



Серия «Математика»

2020. Т. 31. С. 62–77

Онлайн-доступ к журналу:

<http://mathizv.isu.ru>

ИЗВЕСТИЯ

Иркутского
государственного
университета

УДК 517.97

MSC 49J52, 49J53, 49K24, 49M25, 90C30

DOI <https://doi.org/10.26516/1997-7670.2020.31.62>

Optimal Control of Differential Inclusions, II: Sweeping*

B. S. Mordukhovich

Wayne State University, Detroit, USA

In memory of Oleg Vasiliev, a colleague and friend

Abstract. This paper is devoted to optimal control of dynamical systems governed by differential inclusions with discontinuous velocity mappings. This framework mostly concerns a new class of optimal control problems described by various versions of the so-called sweeping/Moreau processes that are very challenging mathematically and highly important in applications to mechanics, engineering, economics, robotics, etc. Our approach is based on developing the method of discrete approximations for optimal control problems of such differential inclusions that addresses both numerical and qualitative aspects of optimal control. In this way we establish necessary optimality conditions for optimal solutions to sweeping differential inclusions and discuss their various applications. Deriving necessary optimality conditions strongly involves advanced tools of first-order and second-order variational analysis and generalized differentiation.

Keywords: optimal control, differential inclusions, variational analysis, sweeping processes, discrete approximations, generalized differentiation.

1. Introduction

We refer the reader to Part I [22] for optimal control problems governed by Lipschitzian differential inclusions. Here we address new classes of dynamic optimization problems described by *discontinuous* differential

* This research was partly supported by the USA National Science Foundation under grants DMS-1512846 and DMS-1808978, by the USA Air Force Office of Scientific Research under grant 15RT04, and by Australian Research Council, Discovery Project under grant DP-190100555.

inclusion with the main emphasis to deriving necessary optimality conditions. The dynamics of such systems is governed by various versions of the *sweeping process*. The original *uncontrolled* version of the sweeping process was introduced by Moreau in the 1970s motivated by applications to problems of elastoplasticity; see [25] and the survey paper [11]. Similar processes were independently considered by Krasnosel'skii and Pokrovskii for dynamical systems with hysteresis; see their book [16]. Later various models of the sweeping type appeared in other areas of applied science and practical modeling; see more discussions and references below.

Recall that the basic sweeping process of Moreau is described by the unbounded differential inclusion

$$\dot{x}(t) \in -N(x(t); C(t)) \quad \text{a.e. } t \in [0, T], \quad x(0) := x_0 \in C(0), \quad (1.1)$$

on the fixed time interval $[0, T]$, where $N(x; \Omega) = N_\Omega(x)$ signifies the normal cone to a convex set $\Omega \subset \mathbb{R}^n$ at x in the standard sense of convex analysis

$$N(x; \Omega) := \begin{cases} \{v \in \mathbb{R}^n \mid \langle v, u - x \rangle \leq 0 \text{ for all } u \in \Omega\} & \text{if } x \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \quad (1.2)$$

and where the moving convex set $C(t)$ continuously depends on time.

Denoting $F(x) := -N(x; C(t))$ in (1.1), we observe that this velocity mapping is discontinuous and hence is never Lipschitzian, i.e., fails to satisfy the crucial assumption of the aforementioned control theory for differential inclusions. There is a more striking thing to say on comparison of (1.1) with Lipschitzian differential inclusions: the Cauchy problem in (1.1) admits a *unique* solution due to the well-known maximal monotonicity of the normal cone mapping $x \mapsto N(x; C(t))$ in convex analysis. This excludes, in contrast to the Lipschitzian theory discussed in Part I [22], the possibility of optimization of the sweeping differential inclusion as given in (1.1) with a fixed moving set $C(t)$.

In [7] we suggested for the first time in the literature to insert *control actions* into the *moving sets*

$$C(t) := C(u(t)) \quad \text{for all } t \in [0, T], \quad (1.3)$$

which makes it possible to change and optimize the *shape* of the right-hand side in (1.1) in order to achieve a desired performance of the controlled sweeping process with respect to a prescribed cost functional. This novel and practically motivated approach led us to a new class of control systems that is essentially different from those considered before in control theory. Besides the discontinuity and the changeable shape of the velocity mapping in (1.1), we unavoidably have the *pointwise mixed control-state constraints*

$$x(t) \in C(u(t)) \quad \text{for a.e. } t \in [0, T],$$

which are intrinsic in (1.1) and (1.3) due to the normal cone construction (1.2). Such constraints are among the most difficult even for standard optimal control of smooth systems while being investigated therein only under restrictive regularity assumptions.

Other classes of dynamic optimization problems for controlled sweeping processes correspond to the appearance of control actions in additive *external perturbations* of the type

$$\dot{x}(t) \in g(x(t), w(t)) - N(x(t); C(t)) \text{ a.e., } x(0) := x_0 \in C(0), \quad (1.4)$$

where control functions $w(\cdot)$ may be applied either together with controls $u(\cdot)$ in the moving sets (1.3), or in the absence of them. Problems of type (1.4) also exhibit new phenomena in control theory and require the development and implementation of advanced tools of variational analysis.

Our recent results discussed below show that the machinery of *discrete approximations* married to powerful tools of first-order and second-order *generalized differentiation* lead us to deriving new necessary optimality conditions for local minimizers in optimal control problems of both types as in (1.1), (1.3) and in (1.4) expressed entirely in terms of the given problem data. Some of the obtained necessary optimality conditions contain conventional Hamiltonian maximization of the Pontryagin maximum principle (PMP) type. On the other hand, we show that in problems with controlled moving sets (1.3) the conventional PMP formalism *fails*, while we are able to establish a new one in terms of a novel Hamiltonian function.

The rest of the paper is organized as follows. In Section 2 we mainly present and discuss basic robust constructions of *second-order* generalized differentiation in variational analysis that are appropriate to study differential inclusions and are widely used in the subsequent sections.

Section 1 addresses some classes of *sweeping control problems* of type (1.4) with controls in *additive perturbations*. First we investigate problems with smooth controls $w(\cdot)$ and $u(\cdot)$ in perturbations and in moving sets, respectively, and then study optimization problems with constrained discontinuous controls only in perturbations. In the first case the method of discrete approximation leads us to deriving *extended Euler-Lagrange conditions* of a new type, while for the sweeping control systems of the second kind we derive optimality conditions extending the *maximum principle*.

Section 4 deals with optimal control problems for sweeping processes with control functions acting in *parameterized moving sets*. Employing discrete approximations, we derive necessary optimality conditions in appropriate Euler-Lagrange and Hamiltonian forms, where the new Hamiltonian function is introduced to establish a *novel version* of the maximum principle. It is observed that the *conventional form* of the maximum principle *fails* to provide necessary optimality conditions for such control systems.

The concluding Section 5 is devoted to *applications* of the obtained necessary optimality conditions to some practical models with smooth and

nonsmooth dynamics. We discuss here recent applications to corridor and planar versions of the crowd motion model and related models of traffic equilibria, to hysteresis systems and elastoplasticity problems, and to typical control models arising in robotics.

Throughout the paper we use the standard notation of variational analysis, generalized differentiation and control theory; see, e.g., [20; 21] and also Part I [22] for details.

2. Second-Order Generalized Differentiation

In this section we briefly review some constructions of second-order generalized differentiation in variational analysis initiated by the author in [18] that are broadly employed in what follows. The reader can find more details in the books [19; 21] and the bibliographies therein.

The first-order generalized differential constructions used below were defined in Part I. They are the basic/limiting *normal cone* $N(\bar{x}; \Omega)$ to a set $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^n$, the *subdifferential* of an extended-real-valued function $\varphi: \mathbb{R}^n \rightarrow \bar{\mathbb{R}} := (-\infty, \infty]$ at $\bar{x} \in \text{dom } \varphi$, and the *coderivative* $D^*F(\bar{x}, \bar{y})(u)$ of a set-valued mapping/multifunction $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \text{gph } F$.

Now we turn to second-order generalized differential constructions for $\varphi: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ by employing the dual “derivative-of-derivative” approach. Given $\bar{x} \in \text{dom } \varphi$, pick $\bar{v} \in \partial\varphi(\bar{x})$ and define the *second-order subdifferential* (or *generalized Hessian*) $\partial^2\varphi(\bar{x}, \bar{v}): \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of φ at \bar{x} relative to \bar{v} as the coderivative of the first-order subgradient mapping by

$$\partial^2\varphi(\bar{x}, \bar{v})(u) := (D^*\partial\varphi)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n, \quad (2.1)$$

where $\bar{v} = \nabla\varphi(\bar{x})$ is dropped φ is differentiable at \bar{x} . If φ is C^2 -smooth around \bar{x} , then (2.1) reduces to the classical (symmetric) Hessian matrix:

$$\partial^2\varphi(\bar{x})(u) = \{\nabla^2\varphi(\bar{x})u\} \quad \text{for all } u \in \mathbb{R}^n.$$

Second-order subdifferential constructions of type (2.1) naturally appear in the study of the sweeping processes defined via the normal cone mappings as in (1.1) and its nonconvex extensions. This is due to the description of adjoint systems in first-order optimality conditions for differential inclusions via coderivatives. In fact, in modeling of a large class of sweeping processes with control-dependent moving sets we use the parameterized normal cone mapping $N: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$N(x, w) := N(x; S(w)) \quad \text{for } x \in S(w) := \{x \in \mathbb{R}^n \mid \theta(x, w) \in \Theta\}. \quad (2.2)$$

To proceed in more detail, consider a function $\varphi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ of two variables and define the *partial second-order subdifferential* of φ with respect to x at (\bar{x}, \bar{w}) relative to \bar{v} by

$$\partial_{\bar{x}}^2\varphi(\bar{x}, \bar{w}, \bar{v})(u) := (D^*\partial_x\varphi)(\bar{x}, \bar{w}, \bar{v})(u) \quad \text{for all } u \in \mathbb{R}^n \quad (2.3)$$

via the coderivative of the first-order partial subdifferential mapping

$$\partial_x \varphi(x, w) := \partial \varphi_w(x) \quad \text{with} \quad \varphi_w(x) := \varphi(x, w).$$

Observe that $N(x, w) = \partial_x \varphi(x, w)$ with $\varphi(x, w) := (\delta_\Theta \circ \theta)(x, w)$, where θ and Θ are taken from (2.2). We clearly have the following coderivative representation for the normal cone mapping (2.2):

$$D^*N(\bar{x}, \bar{w}, \bar{v})(u) = \partial_x^2 \varphi(\bar{x}, \bar{w}, \bar{v})(u) \quad \text{whenever} \quad \bar{v} \in N(\bar{x}, \bar{w}) \quad \text{and} \quad u \in \mathbb{R}^n.$$

Further elaborations of this formula require developing chain rules for the partial second-order subdifferential (2.1) for the composite function φ therein. The following *generalized second-order chain rule* taken from [24, Theorem 3.1] is efficient in our applications to controlled sweeping processes: Let $\theta: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ be C^2 -smooth around (\bar{x}, \bar{w}) with the surjective partial Jacobian operator $\nabla_x \theta(\bar{x}, \bar{w})$. Then for each $\bar{v} \in N(\bar{x}, \bar{w})$ there exists a unique vector $\bar{q} \in N_\Theta(\theta(\bar{x}, \bar{w}))$ such that $\nabla_x \theta(\bar{x}, \bar{w})^* \bar{q} = \bar{v}$ and the coderivative of the normal cone mapping (2.2) is calculated by

$$D^*N(\bar{x}, \bar{w}, \bar{v})(u) = \left[\begin{array}{c} \nabla_{xx}^2 \langle \bar{q}, \theta \rangle(\bar{x}, \bar{w}) \\ \nabla_{xw}^2 \langle \bar{q}, \theta \rangle(\bar{x}, \bar{w}) \end{array} \right] u + \quad (2.4)$$

$$+ \nabla \theta(\bar{x}, \bar{w})^* D^*N_\Theta(\theta(\bar{x}, \bar{w}), \bar{q})(\nabla_x \theta(\bar{x}, \bar{w})u), \quad u \in \mathbb{R}^n.$$

As we see, the second-order chain rule (2.4) reduces the calculation of D^*N to that of D^*N_Θ . Constructive computations of it for various classes of sets Θ , which are largely encountered in optimization, control and their applications, can be found in [21; 23] and the references therein.

3. Sweeping Processes with Controlled Perturbations

In this section we consider the sweeping control system defined in (1.4), where control functions $w(\cdot)$ are acting in the additive perturbations. When the moving set $C(t)$ in (1.4) is given a priori, optimal control problems of the Bolza type were studied in [13; 26] from the viewpoints of the existence of optimal solutions and relaxation stability.

In [3; 4] we considered the perturbed sweeping process in (1.4), where—along with the controls $w(\cdot)$ in perturbations—the other type of controls $u(\cdot)$ were applied to the moving set $C(t)$ given by

$$C(t) := C + u(t), \quad C := \{x \in \mathbb{R}^n \mid \langle x_i^*, x \rangle \leq 0, \quad i = 1, \dots, m\} \quad (3.1)$$

and the fixed vectors x_i^* generating the convex polyhedron C in (3.1). The optimal control problem studied in [3; 4] was as follows: minimize

$$J[x, u, w] := \varphi(x(T)) + \int_0^T \ell(x(t), u(t), w(t), \dot{x}(t), \dot{u}(t), \dot{w}(t)) dt \quad (3.2)$$

over control pairs $u(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$, $w(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ of the controlled sweeping system in (1.4), (3.1). The main attention in [3; 4] was paid to the construction of well-posed discrete approximations of (1.4)–(3.2) and using this approach to derive necessary conditions for local optimal solutions to this problem. The obtained results were then applied in [4] to solving some optimal control problems for the corridor version of the crowd motion model of traffic equilibria; see Section 5 for more details on this model.

One of the strongest motivations for our subsequent paper [5] was to formulate and investigate a class of sweeping control system, which is suitable for applications to the much more realistic planar crowd motion model the dynamic of which was described in [27] as a sweeping process over a nonpolyhedral moving set. To accomplish this goal, we considered in [5] the controlled sweeping process given by (1.4) with the *nonconvex* (and hence nonpolyhedral) moving set in the form

$$C(t) := C + u(t) = \bigcap_{i=1}^m C_i + u(t), \quad (3.3)$$

$$C_i := \{x \in \mathbb{R}^n \mid \xi_i(x) \geq 0\} \text{ for all } i = 1, \dots, m,$$

defined via some convex C^2 -smooth functions $\xi_i: \mathbb{R}^n \rightarrow \mathbb{R}$. Due to the nonconvexity of the set $C(t)$ in (3.3), we replaced therein the normal cone of convex analysis (1.2) by the nonconvex one from Part I [22]. The optimal control problem formulated in [5] reads as follows: minimize the cost functional (3.2) over control pairs $u(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$, $v(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ of (1.4) with the controlled moving set (3.3). Besides the dynamic constraints (1.4), we imposed the pointwise constraints on the u -controls as

$$0 < r_1 \leq \|u(t)\| \leq r_2 \text{ for all } t \in [0, T]$$

with the given constraint bounds r_1, r_2 . Note that (1.4) yields the pointwise *mixed state-control constraints*

$$\xi_i(x(t) - u(t)) \geq 0 \text{ for all } t \in [0, T] \text{ and } i = 1, \dots, m.$$

The method of discrete approximations combined with the machinery of first-order and second-order generalized differentiation from Section 2 led us in [5] to deriving constructive necessary optimality conditions for intermediate local minimizers (as defined in Part I) in the above problem.

Let us now discuss yet another setting of sweeping optimal control, where control actions $w(\cdot)$ entering the additive perturbations in (1.4) are *constrained* and *discontinuous*. Consider the following problem (P):

$$\text{minimize } J[x, u] := \varphi(x(T)) \quad (3.4)$$

over feasible pairs $(x(\cdot), u(\cdot))$ of measurable controls $u(t)$ and absolutely continuous trajectories $x(t)$ on $[0, T]$ satisfying the perturbed controlled sweeping process of type (1.4) written as

$$\dot{x}(t) \in g(x(t), u(t)) - N(x(t); C) \text{ a.e.}, \quad x(0) := x_0 \in C \subset \mathbb{R}^n, \quad (3.5)$$

with the conventional notation for control functions (i.e., using $u(t)$ instead of $w(t)$ as in (1.4), since the set C is uncontrolled now) subject to the pointwise control constraints given by

$$u(t) \in U \subset \mathbb{R}^d \text{ a.e. } t \in [0, T]. \quad (3.6)$$

The set C in (3.5) is a convex polyhedron defined by

$$C := \bigcap_{i=1}^m C_i \text{ with } C_i := \{x \in \mathbb{R}^n \mid \langle x_i^*, x \rangle \leq c_i\}. \quad (3.7)$$

Developing an advanced version of the method of discrete approximations, we recently obtained in [9] a collection of new necessary optimality conditions for (3.4)–(3.7) that includes the maximization condition of the PMP type. Let us first describe the class of local minimizers studied in [9]. We say that a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (3.4)–(3.7) is a $W^{1,2} \times L^2$ -local minimizer for this problem if there exists a number $\varepsilon > 0$ such that $J[\bar{x}, \bar{u}] \leq J[x, u]$ whenever a feasible pair $(x(\cdot), u(\cdot))$ satisfies

$$\int_0^T \left(\|\dot{x}(t) - \dot{\bar{x}}(t)\|^2 + \|u(t) - \bar{u}(t)\|^2 \right) dt < \varepsilon.$$

For the reader's convenience and brevity, we present now the major result of [9] under the following simplified assumptions in comparison with those imposed in [9]. The pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ therein is the fixed $W^{1,2} \times L^2$ -local minimizer under consideration.

- (A1)** The cost function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ in (3.2) is smooth around $\bar{x}(T)$.
(A2) The perturbation mapping $g: \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ in (3.5) is smooth around $(\bar{x}(\cdot), \bar{u}(\cdot))$ and satisfies the sublinear growth condition

$$\|g(x, u)\| \leq \alpha(1 + \|x\|) \text{ for all } u \in U \text{ with some } \alpha > 0.$$

- (A3)** The control set U is compact and convex in \mathbb{R}^d .
(A4) The image set $g(x, U)$ is convex in \mathbb{R}^n .
(A5) The vertices x_i^* of (3.7) satisfy the *positive linear independence constraint qualification* (PLICQ) meaning that

$$\left[\sum_{i \in I(\bar{x})} \alpha_i x_i^* = 0, \alpha_i \geq 0 \right] \implies [\alpha_i = 0 \text{ for all } i \in I(\bar{x})]$$

along $\bar{x} = \bar{x}(t)$ as $t \in [0, T]$, where $I(\bar{x}) := \{i \in \{1, \dots, m\} \mid \langle x_i^*, \bar{x} \rangle = c_i\}$.

The imposed PLICQ assumption is significantly weaker than the *linear independence constraint qualification* (LICQ) corresponding to (A5) with $\alpha_i \in \mathbb{R}$. Note also that the convexity assumptions imposed in (A3) and (A4) can be removed in the necessary optimality conditions presented below for the case of *strong* minimizers by considering a certain relaxation procedure as in [9]. We refer the reader to [13; 26] for various relaxation results (of the Bogolyubov-Young type) for non-Lipschitzian differential inclusions.

Theorem 1. *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a $W^{1,2} \times L^2$ -local minimizer of problem (P) under assumptions (A1)–(A5) such that $\bar{u}(\cdot)$ is of bounded variation (BV) and admits a right continuous representative on $[0, T]$. Then there exist a multiplier $\lambda \geq 0$, a signed vector measure $\gamma = (\gamma^1, \dots, \gamma^s) \in \mathcal{C}^*([0, T]; \mathbb{R}^s)$ as well as adjoint arcs $p(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ and $q(\cdot) \in BV([0, T]; \mathbb{R}^n)$ such that the following conditions are fulfilled:*

(i) *The PRIMAL-DUAL DYNAMIC RELATIONSHIPS consisting of:*

- *The PRIMAL ARC REPRESENTATION*

$$-\dot{\bar{x}}(t) = \sum_{j=1}^s \eta^j(t) x_*^j - g(\bar{x}(t), \bar{u}(t)) \quad \text{for a.e. } t \in [0, T],$$

where the functions $\eta^j(\cdot) \in L^2([0, T]; \mathbb{R}_+)$ are uniquely determined for a.e. $t \in [0, T]$ by this representation.

- *The ADJOINT DYNAMIC SYSTEM*

$$\dot{p}(t) = -\nabla_x g(\bar{x}(t), \bar{u}(t))^* q(t) \quad \text{for a.e. } t \in [0, T],$$

where the right continuous representative of $q(\cdot)$ satisfies

$$q(t) = p(t) - \int_{(t, T]} \sum_{j=1}^s d\gamma^j(\tau) x_*^j d\tau,$$

for all $t \in [0, T]$ except at most a countable subset.

- *The MAXIMIZATION CONDITION*

$$\langle \psi(t), \bar{u}(t) \rangle = \max_{u \in U} \langle \psi(t), u \rangle \quad \text{for a.e. } t \in [0, T],$$

where $\psi(t) := \nabla_u g(\bar{x}(t), \bar{u}(t))^* q(t)$ on $[0, T]$.

- *The DYNAMIC COMPLEMENTARY SLACKNESS CONDITIONS*

$$\langle x_*^j, \bar{x}(t) \rangle < c_j \implies \eta^j(t) = 0 \quad \text{and} \quad \eta^j(t) > 0 \implies \langle x_*^j, q(t) \rangle = c_j$$

for a.e. $t \in [0, T]$ and all $j = 1, \dots, s$ if LICQ holds at $\bar{x}(t)$.

(ii) The ENDPOINT RELATIONSHIPS consisting of:

- The TRANSVERSALITY CONDITIONS: there exist numbers $\eta^j(T) \geq 0$ for $j \in I(\bar{x}(T))$ such that

$$-p(T) - \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j = \lambda \nabla \varphi(\bar{x}(T)) \quad \text{and} \quad \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j \in N(\bar{x}(T); C).$$

- The ENDPOINT COMPLEMENTARY SLACKNESS CONDITIONS

$$\langle x_*^j, \bar{x}(T) \rangle < c_j \implies \eta^j(T) = 0 \quad \text{and} \quad \eta^j(T) > 0 \implies \langle x_*^j, q(T) \rangle = c_j,$$

where the latter holds if LICQ at $\bar{x}(T)$ is additionally imposed.

(iii) The MEASURE NONATOMICITY CONDITION: If $t \in [0, T)$ and $\langle x_*^j, \bar{x}(t) \rangle < c_j$ for all $j = 1, \dots, s$, then there exists a neighborhood V_t of t in $[0, T)$ such that $\gamma(V) = 0$ for all the Borel subsets V of V_t .

(iv) The NONTRIVIALITY RELATIONSHIPS consisting of:

- The GENERAL NONTRIVIALITY CONDITIONS: we always have

$$(\lambda, p, \|\gamma\|_{TV}) \neq 0,$$

which is equivalent to $(\lambda, p, q) \neq 0$ provided that LICQ holds at $\bar{x}(t)$.

- The ENHANCED NONTRIVIALITY CONDITION

$$(\lambda, p) \neq 0$$

holds provided that $\langle x_*^j, \bar{x}(t) \rangle < c_j$ as $t \in [0, T)$ and $j = 1, \dots, s$.

The proof of Theorem 1 is based on the advanced *method of discrete approximations* and the *calculation* of the second-order subdifferential (2.1) for $\varphi = \delta_C$ entirely via the given data of the polyhedron (3.1); see [9].

Note that necessary optimality conditions in sweeping control theory containing the maximization of the corresponding Hamiltonian were first obtained in [2] for (global) optimal solutions to a sweeping process of another type with an uncontrolled strictly smooth, convex and solid set $C(t) \equiv C$ and controls linearly entered an adjacent ordinary differential equation. Further results were derived in the case of the sweeping control system (3.5), where measurable controls $u(t)$ enter the additive smooth term g while the uncontrolled moving set $C(t)$ is compact, convex or mildly nonconvex, and possesses a \mathcal{C}^3 -smooth boundary for each $t \in [0, T]$ along with some additional assumptions. The recent paper [12] also deals with a sweeping control system of type (3.5) and establishes necessary optimality conditions for global minimizers involving the maximization of the standard Hamiltonian function provided that the convex and compact set $C(t) \equiv C$ of nonempty interior given by $C := \{x \in \mathbb{R}^n \mid \vartheta(x) \leq 0\}$ via a \mathcal{C}^2 -smooth function ϑ under some other assumptions, which partly differ from [1].

Certain penalty-type approximation methods developed in [1], [2], and [12] are different from each other, significantly based on the *smoothness* of *uncontrolled moving sets* while being sharply distinct from the method of discrete approximations used in our approach.

4. Sweeping Processes with Controlled Moving Sets

In this section we concentrate on a challenging class of controlled sweeping processes with control functions acting in moving sets. Such control problems were introduced and studied in [7] for the case where the set $C(u)$ in (1.3) was defined by a half-space in \mathbb{R}^n . A more general and involved case of the polyhedral description of $C(u)$ was fully investigated in [8]. The following optimal control problem was considered therein: minimize the cost functional (3.2) over the collection of absolutely continuous controls $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$, $w(\cdot) = (w_1(\cdot), \dots, w_m(\cdot))$ and the corresponding absolutely continuous trajectories $x(\cdot)$ satisfying the sweeping differential inclusion (1.1) with the controlled moving set

$$C(t) := \{x \in \mathbb{R}^n \mid \langle u_i(t), x \rangle \leq w_i(t), \quad i = 1, \dots, m\} \quad (4.1)$$

subject to the control constraints

$$\|u_i(t)\| = 1 \quad \text{for all } t \in [0, T], \quad i = 1, \dots, m, \quad (4.2)$$

By (4.1) we have also the *pointwise state-control constraints*

$$x(t) \in C(u(t), w(t)) \quad \text{for all } t \in [0, T] \quad (4.3)$$

Using the method of discrete approximations and advanced tools of variational analysis, we derived in [7; 8] necessary optimality conditions of the extended Euler-Lagrange type for polyhedral sweeping control problems. In this section we present more general results in this direction taken from [15] and obtained without any polyhedrality assumptions. Besides the conditions of the *Euler-Lagrange type*, the novel optimality conditions of the *extended Hamiltonian type* are also given therein with discovering that the conventional PMP formalism fails for such control systems.

Here we address the following sweeping control problem:

$$\text{minimize } J[x, u] := \varphi(x(T)) + \int_0^T \ell(x(t), u(t), \dot{x}(t), \dot{u}(t)) dt \quad (4.4)$$

over absolutely continuous controls $u: [0, T] \rightarrow \mathbb{R}^m$ and absolutely continuous trajectories $x: [0, T] \rightarrow \mathbb{R}^n$ of the sweeping differential inclusion

$$\dot{x}(t) \in g(x(t)) - N(h(x(t)); C(u(t))) \quad \text{a.e.}, \quad x(0) = x_0 \in C(u(0)) \quad (4.5)$$

with the controlled moving set defined by the inverse images

$$C(u) := \{x \in \mathbb{R}^n \mid \theta(x, u) \in \Theta\}, \quad u \in \mathbb{R}^m, \quad (4.6)$$

where φ , ℓ , g , and h are smooth mappings, and where θ is twice continuously differentiable around the references points with its partial Jacobian matrix $\nabla_x \theta$ of full rank. The set Θ in (4.6) is locally closed and is not assumed to be convex. Hence the set $C(u)$ is generally nonconvex as well, while the normal cone in (4.5) is understood in our basic/limiting sense; see Part I [22].

Consider the following concept of local minimizers for (4.4)–(4.6).

Definition 1. Fix a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ to problem (4.4)–(4.6). Then $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a LOCAL $W^{1,2} \times W^{1,2}$ -MINIMIZER for this problem if

$$J[\bar{x}, \bar{u}] \leq J[x, u] \text{ for } x(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n) \text{ and } u(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^m),$$

which are sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of $W^{1,2}$.

Let us formulate major necessary optimality conditions proved in [15, Theorem 4.3] by using the method of discrete approximations combined with generalized second-order calculus rule from (2.4). For simplicity we present this result in the case where $g(x) := 0$ and $h(x) := x$ for all $x \in \mathbb{R}^n$.

Theorem 2. Let $\bar{z}(t) := (\bar{x}(t), \bar{u}(t))$ be a local $W^{1,2} \times W^{1,2}$ -minimizer for (4.4)–(4.6). Then there exist a multiplier $\lambda \geq 0$, an adjoint arc $p(\cdot) = (p^x, p^u) \in W^{1,2}([0, T]; \mathbb{R}^n \times \mathbb{R}^m)$, and a measure $\gamma \in \mathcal{C}^*([0, T]; \mathbb{R}^d)$ satisfying:

- PRIMAL-DUAL DYNAMIC RELATIONSHIPS:

$$\begin{aligned} \dot{p}(t) &= \lambda \nabla_x \ell(\bar{z}(t), \dot{\bar{z}}(t)) + \\ &+ \left[\begin{array}{c} \nabla_{xx}^2 \langle \eta(t), \theta \rangle (\bar{x}(t), \bar{u}(t)) \\ \nabla_{xw}^2 \langle \eta(t), \theta \rangle (\bar{x}(t), \bar{u}(t)) \end{array} \right] \left(-\lambda \ell_{\dot{x}}(\bar{z}(t), \dot{\bar{z}}(t)) + q^x(t) \right), \\ q^u(t) &= \lambda \nabla_{\dot{u}} \ell(\bar{z}(t), \dot{\bar{z}}(t)) \quad \text{a.e. } t \in [0, T], \end{aligned}$$

where $\eta(\cdot) \in L^2([0, T]; \mathbb{R}^s)$ is uniquely defined by the representation

$$\dot{\bar{x}}(t) = -\nabla_x \theta(\bar{x}(t), \bar{u}(t))^* \eta(t) \quad \text{a.e. } t \in [0, T]$$

with $\eta(t) \in N(\theta(\bar{x}(t), \bar{u}(t)); \Theta)$, and where $q: [0, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ is a function of bounded variation on $[0, T]$ with its left-continuous representative given, for all $t \in [0, T]$ except at most a countable subset, by

$$q(t) = p(t) - \int_{[t, T]} \nabla \theta(\bar{x}(\tau), \bar{u}(\tau))^* d\gamma(\tau).$$

- MEASURED CODERIVATIVE CONDITION: Consider the outer limit

$$\text{Lim sup}_{|B| \rightarrow 0} \frac{\gamma(B)}{|B|}(t) := \left\{ y \in \mathbb{R}^s \mid \exists \text{ sequence } B_k \subset [0, T] \right.$$

$$\left. \text{with } t \in B_k, |B_k| \rightarrow 0, \frac{\gamma(B_k)}{|B_k|} \rightarrow y \right\}$$

over Borel subsets $B \subset [0, 1]$ with the Lebesgue measure $|B|$. Then for a.e. $t \in [0, T]$ this set has common points with

$$D^*N_\Theta\left(\theta(\bar{x}(t), \bar{u}(t)), \eta(t)\right)\left(\nabla_x\theta(\bar{x}(t), \bar{u}(t))(q^x(t) - \lambda\nabla_{\dot{u}}\ell(\bar{z}(t), \dot{\bar{z}}(t)))\right).$$

- TRANSVERSALITY CONDITION at the right endpoint:

$$-(p^x(T), p^u(T)) \in \lambda(\nabla\varphi(\bar{x}(T)), 0) + \nabla\theta(\bar{x}(T), \bar{u}(T))N_\Theta((\bar{x}(T), \bar{u}(T))).$$

- MEASURE NONATOMICITY CONDITION:

If $t \in [0, T)$ with $\theta(\bar{x}(t), \bar{u}(t)) \in \text{int } \Theta$, then there is a neighborhood V_t of t in $[0, T]$ such that $\gamma(V) = 0$ for any Borel subset V of V_t .

- NONTRIVIALITY CONDITION:

$$\lambda + \sup_{t \in [0, T]} \|p(t)\| + \|\gamma\| \neq 0 \quad \text{with} \quad \|\gamma\| := \sup_{\|x\|_{C([0, T])} = 1} \int_{[0, T]} x(s) d\gamma.$$

It is worth mentioning that Theorem 2 does not contain a maximization condition of the PMP. Next we present necessary optimality conditions in the novel *Hamiltonian form*, which is complemented to Theorem 2 and does contain a maximization condition of the new type as the first version of the maximum principle for sweeping process with controlled moving sets. To proceed, consider problem (4.4)–(4.6) with $\Theta = \mathbb{R}^d_-$. The proof from [15] is based on the generalized second-order chain rule given in Section 2 and the precise calculation of the second-order construction $D^*N_{\mathbb{R}^d_-}$ taken from [23]. For $\Theta = \mathbb{R}^d_-$, consider the active index set

$$I(x, u) := \{i \in \{1, \dots, d\} \mid \theta_i(x, u) = 0\}$$

and observe that under the standing surjectivity assumption on $\nabla_x\theta$ for each $v \in -N(x; C(u))$ there exists a unique collection $\{\alpha_i\}_{i \in I(x, u)}$ with $\alpha_i \leq 0$ and $v = \sum_{i \in I(x, u)} \alpha_i [\nabla_x\theta(x, u)]_i$. Given $\nu \in \mathbb{R}^d$, define

$$[\nu, v] := \sum_{i \in I(x, u)} \nu_i \alpha_i [\nabla_x\theta(x, u)]_i$$

and introduce the *new Hamiltonian function* by

$$H_\nu(x, u, p) := \sup \{ \langle [\nu, v], p \rangle \mid v \in -N(x; C(u)) \}. \tag{4.7}$$

Theorem 3. *Let $\Theta = \mathbb{R}^d_-$ in (4.4)–(4.6). Then in addition to Theorem 2 we have the MAXIMIZATION CONDITION*

$$\langle [\nu(t), \dot{\bar{x}}(t)], q^x(t) - \lambda v^x(t) \rangle = H_{\nu(t)}(\bar{x}(t), \bar{u}(t), q^x(t) - \lambda v^x(t)) = 0 \text{ a.e.}$$

holds with a measurable mapping $\nu: [0, T] \rightarrow \mathbb{R}^d$ that for a.e. $t \in [0, T]$ belongs to both sets in the measured coderivative condition of Theorem 2.

Furthermore, it is shown in [15] that a conventional form of the maximum principle with replacing the new Hamiltonian function (4.7) by

$$H(x, u, p) := \sup \{ \langle p, v \rangle \mid v \in -N(x; C(u)) \}$$

fails as a necessary optimality condition in (4.4)–(4.6).

5. Selected Applications

In this final section we discuss selected applications of the necessary optimality conditions for sweeping control systems presented above as well as some directions of future research. The original sweeping control applications in [7;8] concerned some models from *elastoplasticity*. In particular, the reader can find in [8] the results for the *quasistatic elastoplasticity models with hardening*, which are described via the sweeping dynamics.

The applications in [4;6] addressed the corridor and planar versions of the *crowd motion model*. The original developments on the crowd motion model concern local interactions between participants in order to describe the dynamics of pedestrian traffic. Nowadays this model is successfully used to study more general classes of problems in socioeconomics, mechanics, operations research, etc. The microscopic form of the crowd motion model is based on the following two postulates. Firstly, each individual has a *spontaneous* velocity that he/she intends to implement in the absence of other participants. However, in reality the *actual* velocity must be considered. The latter one is incorporated via a projection of the spontaneous velocity into the set of admissible velocities, i.e., those which do not violate certain *nonoverlapping constraints*. A mathematical description of the *uncontrolled* microscopic crowd motion model was given in [27] as a *sweeping process*, and then it was for numerical simulations and various applications.

In [4] we formulated an optimal control problem for the *corridor version* of the crowd motion model, which was described via the sweeping dynamic (1.4) with a *polyhedral* moving of type (3.1). Control problems for the more realistic *planar version* was modeled in [6] in form (1.4) with a nonconvex while *prox-regular* moving set. The obtained necessary optimality conditions led us to developing constructive algorithms to solve such problems with obtaining precise solutions in the case of lower numbers of participants.

The necessary optimality conditions for the sweeping optimal control problem presented in Theorem 1, which is based on [9], were applied in [10] to two practical models written therein in the form of the constrained controlled sweeping process (3.4)–(3.7). The first model is an optimal control version of the *mobile robot model with obstacles* the dynamics of which was described as a sweeping process in [14]. The second one is a continuous-time, deterministic, and optimal control version of the *pedestrian traffic flow model through a doorway* for which a stochastic, discrete-time, and simulation (uncontrolled) counterpart was originated in [17]. The application of

Theorem 1 led us in [10] to complete calculations of optimal solutions for both models in several important settings, but many unsolved issues still remain in further numerical implementations and applications.

The obtained necessary optimality conditions for the sweeping control problem (4.4)–(4.6) and its specifications presented in Section 4 also admit various applications to practical models. We refer the reader to [15] for some applications to *nonpolyhedral* models of *elastoplasticity* and *hysteresis*. The necessary optimality conditions given in Theorems 2 and 3 are used therein for complete calculations of optimal solutions in the controlled hysteresis model the dynamics of which dynamics is described in the sweeping form (4.5). Subsequent applications in this direction, including hysteresis models that arise in problems of *contact and nonsmooth mechanics*, require further elaborations of the results obtained in [15]. Among other future developments we mention *rate-independent* systems arising in hysteresis and related areas. Some of such (uncontrolled) models are formulated in [2; 16] with sweeping process descriptions of their dynamics.

References

1. Arround C.E., Colombo G. A maximum principle for the controlled sweeping process. *Set-Valued Var. Anal.*, 2018, vol. 26, pp. 607-629. <https://doi.org/10.1007/s11228-017-0400-4>
2. Brokate M., Krejčí P. Optimal control of ODE systems involving a rate independent variational inequality. *Disc. Contin. Dyn. Syst. Ser. B*, 2013, vol. 18, pp. 331-348. <https://doi.org/10.3934/dcdsb.2013.18.331>
3. Cao T.H., Mordukhovich B.S. Optimal control of a perturbed sweeping process via discrete approximations. *Disc. Contin. Dyn. Syst. Ser. B*, 2016, vol. 21, pp. 3331-3358. <https://doi.org/10.3934/dcdsb.2016100>
4. Cao T.H., Mordukhovich B.S. Optimality conditions for a controlled sweeping process with applications to the crowd motion model. *Disc. Contin. Dyn. Syst. Ser. B*, 2017, vol. 22, pp. 267-306. https://doi.org/10.1111/resp.13207_486
5. Cao T.H., Mordukhovich B.S. Optimal control of a nonconvex perturbed sweeping process. *J. Diff. Eqs.*, 2019, vol. 266, pp. 1003-1050. <https://doi.org/10.1016/j.jde.2018.07.066>
6. Cao T.H., Mordukhovich B.S. Applications of optimal control of a nonconvex sweeping process to optimization of the planar crowd motion model. *Disc. Contin. Dyn. Syst., Ser. B*, 2019, vol.24, pp. 4191-4216.
7. Colombo G., Henrion R., Hoang N.D., Mordukhovich B.S. Optimal control of the sweeping process. *Dyn. Contin. Discrete Impuls. Syst. Ser. B*, 2012, vol. 19, pp. 117-159.
8. Colombo G., Henrion R., Hoang N.D., Mordukhovich B.S. Optimal control of the sweeping process over polyhedral controlled sets. *J. Diff. Eqs.*, 2016, vol. 260, pp. 3397-3447.
9. Colombo G., Mordukhovich B.S., Nguyen D. Optimization of a perturbed sweeping process by discontinuous controls, to appear in *SIAM J. Control Optim.*; arXiv:1808.04041.
10. Colombo G., Mordukhovich B.S., Nguyen D. Optimal control of sweeping processes in robotics and traffic flow models. *J. Optim. Theory Appl.*, 2019, vol. 182, pp. 439-472.

11. Colombo G., Thibault L. Prox-regular sets and applications. *Handbook of Non-convex Analysis*. D.Y. Gao and D. Motreanu (eds.). Boston, International Press, 2010, pp. 99-182.
12. De Pinho M.d.R., Ferreira M.M.A., Smirnov G.V. Optimal control involving sweeping processes. *Set-Valued Var. Anal.*, 2019, vol. 27, pp. 523-548. <https://doi.org/10.1007/s11228-018-0501-8>
13. Edmond J.F., Thibault L. Relaxation of an optimal control problem involving a perturbed sweeping process. *Math. Program.*, 2005, vol. 104, pp. 347-373. <https://doi.org/10.1007/s10107-005-0619-y>
14. Hedjar R., Bounkhel M. Real-time obstacle avoidance for a swarm of autonomous mobile robots. *Int. J. Adv. Robot. Syst.*, 2014, vol. 11, pp. 1-12. <https://doi.org/10.1016/B978-0-12-417049-0.00001-8>
15. Hoang N.D., Mordukhovich B.S. Extended Euler-Lagrange and Hamiltonian formalisms in optimal control of sweeping processes with controlled sweeping sets. *J. Optim. Theory Appl.*, 2019, vol. 180, pp. 256-289. <https://doi.org/10.1007/s10957-018-1384-4>
16. Krasnosel'skii A.M., Pokrovskii A.V. *Systems with Hysteresis*, Springer, Berlin, 1989. <https://doi.org/10.1007/978-3-642-61302-9>
17. Lovas G.G. Modeling and simulation of pedestrian traffic flow. *Transpn. Res.-B*, 1994, vol. 28B, pp. 429-443. [https://doi.org/10.1016/0191-2615\(94\)90013-2](https://doi.org/10.1016/0191-2615(94)90013-2)
18. Mordukhovich B.S. Sensitivity analysis in nonsmooth optimization, in *Theoretical Aspects of Industrial Design*. Edited by D. A. Field and V. Komkov, *SIAM Proc. Appl. Math.*, 58, pp. 32-46, Philadelphia, Pennsylvania.
19. Mordukhovich B.S. *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Berlin, Springer, 2006. <https://doi.org/10.1007/3-540-31246-3>
20. Mordukhovich B.S. *Variational Analysis and Generalized Differentiation, II: Applications*. Berlin, Springer, 2006. <https://doi.org/10.1007/3-540-31246-3>
21. Mordukhovich B.S. *Variational Analysis and Applications*. New York, Springer, 2018. <https://doi.org/10.1007/978-3-319-92775-6>
22. Mordukhovich B.S. Optimal control of differential inclusions, I: Lipschitzian case. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 30, pp. 45-58. <https://doi.org/10.26516/1997-7670.2019.30.45>
23. Mordukhovich B.S., Outrata J.V. Coderivative analysis of quasi-variational inequalities with applications to stability and optimization. *SIAM J. Optim.*, 2007, vol. 18, pp. 389-412.
24. Mordukhovich B.S., Rockafellar R.T. Second-order subdifferential calculus with applications to tilt stability in optimization. *SIAM J. Optim.*, 2012, vol. 22, pp. 953-986.
25. Moreau J.J. On unilateral constraints, friction and plasticity. *New Variational Techniques in Mathematical Physics, Proceedings from CIME*. G. Capriz and G. Stampacchia (eds.). Cremonese, Rome, 1974, pp. 173-322.
26. Tolstonogov A.A. Control sweeping process. *J. Convex Anal.*, 2016, vol. 23, pp. 1099-1123.
27. Venel J. A numerical scheme for a class of sweeping process. *Numerische Mathematik*, 2011, vol. 118, pp. 451-484. <https://doi.org/10.1182/blood-2011-05-355156>

Boris S. Mordukhovich, Ph.D. in Applied Mathematics, Distinguished University Professor, Department of Mathematics, Wayne State University, Detroit, Michigan, 48202, USA, e-mail: boris@math.wayne.edu,
ORCID iD <https://orcid.org/0000-0002-6071-6049>.

Received 04.11.19

Оптимальное управление дифференциальными включениями, II: процессы выметания

Б. Ш. Мордухович

Государственный университет Уэйна, Детройт, США

Аннотация. Статья посвящена оптимальному управлению динамическими системами, управляемыми дифференциальными включениями с разрывными отображениями скорости. Эта структура в основном касается нового класса задач оптимального управления, описываемых различными версиями так называемых процессов выметания/Моро, которые являются математически очень сложными и очень важными в приложениях к механике, технике, экономике, робототехнике и т. д. Наш подход основан на разработке метода дискретных приближений для задач оптимального управления такими дифференциальными включениями, который затрагивает как численные, так и качественные аспекты оптимального управления. Таким образом, мы устанавливаем необходимые условия оптимальности для оптимальных решений дифференциальных включений и обсуждаем их различные применения. Для получения необходимых условий оптимальности активно используются продвинутые инструменты вариационного анализа первого и второго порядка и обобщенного дифференцирования.

Ключевые слова: оптимальное управление, дифференциальные включения, вариационный анализ, процессы выметания, дискретные аппроксимации, обобщенное дифференцирование.

Борис Шолимович Мордухович, доктор физико-математических наук, профессор, математический факультет, Государственный университет Уэйна, США, 48202, Мичиган, Детройт, e-mail: boris@math.wayne.edu, ORCID iD <https://orcid.org/0000-0002-6071-6049>.

Поступила в редакцию 04.11.2019