obtain a value $\mathrm{RS}'(\mathcal{T}')$ following steps witnessing $\mathrm{RS}((\mathcal{T})$. We start with $\mathrm{RS}'(\mathcal{T}') = 0$ for isolated elements in $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ and step-by-step increase the values till $\alpha + 1$ for neighbourhoods $\mathcal{T}' = \mathcal{T}_\varphi$ in appliance with steps uniting infinitely many disjoint neighbourhoods $\mathcal{T}_\psi \subset \mathcal{T}_\varphi$ with $\mathrm{RS}'(\mathcal{T}_\psi) \leq \alpha$. We also unite $\mathcal{T}_\psi \subset \mathcal{T}_\varphi$ with $\mathrm{RS}'(\mathcal{T}_\psi) = \beta$ for $\beta < \alpha$ obtaining $\mathrm{RS}'(\mathcal{T}_\psi) = \alpha$, if α is limit. Finally, if $\mathcal{T}$ is not e-totally transcendental, it is witnessed by elements $\mathcal{T}'$ of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ with $\mathrm{RS}'(\mathcal{T}') = \infty$.

Having the values $\mathrm{RS}'(\mathcal{T}')$ for elements $\mathcal{T}'$ of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ we form, for any ordinal α, the subfamilies $\mathcal{F}^{\leq\alpha}_{\mathrm{RS}}(\mathcal{T})$ and $\mathcal{F}^{\geq\alpha}_{\mathrm{RS}}(\mathcal{T})$ of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ consisting of all elements $\mathcal{T}'$ with $\mathrm{RS}'(\mathcal{T}') \leq \alpha$ and $\mathrm{RS}'(\mathcal{T}') \geq \alpha$, respectively. Now $\mathcal{F}^{\geq\alpha}_{\mathrm{RS}}(\mathcal{T})$ admits β steps according with the process for its RS-value, where $\alpha + \beta = \mathrm{RS}(\mathcal{T})$. Thus, we obtain the following *additivity formula* in accordance with a decomposition of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ into $\mathcal{F}^{\leq\alpha}_{\mathrm{RS}}(\mathcal{T})$ and $\mathcal{F}^{\geq\alpha}_{\mathrm{RS}}(\mathcal{T})$:

$$\mathrm{RS}(\mathcal{T}) = \mathrm{RS}\left(\mathcal{F}^{\leq\alpha}_{\mathrm{RS}}(\mathcal{T})\right) + \mathrm{RS}\left(\mathcal{F}^{\geq\alpha}_{\mathrm{RS}}(\mathcal{T})\right). \tag{4.1}$$

The decomposition formula holds both for an ordinal $\mathrm{RS}(\mathcal{T})$ and for the case $\mathrm{RS}(\mathcal{T}) = \infty$. In the latter case $\mathrm{RS}\left(\mathcal{F}^{\geq\alpha}_{\mathrm{RS}}(\mathcal{T})\right) = \infty$.

Thus we obtain the following:

Theorem 4.4. *For any nonempty family $\mathcal{T}$ and an ordinal α the decomposition formula* (4.1) *holds.*

This decomposition allows to divide, into several parts, steps for construction witnessing the value $\mathrm{RS}(\mathcal{T})$.

Remark 4.3 and Theorem 4.4 immediately imply the following:

Corollary 4.5. *If* $\mathrm{RS}(\mathcal{T}) = \alpha \in \mathrm{Ord}$ *and* $n \in \omega$ *then there are subfamilies* $\mathcal{T}_1, \dots, \mathcal{T}_n$ *of* $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ *such that* $\bigcup\limits_{k=1}^{n} \mathcal{T}_k = \mathcal{F}_{\mathrm{RS}}(\mathcal{T})$, $\mathcal{T}_k$ *consists of elements with* RS'*-ranks* $\beta \in [\alpha_{k-1}, \alpha_k]$, $k \leq n$, $0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n = \alpha$, *and*

$$\mathrm{RS}(\mathcal{T}) = \sum_{k=1}^{n} \mathrm{RS}(\mathcal{T}_k).$$

The families $\mathcal{T}_k$ in Corollary 4.5 are called $\mathcal{T}$*-interval*, and the family $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ is called *sequentially complete* $\mathcal{T}$*-interval*.

Remark 4.6. The notion of sequentially complete $\mathcal{T}$-interval family can be naturally extended by infinite $\{\mathcal{T}_i \mid i \in I\}$, where I is formed by an increasing discrete well-ordered chain of correspondent increasing ordinals $\alpha_i > 0$, $i \in I$, $\alpha_0 = 0$, $\bigcup\limits_{i \in I} \alpha_i = \mathrm{RS}(\mathcal{T})$, and each $\mathcal{T}_i$ consists of elements of $\mathcal{F}_{\mathrm{RS}}(\mathcal{T})$ with RS'-ranks $\beta \in [\alpha_j, \alpha_i]$, where j is the predecessor of i, and $j = 0$ for the least element i of I. In such a case we obtain the following generalized decomposition formula connecting RS-ranks:

$$\mathrm{RS}(\mathcal{T}) = \sum_{i \in I} \mathrm{RS}(\mathcal{T}_i).$$

Remark 4.7. The notions and assertions above can be naturally spread both for $\mathrm{RS}^{\forall}$ and $\mathrm{RS}^{\exists}$, as well as for degrees and results of replacements of s-definable subfamilies by some d-definable ones.

5. Conclusion

We introduced and described a hierarchy of families of theories and their rank characteristics including dynamics of ranks. We considered regular families based on a family of urelements — theories in a given language, and a step-by-step process producing a required hierarchy. We introduced ranks $\mathrm{RS}^{\forall}$, $\mathrm{RS}^{\exists}$, RS with respect to sentence definable subfamilies and generalizing the known RS-rank for families of urelements, as well as their degrees. Links and dynamics for these ranks and degrees are described. Graphs and families of neighbourhoods witnessing ranks are introduced and characterized. Decompositions of families of neighbourhoods and their rank links produce the additivity and the possibility to reduce complexity measures for families into simpler subfamilies.

References

1. Barwise J. *Admissible sets and structures. An approach to definability theory.* Perspectivities in Mathematical Logic. Berlin, Heidelberg, New York, Springer-Verlag, 1975. 396 p.
2. Engelking R. *General topology.* Berlin, Heldermann Verlag, 1989. 529 p.
3. *Ershov Yu.L., Palyutin E.A. Mathematical logic.* Moscow, Fizmatlit Publ., 2011. 356 p. [in Russian]
4. Handbook of mathematical logic. Vol. 1. Model Theory, J. Barwise (ed.). Moscow, Nauka, 1982, 392 p. [in Russian]
5. Emelichev V.A., Mel'nikov O.I., Sarvanov V.I., Tyshkevich R.I. *Lectures on the graph theory.* Moscow, Nauka, 1990. 384 p. [in Russian]
6. Markhabatov N.D., Sudoplatov S.V. Definable subfamilies of theories, related calculi and ranks. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 700–714. https://doi.org/10.33048/semi.2020.17.048
7. Markhabatov N.D., Sudoplatov S.V. Algebras for definable families of theories. *Siberian Electronic Mathematical Reports*, 2019, vol. 16, pp. 600–608. https://doi.org/10.33048/semi.2019.16.038
8. Markhabatov N.D. Ranks for families of permutation theories. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2019, vol. 28, pp. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
9. Pavlyuk In.I., Sudoplatov S.V. Ranks for families of theories of abelian groups. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2019, vol. 28, pp. 96–113. https://doi.org/10.26516/1997-7670.2019.28.95
10. Pavlyuk In.I., Sudoplatov S.V. Approximations for theories of abelian groups. *Mathematics and Statistics*, 2020, vol. 8, no. 2, pp. 220–224. https://doi.org/10.13189/ms.2020.080218
11. Sudoplatov S.V. Combinations of structures. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2018, vol. 24, pp. 65–84. https://doi.org/10.26516/1997-7670.2018.24.82
12. Sudoplatov S.V. Closures and generating sets related to combinations of structures. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2016, vol. 16, pp. 131–144.
13. Sudoplatov S.V. Combinations related to classes of finite and countably categorical structures and their theories. *Siberian Electronic Mathematical Reports*, 2017, vol. 14, pp. 135–150. https://doi.org/10.17377/semi.2017.14.014
14. Sudoplatov S.V. Approximations of theories. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 715–725. https://doi.org/10.33048/semi.2020.17.049
15. Sudoplatov S.V. Ranks for families of theories and their spectra. em arXiv:1901.08464v1 [math.LO], 2019, 17 p.

Sergey Sudoplatov, Doctor of Sciences (Physics and Mathematics), docent; Leading researcher, Sobolev Institute of Mathematics SB RAS, 4, Academician Koptyug Avenue, Novosibirsk, 630090, Russian Federation, tel.: (383)3297586; Head of Chair, Novosibirsk State Technical University, 20, K. Marx Avenue, Novosibirsk, 630073, Russian Federation, tel.: (383)3461166; Professor, Novosibirsk State University, 1, Pirogova street, Novosibirsk, 630090, Russian Federation, tel.: (383)3634020, email: sudoplat@math.nsc.ru, ORCID iD https://orcid.org/0000-0002-3268-9389.

Received 08.07.20

Иерархия семейств теорий и их ранговые характеристики

С. В. Судоплатов[123]

[1] *Институт математики им. С. Л. Соболева СО РАН, Новосибирск, Российская Федерация*

[2] *Новосибирский государственный технический университет, Новосибирск, Российская Федерация*

[3] *Новосибирский государственный университет, Новосибирск, Российская Федерация*

Аннотация. Изучение семейств элементарных теорий дает информацию о поведении и взаимосвязях теорий внутри семейств, возможности порождения и их сложности. Эта сложность выражается ранговыми характеристиками как для семейств, так и для их элементов внутри семейств.

В работе вводится и описывается иерархия семейств теорий и их ранговые характеристики, включая динамику рангов. Рассматриваются регулярные семейства, базирующиеся на основе семейства праэлементов — теорий данной сигнатуры, и пошагового процесса, задающего искомую иерархию. Для отражения шагов этого процесса используется ординально-значный теоретико-множественный ранг. Вводится ранг RS и связанные с ним ранги для регулярных семейств относительно определимых предложениями подсемейств, обобщается известный RS-ранг для семейств праэлементов, а также их степень. На основе отделимости множеств праэлементов описываются связи и динамика для этих рангов и степеней. Вводятся и характеризуются графы и семейства окрестностей, свидетельствующие о рангах. Показано, что декомпозиции семейств окрестностей и ранговые связи, для дискретных разложений, задают аддитивность и возможность сведения меры сложности для семейств к более простым подсемействам.

Ключевые слова: семейство теорий, замыкание, праэлемент, иерархия, ранг, декомпозиция.

Список литературы

1. Barwise J. Admissible sets and structures. An approach to definability theory. Perspectivities in Mathematical Logic. Berlin, Heidelberg, New York, Springer-Verlag, 1975. 396 p.
2. Энгелькинг Р. Общая топология. М. : Мир, 1986. 752 с.
3. *Ершов Ю. Л., Палютин Е. А.* Математическая логика. М. : Физматлит, 2011, 356 с.
4. Справочная книга по математической логике / Под ред. Дж. Барвайса. М. : Наука, 1982. Ч. 1. Теория моделей. 392 с.
5. Емеличев В.А., Мельников О.И., Сарванов В.И., Тышкевич Р.И. Лекции по теории графов. М. : Наука, 1990. 384 с.
6. Markhabatov N.D., Sudoplatov S.V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 700–714. https://doi.org/10.33048/semi.2020.17.048

Известия Иркутского государственного университета.
2020. Т. 33. Серия «Математика». С. 80–95

7. Markhabatov N.D., Sudoplatov S.V. Algebras for definable families of theories // Siberian Electronic Mathematical Reports. 2019. Vol. 16. P. 600–608. https://doi.org/10.33048/semi.2019.16.038
8. Markhabatov N.D. Ranks for families of permutation theories // The Bulletin of Irkutsk State University. Series "Mathematics". 2019. Vol. 28. P. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
9. Pavlyuk In.I., Sudoplatov S.V. Ranks for families of theories of abelian groups // The Bulletin of Irkutsk State University. Series "Mathematics". 2019. Vol. 28. P. 96–113. https://doi.org/10.26516/1997-7670.2019.28.95
10. Pavlyuk In.I., Sudoplatov S.V. Approximations for theories of abelian groups // Mathematics and Statistics. 2020. Vol. 8, no. 2. P. 220–224. https://doi.org/10.13189/ms.2020.080218
11. Sudoplatov S.V. Combinations of structures // The Bulletin of Irkutsk State University. Series "Mathematics". 2018. Vol. 24. P. 65–84. https://doi.org/10.26516/1997-7670.2018.24.82
12. Sudoplatov S.V. Closures and generating sets related to combinations of structures // The Bulletin of Irkutsk State University. Series "Mathematics". 2016. Vol. 16. P. 131–144.
13. Sudoplatov S.V. Combinations related to classes of finite and countably categorical structures and their theories // Siberian Electronic Mathematical Reports. 2017. Vol. 14. P. 135–150. https://doi.org/10.17377/semi.2017.14.014
14. Sudoplatov S.V. Approximations of theories // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 715–725. https://doi.org/10.33048/semi.2020.17.049
15. Sudoplatov S.V. Ranks for families of theories and their spectra // arXiv:1901.08464v1 [math.LO], 2019, 17 p.

Сергей Владимирович Судоплатов, доктор физико-математических наук, доцент, ведущий научный сотрудник, Институт математики им. С. Л. Соболева СО РАН, Российская Федерация, 630090, Новосибирск, пр. Академика Коптюга, 4, тел.: (383)3297586; заведующий кафедрой алгебры и математической логики, Новосибирский государственный технический университет, Российская Федерация, 630073, Новосибирск, пр. К. Маркса, 20, тел. (383)3461166; профессор кафедры алгебры и математической логики, Новосибирский государственный университет, Российская Федерация, 630090, Новосибирск, ул. Пирогова, 1, тел. (383)3634020, email: sudoplat@math.nsc.ru,
ORCID iD https://orcid.org/0000-0002-3268-9389.

Поступила в редакцию 08.07.20