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The Completeness Criterion for Closure Operator with the Equality Predicate Branching on the Set of Multioperations on Two-Element Set

V. I. Panteleev
Irkutsk State University, Irkutsk, Russian Federation

L. V. Riabets
Irkutsk State University, Irkutsk, Russian Federation

Abstract. Multioperations are operations from a finite set A to set of all subsets of A. The usual composition operator leads to a continuum of closed sets. Therefore, the research of closure operators, which contain composition and other operations becomes necessary. In the paper, the closure of multioperations that can be obtained using the operations of adding dummy variables, identifying variables, composition operator, and operator with the equality predicate branching is studied. We obtain eleven precomplete closed classes of multioperations of rank 2 and prove the completeness criterion. The diagram of inclusions for one of the precomplete class is presented.

Keywords: closure, equality predicate, multioperation, closed set, composition, completeness criterion.

1. Introduction

Discrete functions defined on a finite set A and taking values in the set of subsets of A are widely considered as a generalization of classical functional systems of k-valued logic. Partial Boolean functions, hyperfunctions, and multifunctions with respect to the composition operator were studied in [2–6; 12; 13; 15].

The superposition operator leads, as a rule, to a countable or continuous classification; therefore, closure operators that generate finite classifications of functions are of interest. Such operators, in particular, include the parametric and positive closure operators [7], the operator with the equality predicate branching (E-closure operator) [8]. An investigation of the last operator on the set of Boolean functions, partial Boolean functions and on the set of functions of k-valued logic can be found in [8–10]. All E-closed classes for the set of partial Boolean functions were obtained in [11]. The completeness criterion for the E-closing operator on the set of hyperfunctions of rank two was proved in [14].

Suppose $E_2 = \{0,1\}$ and $\alpha_i \in E_2,\ i \in \{1,\dots,n\}$; then $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is called a binary set or just a set and is denoted by $\tilde{\alpha}$. Let n be a length of $\tilde{\alpha}$. If the length of the binary set $\tilde{\alpha}$ is not indicated, it is determined by context.

By M_2 denote a set of all rank two multioperations, and it defines as follows

$$M_{2,n} = \left\{f \mid f : E_2^n \to 2^{E_2}\right\},\ M_2 = \bigcup_n M_{2,n}.$$

In what follows, we will not distinguish between a set of one element and an element of this set. For the set E_2, we will use the notation "$-$" (dash) and empty set we will denote as "$*$". Instead of term the "multioperation", sometimes, we will use the word "operation" if this does not confuse.

The set M_2 contains the set of hyperfunctions (H_2), the set of partial Boolean functions (O_2^*), and all Boolean functions (O_2):

$$H_{2,n} = \left\{f \mid f : E_2^n \to 2^{E_2} \setminus \{\varnothing\}\right\},\ H_2 = \bigcup_n H_{2,n};$$

$$\mathrm{O}_{2,n}^* = \{f \mid f : E_2^n \to E_2 \cup \{\varnothing\}\},\ \mathrm{O}_2^* = \bigcup_n \mathrm{O}_{2,n}^*;$$

$$\mathrm{O}_{2,n} = \{f \mid f : E_2^n \to E_2\},\ \mathrm{O}_2 = \bigcup_n \mathrm{O}_{2,n}.$$

The n-variable multioperation f will be represented as a vector $(\tau_{\tilde{0}}, \dots, \tau_{\tilde{1}})$, where $\tau_{\tilde{\sigma}}$ equals to $f(\tilde{\sigma})$. Such vectors have the form $\big(f(0)\, f(1)\big)$ and $\big(f(0,0)\ f(0,1)\ f(1,0)\ f(1,1)\big)$ for unary and binary multioperations respectively.

Suppose $f(x_1, \dots, x_n),\ f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$ are multioperations.

The $g(x_1, \dots, x_m)$ is said to be S_U-composition of

$$f\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

if

$$g(\alpha_1, \dots, \alpha_m) = \bigcup_{\beta_i \in f_i(\alpha_1, \dots, \alpha_m)} f(\beta_1, \dots, \beta_n),$$

where $(\alpha_1, \dots, \alpha_m) \in E_2^m$.

The composition operator defined in that way allows us to find multioperation values on the subsets of 2^{E_2}. Moreover, we consider the element of such sets as a constant function.

Example 1. $f(0,-,1) = f(0,0,1) \cup f(0,1,1)$ and $f(0,*,1) = *$.

We say that the multioperation $g(x_1,\dots,x_n)$ is obtained from the functions $f_1(x_1,\dots,x_n)$, $f_2(x_1,\dots,x_n)$ using the operator with the equality predicate branching (E-operator) if for some $i,j \in \{1,\dots,n\}$ the following relation holds:

$$g(x_1,\dots,x_n) = \begin{cases} f_1(x_1,\dots,x_n), & \text{if } x_i = x_j; \\ f_2(x_1,\dots,x_n), & \text{otherwise.} \end{cases}$$

The set of all multioperations that can be obtained from the set $Q \subseteq M_2$ using the operations of adding dummy variables, identifying variables, S_U-composition and E-operator is called ES_U-closure of set Q and is denoted by $[Q]$.

The function obtained by adding dummy variables will be denoted by the same symbol as the original one. Thus, if $g(x)$ is some multioperation, then $g(x,y)$ is the operation obtained by adding the dummy variable y. In the future, we will not discuss this separately.

A set of multioperations that coincides with its closure is called an ES_U-closed class. We say that the set $P \subseteq Q$ generates an ES_U-closed class Q if $[P] = Q$. Therefore P is ES_U-complete in Q.

The $P \subset M_2$ is said to be the precomplete set in M_2 if $[P] \subset M_2$, but $[P \cup f] = M_2$ for any $f \notin P$.

Let R^m be an m-place predicate on 2^{E_2} of the form

$$R^m = \{(\alpha_{11},\dots,\alpha_{1m}), (\alpha_{21},\dots,\alpha_{2m}), \dots (\alpha_{p1},\dots,\alpha_{pm})\}.$$

The multioperation $f(x_1,...,x_n)$ preserves predicate R^m if for any n sets

$$(\beta_{11},\dots,\beta_{1m}),\dots,(\beta_{n1},\dots,\beta_{nm})$$

from the predicate R^m, the set

$$\big(f(\beta_{11},\dots,\beta_{1m}),\dots,f(\beta_{n1},\dots,\beta_{nm})\big)$$

belongs to R^m.

Pol R denotes the set of multioperations preserving R. Moreover, m-place predicate containing n sets will be set by a $m \times n$ matrix, in which the columns are sets from the predicate.

In the general case, the set of multioperations preserving a particular predicate is not necessarily closed with respect to composition. But the following lemma holds.

Lemma 1. *If the operation f is obtained by a composition of operations g, g_1, …, g_m that preserve some predicate R, then operation f on binary sets from R will necessarily return a set (not necessarily binary) from the predicate.*

2. ES_U-closed classes

Consider the following 11 sets of multioperations:
$K_1 = \{f \mid f(0,\dots,0) \in \{0,-\}\}$; $K_2 = \{f \mid f(1,\dots,1) \in \{1,-\}\}$;
$K_3 = \{f \mid f(0,\dots,0) \in \{0,*\}\}$; $K_4 = \{f \mid f(1,\dots,1) \in \{1,*\}\}$;
$K_5 = O_2^*$; $K_6 = H_2$; $K_7 = \{f \mid f(\tilde{\alpha}) \in \{*,1,-\}\}$;
$K_8 = \{f \mid f(\tilde{\alpha}) \in \{*,0,-\}\}$; $K_9 = PolR_9$; $R_9 = \begin{pmatrix} 0 & 1 & * & - \\ 1 & 0 & * & - \end{pmatrix}$;
$K_{10} = PolR_{10}$; $R_{10} = \begin{pmatrix} 0 & 1 & * & * & * & * & 0 & 1 & - \\ 1 & 0 & 0 & 1 & - & * & * & * & * \end{pmatrix}$;
$K_{11} = \{f \mid * \in f(0,\dots,0) \cup f(1,\dots,1)$ or $f(0,\dots,0) = 0$ and $f(1,\dots,1) = 1\}$.

Theorem 1. *The sets $K_1 - K_{11}$ are ES_U-closed.*

Proof. It is easily proved that the sets $K_1 - K_8$ are ES_U-closed.

Consider the set K_9. Let multioperations $f, f_1, \dots, f_m$ preserve R_9 and multioperation

$$g(x_1,\dots,x_m) = f(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$

does not preserve the predicate R_9. Then

$$g\begin{pmatrix} 0 & 1 & - \\ 1 & 0 & - \end{pmatrix} \cap \left\{ \begin{matrix} 0 & 1 & * & * & * & - & - & 0 & 1 & - & 0 & 1 \\ 0 & 1 & 0 & 1 & - & 0 & 1 & * & * & * & - & - \end{matrix} \right\} \neq \varnothing.$$

Therefore

$$g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cap \left\{ \begin{matrix} 0 & 1 & * & * & * & - & - & 0 & 1 & - & 0 & 1 \\ 0 & 1 & 0 & 1 & - & 0 & 1 & * & * & * & - & - \end{matrix} \right\} \neq \varnothing.$$

But this contradicts Lemma 1.

Consider the operator of equality predicate branching for K_9.

Let

$$g(x_1,\dots,x_m) = \begin{cases} f_1(x_1,\dots,x_m), & \text{if } x_i = x_j; \\ f_2(x_1,\dots,x_m), & \text{otherwise.} \end{cases}$$

Suppose that g does not preserve the predicate R_9, so

$$g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cap \left\{ \begin{matrix} 0 & 1 & * & * & * & - & - & 0 & 1 & - & 0 & 1 \\ 0 & 1 & 0 & 1 & - & 0 & 1 & * & * & * & - & - \end{matrix} \right\} \neq \varnothing.$$

We see that in 2×12 matrix the elements in $(1, i)$ and $(1, j)$ positions coincide if and only if they coincide in $(2, i)$ and $(2, j)$ positions.

Thus, $g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ coincides with $f_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or coincides with $f_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It contradicts the fact that f_1 and f_2 preserve R_9.

The ES_U-closure of the remaining sets is verified similarly. □

Theorem 2. *For all $K_1, \ldots, K_{11}$ if $i \neq j$, then $K_i \not\subseteq K_j$.*

Proof. The validity of the statement follows from Table 1. There is an unary multioperation f at the i-th row and the j-th column in the table such that $f \in K_i$ and $f \notin K_j$. □

Table 1

The pairwise difference of sets $K_1 - K_{11}$

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}
K_1	$\times$	$0*$	$--$	$--$	$--$	$0*$	00	01	00	00	00
K_2	$*1$	$\times$	11	$--$	$--$	$*1$	01	01	11	$--$	$--$
K_3	$**$	00	$\times$	$0-$	$0-$	$**$	00	01	00	$0-$	00
K_4	$**$	$**$	11	$\times$	-1	$**$	01	01	11	11	11
K_5	$**$	$**$	11	00	$\times$	$**$	01	01	11	11	11
K_6	11	00	11	00	$--$	$\times$	00	11	00	$--$	$--$
K_7	$**$	$**$	$--$	$--$	$--$	$**$	$\times$	11	11	11	-1
K_8	$**$	00	$--$	$--$	$--$	$**$	00	$\times$	00	00	00
K_9	10	10	10	10	$--$	$**$	01	01	$\times$	$--$	$--$
K_{10}	$**$	$**$	10	10	$*-$	$**$	01	01	$-*$	$\times$	10
K_{11}	$**$	$**$	$1*$	$*0$	$*-$	$**$	01	01	$-*$	0111	$\times$

3. Completeness criterion

Let f_{K_i} be a multioperation that does not belong to K_i $(i \in \{1, \ldots, 11\})$.

Lemma 2. $[0, 1, f_{K_5}, f_{K_6}] = M_2$.

Proof. Obviously, $[0, 1, *, -] \subseteq [0, 1, f_{K_5}, f_{K_6}]$. In paper [9] it is proved that $[0, 1, *] = O_2^*$. In [1] it is shown that the set O_2^* is precomplete in M_2 with respect to the composition operator. Thus, the lemma is proved. □

Lemma 3. *If $g_1(x) = (--)$, $g_2(x) = (10)$, then $[g_1, g_2, f_{K_5}, f_{K_6}, f_{K_9}] = M_2$.*

Proof. Substitute $g_1(x)$ and $g_2(x)$ into the multioperation f_{K_9}. Consider the cases when we obtain multioperations $u_1(x) = (0-)$ and $u_2(x) = (-*)$. The other cases reduce to this one or Lemma 2.

Let $v(x_1, x_2) = g_2(x_2, x_2)$. Substituting u_2 and g_2 in v we get

$$v(u_2(x_2), g_2(x_2)) = g_2(u_2(x_2), g_2(x_2)) = (0*).$$

Using the multioperations g_1 and $(0*)$ it is easy to get a constant 0.

Substitute $g_2(x)$ into the multioperation f_{K_6}. We get four operations $s_1(x) = (0*)$, $s_2(x) = (1*)$, $s_3(x) = (-*)$, $s_4(x) = (**)$. Consider the last one. Using the equality predicate branching operator we get

$$t(x_1, x_2) = \begin{cases} g_2(x_1), & \text{if } x_1 = x_2; \\ s_4(x_1), & \text{otherwise}; \end{cases} = (1 * * 0).$$

Thus we have $t((0-)(x_2), (-0)(x_2)) = (11)$. Now, by Lemma 2, we obtain that $[g_1, g_2, f_{K_5}, f_{K_6}, f_{K_9}]$ is complete in M_2. □

Lemma 4. *If* $g_1(x) = (--), g_2(x) = (11)$. *Then* $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_7}]$ *is complete in* M_2.

Proof. It is enough to obtain the constant 0 and use Lemma 2.

The composition $f_{K_2}(1, \ldots, 1)$ defines the unary operation $v_1 = (00)$ or $v_2 = (**)$. There is a binary set $(\alpha_1, \ldots, \alpha_n)$ for the multioperation f_{K_7} such that $f_{K_7}(\alpha_1, \ldots, \alpha_n) = 0$.

Let $h(x) = f_{K_7}(u_1(x), \ldots, u_n(x))$ and

$$u_i(x) = \begin{cases} x_i, \text{ if } \alpha_i = 0; \\ 1, \text{ if } \alpha_i = 1. \end{cases}$$

Then $h(x)$ is one of the following operations:

$$t_1 = (00),\ t_2 = (0*),\ t_3 = (01),\ t_4 = (0-).$$

Clearly, $t_2(g_1(x)) = 0$, therefore, it remains to consider two cases for t_3 and t_4.

Let

$$p(x_1, x_2) = \begin{cases} v_2(x_1), \text{ if } x_1 = x_2; \\ t_3(x_1)\,(\text{or } t_4(x_1)), \text{ otherwise.} \end{cases}$$

The composition $p(g_1(x_2), g_2(x_2))$ defines the constant 0. □

Lemma 5. *If* $g_1(x) = (--), g_2(x) = (1-)$, *then* $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_7}]$ *is complete in* M_2.

Proof. It is enough to obtain constant 0 or constant 1. One of 8 unary multioperations can be obtained by identifying variables in f_{K_2}:

$$h_1 = (00), \quad h_2 = (*0), \quad h_3 = (0*), \quad h_4 = (1*),$$
$$h_5 = (10), \quad h_6 = (-0), \quad h_7 = (**), \quad h_8 = (-*).$$

The first four cases allow us to obtain the necessary constant.

Consider $h_5 = (10)$. In this case, we have following operations $g_3(x) = (0-)$, $g_4(x) = (-0)$, and $g_5(x) = (-1)$. There is a binary set $(\alpha_1, \dots, \alpha_n)$ for the multioperation f_{K_6} such that $f_{K_6}(\alpha_1, \dots, \alpha_n) = *$. Let us consider cases when a composition with an external operation f_{K_6} and internal h_5 (or variables identification) defines unary operations $t_1(x) = (**)$ or $t_2(x) = (*-)$.

Using the equality predicate branching operator from multioperations (1010) and $(* * * *)$ we obtain $u(x_1, x_2) = (1 * * 0)$. Superposition $u(g_3(x_2), g_4(x_2))$ defines the constant 1. To conclude the proof, it remains to note that $h_5(t_2(x_2), g_5(x_2)) = (*0)$. □

Lemma 6. *If* $g_1(x) = (--), g_2(x) = (00)$, *then* $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_9}]$ *is complete in* M_2.

Proof. The proof of this statement is similar to the proof of Lemma 4. □

Lemma 7. *If* $g_1(x) = (--), g_2(x) = (**)$, *then* $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_7}]$ *is complete in* M_2.

Proof. By identifying variables in f_{K_7}, we can obtain a multioperation $h(x_1, x_2)$ such that on the binary sets (01) and (10) it takes one of the following four values: (00), (01), $(0*)$, $(0-)$.

Let

$$t(x_1, x_2) = \begin{cases} *, & \text{if } x_1 = x_2; \\ h(x_1, x_2), & \text{otherwise.} \end{cases}$$

Superposition $t(x_1, g_1(x_1))$ defines unary multioperation $p(x) = (0\beta)$, where $\beta \in \{0, *, 1, -\}$. In the first two cases the validity of the statement follows from Lemma 6. Consider $p_1(x) = (01)$ and $p_2(x) = (0-)$.

Using the equality predicate branching operator from multioperations $p_1(x_2)$ and $(* * * *)$ we obtain $(*10*)$. Superposition $(*10*)(p_1(x_1), g_1(x_1))$ defines multioperation (10). Hence, the set $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_7}]$ is complete by Lemma 3.

Similarly, from p_2 and $*$ we get multioperation $(* - 0 *)$. Superposition $(* - 0 *)(x_1, p_2(x_1))$ defines multioperation $(*0)$. Then we obtain $(* 0 * 0)(p_2(x_2), g_1(x_2)) = (00)$. Finally, we have Lemma 6. □

Lemma 8. *If* $g_1(x) = (--), g_2(x) = (*-)$, *then* $[g_1, g_2, f_{K_2}, f_{K_5}, f_{K_6}, f_{K_7}]$ *is complete in* M_2.

Proof. Using the ideas and techniques from Lemma 7, we can obtain multioperation $(0-)$. □

Theorem 3. *A set of multioperations* R *is* ES_U*-complete if and only if it is not contained entirely in any of the classes* $K_1 - K_{11}$.

Proof. One of 8 unary multioperations can be obtained by identifying variables in $f_{K_{11}}$:

$$f^1_{K_{11}} = (--), \quad f^2_{K_{11}} = (00), \quad f^3_{K_{11}} = (11), \quad f^4_{K_{11}} = (10),$$
$$f^5_{K_{11}} = (0-), \quad f^6_{K_{11}} = (-1), \quad f^7_{K_{11}} = (-0), \quad f^8_{K_{11}} = (1-).$$

Since $f^7_{K_{11}}(f^7_{K_{11}}(x)) = (--)$ and $f^8_{K_{11}}(f^8_{K_{11}}(x)) = (--)$, then it is enough to consider the first six operations.

Case 1. Consider $f^1_{K_{11}} = (--)$. By identifying variables in f_{K_1}, we can obtain one of the following unary multioperations: (10), (11), $(1*)$, $(1-)$, $(*0)$, $(*1)$, $(**)$, $(*-)$.

It is clearly that it is enough to consider only six pairs of multioperations:

$$\{(10),(--)\} \ \{(11),(--)\} \ \{(1-),(--)\}$$
$$\{(00),(--)\} \ \{(**),(--)\} \ \{(*-),(--)\}.$$

However, we have Lemmas 3–8 for all these pairs.

Let us remark that $(*1)(--)(x) = (11)$ and $(*0)(--)(x) = (00)$.

Case 2. Consider $f^2_{K_{11}} = (00)$. Superposition f_{K_3} and $f^2_{K_{11}}$ defines multioperation (11) or $(--)$.

It follows from Lemma 2 that the set $\{(00),(11)\}$ is complete in M_2. For the set of multioperations $\{(00),(--)\}$ we use Lemma 6.

Case 3. Consider $f^3_{K_{11}} = (11)$. Superposition f_{K_4} and $f^3_{K_{11}}$ define multioperation (00) or $(--)$.

For sets $\{(00),(11)\}$ and $\{(11),(--)\}$ we use Lemma 2 and Lemma 4 respectively .

Case 4. Consider $f^4_{K_{11}} = (10)$. There are a pair of binary sets $(\tilde{\alpha})$ and $(\overline{\tilde{\alpha}})$ for the multioperation $f_{K_{10}}$ such that $\big(f_{K_{10}}(\tilde{\alpha})f_{K_{10}}(\overline{\tilde{\alpha}})\big) \in \{(00), (11), (0-), (-0), (1-), (-1), (11)\}$.

Substitute (10) into the multioperation $f_{K_{10}}$. We obtain one of the following unary multioperations: (00), (11), $(0-)$, (-0), $(1-)$, (-1), $(--)$. Using ideas from cases 1–3 for these multioperations, we complete the proof.

Case 5. Consider $f^5_{K_{11}} = (0-)$. Note that $f_{K_3}(0,\dots,0) \in \{1,-\}$. Therefore we have $f_{K_3}(f^5_{K_{11}}) \in \{(11),(1-),(--)\}$. As above, we use ideas from cases 1–3.

Case 6. Consider $f^6_{K_{11}} = (-1)$. We have $f_{K_4}(1,\dots,1) \in \{0,-\}$. Then we can obtain $f_{K_4}(f^6_{K_{11}}) \in \{(00),(-0),(--)\}$ and use cases 1–3.

This completes the proof of the main theorem. □

4. ES_U-closed subsets of K_9

In this section, we show that precomplete set K_9 consists of 20 ES_U-closed subsets.

Lemma 9. *Any ES_U-closed class from M_2 is generated by the set of all its multioperations depending on at most two variables.*

Proof. This Lemma can be proved by methods of the corresponding statement from [9]. □

According to the previous Lemma, it can be easily checked that K_9 contains the following 16 multioperations only:

$$K_9 = \big\{(****), (*--*), (*01*), (*10*), (-**-), (----), (-01-), (-10-), (0**1), (0--1), (0011), (0101), (1**0), (1--0), (1010), (1100)\big\}$$

Lemma 10. $[(****), (1--0)] = K_9$.

Proof. To prove Lemma, we obtain the remaining 14 multioperations using $g_1(x,y) = (****)$ and $g_2(x,y) = (1--0)$.

We get

$$(----) = g_2(x, g_2(x,y)); \quad (10) = g_2(x,x); \quad (01) = g_2(g_2(x,x)).$$

All other multioperations in K_9 are obtained from the following consideration. Let $f_1 = (\alpha_1\alpha_2\alpha_3\alpha_4)$ and $f_2 = (\beta_1\beta_2\beta_3\beta_4)$. If $f_1, f_2 \in K_9$, then we can obtain $f_3 = (\alpha_1\beta_2\beta_3\alpha_4)$ and $f_4 = (\alpha_1\alpha_3\alpha_2\alpha_4)$, where $f_3, f_4 \in K_9$. □

Let $P_1 = \{f \mid f(0,\dots,0) \in \{*\}$ and $P_2 = \{f \mid f(0,\dots,0) \in \{-\}\}$. It is easy to prove that P_1 and P_2 are ES_U-closed sets.

Now consider the following 20 subsets of K_9.

$S_1 = \{(****)\}$; $S_2 = \big\{(----)\big\}$; $S_3 = \big\{(****), (*--*)\big\}$;
$S_4 = \big\{(0011), (0101)\big\}$; $S_5 = \big\{(-**-), (----)\big\}$;
$S_6 = \big\{(****), (*01*), (*10*)\big\}$; $S_7 = \big\{(0**1), (0011), (0101)\big\}$;
$S_8 = \big\{(0--1), (0011), (0101)\big\}$; $S_9 = \big\{(----), (-01-), (-10-)\big\}$;
$S_{10} = \big\{(0011), (0101), (1010), (1100)\big\}$;
$S_{11} = \big\{(****), (*--*), (-**-), (----)\big\}$;
$S_{12} = \big\{(****), (*--*), (*01*), (*10*)\big\}$;
$S_{13} = \big\{(0**1), (0--1), (0011), (0101)\big\}$;
$S_{14} = \big\{(-**-), (----), (-01-), (-10-)\big\}$;
$S_{15} = \big\{(****), (*01*), (*10*), (0**1), (0011), (0101)\big\}$;
$S_{16} = \big\{(----), (-01-), (-10-), (0--1), (0011), (0101)\big\}$;
$S_{17} = \big\{(****), (*--*), (*01*), (*10*), (0**1), (0--1), (0011), (0101)\big\}$;
$S_{18} = \big\{(-**-), (----), (-01-), (-10-), (0**1), (0--1), (0011), (0101)\big\}$;
$S_{19} = \big\{(****), (*01*), (*10*), (0**1), (0011), (0101), (1**0), (1010), (1100)\big\}$;
$S_{20} = \big\{(----), (-01-), (-10-), (0--1), (0011), (0101), (1--0), (1010), (1100)\big\}$;

Theorem 4. *The sets $S_1 - S_{20}$ are ES_U-closed.*

Proof. It is evident that S_1, S_2, S_3, S_4, and S_5 are ES_U-closed. Matveev in [11] proved the ES_U-closure of S_6, S_7, S_{10}, S_{15}, and S_{19}. We define the remaining sets as intersections of known ES_U-closed classes.

$$\begin{array}{ll} S_8 = K_1 \cap K_3 \cap K_6 \cap K_9; & S_9 = P_2 \cap K_6 \cap K_9; \\ S_{11} = K_7 \cap K_9; & S_{12} = P_1 \cap K_9; \\ S_{13} = K_1 \cap K_3 \cap K_9; & S_{14} = P_2 \cap K_9; \\ S_{16} = K_1 \cap K_6 \cap K_9; & S_{17} = K_3 \cap K_9; \\ S_{18} = K_1 \cap K_9; & S_{20} = K_6 \cap K_9. \end{array}$$

It proves that all $S_1 - S_{20}$ are ES_U-closed. □

Remark 1. Closed sets from [11] can be represented as follows.

$$\begin{array}{ll} S_6 = P_1 \cap K_3 \cap K_5 \cap K_9; & S_7 = K_1 \cap K_3 \cap K_5 \cap K_9; \\ S_{10} = K_5 \cap K_6 \cap K_9; & S_{15} = K_3 \cap K_5 \cap K_9; \\ S_{19} = K_5 \cap K_9. & \end{array}$$

Lemma 11. *Let Q be an any ES_U-closed subset of K_9. Let $g(x)$ be an unary multioperation obtained by identifying variables of some*

$$f(x_1, \dots, x_n) \in Q.$$

Then $g(x)$ can be one of the following three multioperations:

$$g(x) = (01),\ g(x) = (**),\ g(x) = (--).$$

Proof. Consider $f(x_1, \dots, x_n) \in Q$. Since $Q \subset K_9$, it follows that $f \in Pol\ R_9$. Therefore,

$$f(x, \dots, x) \in \big\{(**), (--), (01), (10)\big\}.$$

However $(01)(x) = (10)(10)(x)$. □

Corollary 1. *There are three minimal subsets of K_9 only.*

In the following lemmas, we will keep in mind that the sets under consideration are precomplete only in given sets.

Lemma 12. *S_{11} is an ES_U-precomplete set in Q iff $[Q] = K_9$.*

Proof. Since $(****) \in S_{11}$, we need to obtain the multioperation $(1--0)$. We consider various cases for $f_{S_{11}}$.

Let $f_{S_{11}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Hence,

$$(1--0) = \begin{cases} f_{S_{11}}(x, y), & \text{if } x = y; \\ (----)(x, y), & \text{otherwise.} \end{cases}$$

For the set of multioperations $\{(****), (1--0)\}$ we use Lemma 10.

Let $f_{S_{11}} = (0\alpha_1\alpha_2 1)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Now we obtain

$$(*01*) = \begin{cases} (****)(x, y), & \text{if } x = y; \\ f_{S_{11}}(x, x), & \text{otherwise}; \end{cases}$$

and

$$(1\beta_1\beta_2 0) = (*01*)\big((----)(x, y), f_{S_{11}}(x, x)\big),$$

where $\beta_1, \beta_2 \in \{*, 0, 1, -\}$. Then we use the previous case.

Let $f_{S_{11}} = (\alpha_1 01\alpha_2)$, where $\alpha_1, \alpha_2 \in \{*, -\}$. We obtain

$$(*01*) = \begin{cases} (****)(x, y), & \text{if } x = y; \\ f_{S_{11}}(x, y), & \text{otherwise}; \end{cases}$$

and

$$(1**0) = (*01*)\big((-**-)(x, y), x\big).$$

Then we use the previous case.

Finally, let $f_{S_{11}} = (\alpha_1 10\alpha_2)$, where $\alpha_1, \alpha_2 \in \{*, -\}$. We can obtain

$$(\alpha_1 01\alpha_2) = f_{S_{11}}\big(y, x\big).$$

This completes the proof. □

Lemma 13. S_{17} *is an* ES_U*-precomplete set in* Q *iff* $[Q] = K_9$.

Proof. Since $(****) \in S_{17}$, we need to obtain the multioperation $(1--0)$. Let $f_{S_{17}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Then

$$(1--0) = \begin{cases} f_{S_{17}}(x, y), & \text{if } x = y; \\ (*--*)(x, y), & \text{otherwise}. \end{cases}$$

Now we get ES_U-complete set in K_9.

Let $f_{S_{17}} = (-\alpha_1\alpha_2-)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Using the operator with the equality predicate branching, we get $(-**-)$ and then

$$(1**0) = (*01*)\big((-**-)(x, y), x\big).$$

Finally, we use ideas from Lemma 12. □

Lemma 14. S_{18} *is an* ES_U*-precomplete set in* Q *iff* $[Q] = K_9$.

Proof. Let $f_{S_{18}} = (*\alpha_1\alpha_2*)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. The multioperation $(**)$ can be obtained by identifying variables in $f_{S_{18}}$. Using the operator with the equality predicate branching, we obtain $(*01*)$ and then we use ideas from the final part of Lemma 12.

Let $f_{S_{18}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Then we obtain

$$(*\beta_1\beta_2*) = (0**1)\big(x, f_{S_{18}}(x, y)\big)$$

and use the previous case. □

Lemma 15. *S_{19} is an ES_U-precomplete set in Q iff $[Q] = K_9$.*

Proof. The multioperation $(--)$ can be obtained by identifying variables in $f_{S_{19}}$. Using $(1**0)$, we get

$$(1--0) = \begin{cases} (1**0), & \text{if } x = y; \\ (--)(x), & \text{otherwise.} \end{cases} \qquad \square$$

Lemma 16. *S_{20} is an ES_U-precomplete set in Q iff $[Q] = K_9$.*

Proof. The proof is trivial. $\square$

Lemma 17. *S_{12} is an ES_U-precomplete set in Q iff $[Q] = S_{17}$.*

Proof. Note that S_{12} contains four multioperations from S_{17}. Let $f_{S_{12}} = (0\alpha_1\alpha_2 1)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Using the operator with the equality predicate branching, we can get the remaining four multioperations.

If $f_{S_{12}} = (-\alpha_1\alpha_2-)$, then using E-operator, we obtain the whole set of S_{11} and some other operations. By Lemma 12, we get K_9.

If $f_{S_{12}} = (1\alpha_1\alpha_2 0)$, then using E-operator, we obtain the whole set of S_{19} and some other operations. So, by Lemma 15, we get K_9. $\square$

Lemma 18. *S_{15} is an ES_U-precomplete set in Q iff $[Q] = S_{17}$ or $[Q] = S_{19}$.*

Proof. Consider cases for $f_{S_{15}}$.

Case 1. Let $f_{S_{15}} \notin K_3$ and $f_{S_{15}} \notin K_5$. If $f_{S_{15}} = (1--0)$, then we have a complete set for K_9. If $f_{S_{15}} = (-\alpha_1\alpha_2-)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$, then using E-operator, we obtain the whole set of S_{11}. Therefore we get K_9.

Case 2. Let $f_{S_{15}} \in K_3$ and $f_{S_{15}} \notin K_5$. If $f_{S_{15}} = (0--1)$, then we obtain $(*--*)$. It gives us S_{17}. Similarly, for $f_{S_{15}} = (*--*)$ we obtain S_{17}.

Case 3. Let $f_{S_{15}} \notin K_3$ and $f_{S_{15}} \in K_5$. Let $f_{S_{15}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1\}$. Using E-operator, we obtain the whole set of S_{19}. $\square$

Lemma 19. *S_{13} is an ES_U-precomplete set in Q iff $[Q] = S_{17}$ or $[Q] = S_{18}$.*

Proof. Case 1. Let $f_{S_{13}} \notin K_1$ and $f_{S_{13}} \notin K_3$. Let $f_{S_{13}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$, then we obtain

$$(*\beta_1\beta_2*) = (0**1)\big(x, f_{S_{13}}(x, y)\big)$$

and

$$(1--0) = \begin{cases} f_{S_{13}}(x, y), & \text{if } x = y; \\ (0--1)(x, y), & \text{otherwise;} \end{cases}$$

Therefore we get K_9.

Case 2. Let $f_{S_{13}} \in K_1$ and $f_{S_{13}} \notin K_3$. Let $f_{S_{13}} = (-\alpha_1\alpha_2-)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Using E-operator, we can get the remaining four multioperations for S_{17}.

Case 3. Let $f_{S_{13}} \notin K_1$ and $f_{S_{13}} \in K_3$. Let $f_{S_{13}} = (*\alpha_1\alpha_2*)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Using E-operator, we can get the remaining four multioperations for S_{18}. □

Lemma 20. *S_{14} is an ES_U-precomplete set in Q iff $[Q] = S_{18}$.*

Proof. Case 1. Let $f_{S_{14}} \notin P_2$ and $f_{S_{14}} \in K_1$. Let $f_{S_{14}} = (0\alpha_1\alpha_2 1)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$. Using E-operator, we can get the remaining four multioperations for S_{18}.

Case 2. Let $f_{S_{14}} \notin K_1$. If $f_{S_{14}} = (*\alpha_1\alpha_2*)$, then using E-operator we obtain the whole set of S_{11}. Adding $(-01-)$ to S_{11}, we get K_9.

If $f_{S_{14}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1, -\}$, then we obtain

$$(*\beta_1\beta_2*) = (- * * -)\big(x, f_{S_{14}}(x, y)\big)$$

and

$$(1 - - 0) = \begin{cases} f_{S_{14}}(x, y), & \text{if } x = y; \\ (- - - -)(x, y), & \text{otherwise.} \end{cases}$$

Therefore we get K_9. □

Lemma 21. *S_{16} is an ES_U-precomplete set in Q iff $[Q] = S_{18}$ or $[Q] = S_{20}$.*

Proof. Consider the various cases for $f_{S_{16}}$.

Case 1. Let $f_{S_{16}} \notin K_1$ and $f_{S_{16}} \notin K_6$. If $f_{S_{16}} = (1 * * 0)$, then we obtain

$$(* * * *) = f_{S_{16}}\big(x, f_{S_{16}}(x, y)\big)$$

and

$$(1 - - 0) = \begin{cases} f_{S_{16}}(x, y), & \text{if } x = y; \\ (0 - -1)(x, y), & \text{otherwise.} \end{cases}$$

Therefore we get K_9.

If $f_{S_{16}} = (*\alpha_1\alpha_2*)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1\}$. The multioperation $(**)$ can be obtained by identifying variables in $f_{S_{16}}$. Using (0101), we get

$$(*10*) = \begin{cases} f_{S_{16}}(x, y), & \text{if } x = y; \\ (0101)(x, y), & \text{otherwise;} \end{cases}$$

and

$$(1 - - 0) = (*10*)\big((0 - - 1)(x, y), (- - - -)(x, y)\big).$$

Now we get K_9.

Case 2. Let $f_{S_{16}} \in K_1$ and $f_{S_{16}} \notin K_6$. Let $f_{S_{16}} = (0{*}{*}1)$, then we obtain $(-{*}{*}-)$. It gives us S_{18}.

Case 3. Let $f_{S_{16}} \notin K_1$ and $f_{S_{16}} \in K_6$. Let $f_{S_{13}} = (1\alpha_1\alpha_2 0)$, where $\alpha_1, \alpha_2 \in \{0, 1, -\}$. Using E-operator, we can get the remaining three multioperations for S_{20}. □

Proposition 1. $[(1 * *0)] = S_{19}$.

Proposition 2. $[(1 - -0)] = S_{20}$.

Propositions 1 and 2 were proved in [11] and [14] respectively.

Lemma 22. *S_{10} is an ES_U-precomplete set in Q iff $[Q] = S_{19}$ or $[Q] = S_{20}$.*

Proof. Consider the various cases for $f_{S_{10}}$.

Case 1. Let $f_{S_{10}} \notin K_5$ and $f_{S_{10}} \notin K_6$. If $f_{S_{10}} = (- * * -)$, then we obtain

$$(* * * *) = f_{S_{10}}\big(x, (1100)(x, y)\big).$$

Moreover,

$$(-10-) = \begin{cases} f_{S_{10}}(x, y), & \text{if } x = y; \\ (1100)(x, y), & \text{otherwise}; \end{cases}$$

and

$$(1 - - 0) = (-10-)\big(x, (1010)(x, y)\big).$$

It follows that we get K_9.

If $f_{S_{10}} = (* - - *)$. The multioperation $(**)$ can be obtained by identifying variables in $f_{S_{10}}$. Using (1010), we get

$$(1 - - 0) = \begin{cases} (1010)(x, y), & \text{if } x = y; \\ f_{S_{10}}(x, y), & \text{otherwise}. \end{cases}$$

Now we get K_9.

Case 2. Let $f_{S_{10}} \in K_5$ and $f_{S_{10}} \notin K_6$. If $f_{S_{10}} = (*\alpha_1\alpha_2*)$, where $\alpha_1, \alpha_2 \in \{*, 0, 1\}$. The multioperation $(**)$ can be obtained by identifying variables in $f_{S_{10}}$. Moreover,

$$(1 * * 0) = \begin{cases} (1010)(x, y), & \text{if } x = y; \\ (**)(x), & \text{otherwise}. \end{cases}$$

Thus we have ES_U-complete set for S_{19} (see Proposition 1).

Trivially, if $f_{S_{10}} = (0 * *1)$ or $f_{S_{10}} = (1 * *0)$ we get S_{19}.

Case 3. Let $f_{S_{10}} \notin K_5$ and $f_{S_{10}} \in K_6$. This case is similar to the previous one. Using Proposition 2, we get S_{20}. □

Lemma 23. *The following statements are true.*

1) S_1 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_3$ *or* $[Q] = S_6$;
2) S_2 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_5$ *or* $[Q] = S_9$;
3) S_3 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{11}$ *or* $[Q] = S_{12}$;
4) S_4 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_7$, *or* $[Q] = S_8$, *or* $[Q] = S_{10}$;
5) S_5 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{11}$ *or* $[Q] = S_{14}$;
6) S_6 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{12}$ *or* $[Q] = S_{15}$;
7) S_7 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{13}$ *or* $[Q] = S_{15}$;
8) S_8 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{13}$ *or* $[Q] = S_{16}$;
9) S_9 *is an* ES_U*-precomplete set in* Q *iff* $[Q] = S_{14}$ *or* $[Q] = S_{16}$.

The proof is omitted. It uses the ideas and techniques from previous Lemmas.

Theorem 5. *The* ES_U*-precomplete set* K_9 *consists of 20* ES_U*-closed subsets.*

In conclusion, we present the structure of K_9 in diagram 1.

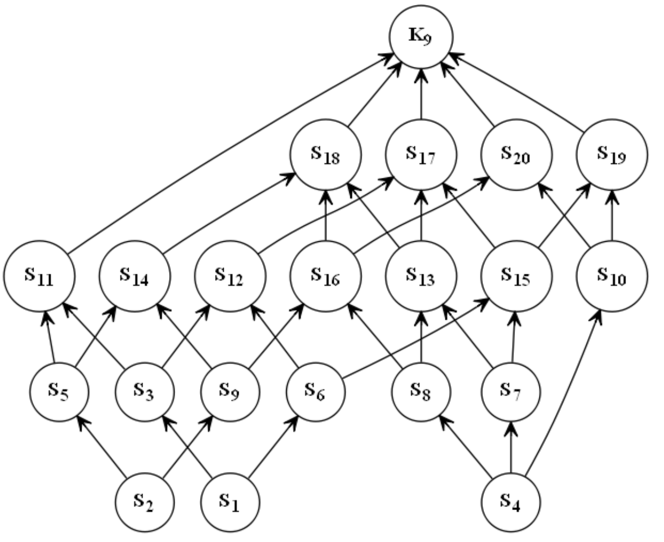

Figure 1. ES_U-closed subsets of K_9

5. Conclusion

In this paper, we considered precomplete ES_U-closed classes of multioperations defined on a 2-element set. The next steps are to obtain all ES_U-closed classes of M_2 and to determine properties of multioperations defined on a 3-element set.

References

1. Doroslovački R., Pantović J., Vojvodić G. One Interval in the Lattice of Partial Hyperclones. *Chechoslovak Mathematical Journal*, 2005, no. 55(3), pp. 719-724. https://doi.org/10.1007/s10587-005-0059-0.
2. Lau D. *Function Algebras on Finite Sets. A basic course on many-valued logic and clone theory.* Berlin, Springer-Verlag, 2006, 668 p.
3. Lo Czu Kai. Maximal closed classes on the Set of Partial Many-valued Logic Functions. *Kiberneticheskiy Sbornik*, Moscow, Mir Publ., 1988, vol. 25, pp. 131–141. (in Russian)
4. Lo Czu Kai. Completeness theory on Partial Many-valued Logic Functions. *Kiberneticheskiy Sbornik*, Moscow, Mir Publ., 1988, vol. 25, pp. 142-157. (in Russian)
5. Machida H. Hyperclones on a Two-Element Set. *Multiple-Valued Logic. An International Journal*, 2002, no. 8(4), pp. 495–501. https://doi.org/10.1080/10236620215294.
6. Machida H., Pantovic J. On Maximal Hyperclones on $\{0, 1\}$ — a new approach. *Proceedings of 38th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2008)*, 2008, pp. 32-37.
7. Marchenkov S.S. On the Expressibility of Functions of Many-Valued Logic in Some Logical-Functional Classes. *Discrete Math. Appl.*, 1999, vol. 9, no. 6, pp. 563-581.
8. Marchenkov S.S. Closure Operators with Predicate Branching. *Bulletin of Moscow State University. Series 1. Mathematics and Mechanics*, 2003, no. 6, pp. 37–39. (in Russian)
9. Marchenkov S.S. The Closure Operator with the Equality Predicate Branching on the Set of Partial Boolean Functions. *Discrete Math. Appl.*, 2008, vol. 18, no. 4, pp. 381-389. https://doi.org/10.1515/DMA.2008.028.
10. Marchenkov S.S. The E-closure Operator on the Set of Partial Many-Valued Logic Functions. *Mathematical problems in cybernetics*, Moscow, Fizmatlit, 2013, vol. 19, pp. 227–238. (in Russian)
11. Matveev S.S. Construction of All E-closed Classes of Partial Boolean Functions. *Mathematical problems in cybernetics*, Moscow, Fizmatlit Publ., 2013, vol. 18, pp. 239-244. (in Russian)
12. Panteleev V.I. The Completeness Criterion for Depredating Boolean Functions. *Vestnik of Samara State University. Natural Science Series.*, 2009, no. 2 (68), pp. 60-79. (in Russian)
13. Panteleev V.I. Completeness Criterion for Underdetermined Partial Boolean Functions. *Vestnik Novosibirsk State University. Series Mathematics*, 2009, vol. 9, no. 3, pp. 95-114. (in Russian)
14. Panteleev V.I., Riabets L.V. The Closure Operator with the Equality Predicate Branching on the Set of Hyperfunctions on Two-Element Set. *The Bulletin of Irkutsk State University. Series Mathematics*, 2014, vol. 10, pp. 93-105. (in Russian)

15. Romov B.A. Hyperclones on a Finite Set. *Multiple-Valued Logic. An International Journal*, 1998, vol. 3(2), pp. 285-300.

Vladimir Panteleev, Doctor of Sciences (Physics and Mathematics), Professor, Institute of Mathematics, Economics and Informatics, Irkutsk State University, Russian Federation, 664000, Irkutsk, K. Marx st., 1, tel.:+7(3952)242214 (e-mail: vl.panteleyev@gmail.com)

Leonid Riabets, Candidate of Sciences (Physics and Mathematics), Associate Professor, Institute of Mathematics, Economics and Informatics, Irkutsk State University, Russian Federation, 664000, Irkutsk, K. Marx st., 1, tel.:+7(3952)242214 (e-mail: l.riabets@gmail.com)

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Критерий полноты для оператора замыкания с разветвлением по предикату равенства на множестве мультиопераций ранга 2

В. И. Пантелеев, Л. В. Рябец

Аннотация. Мультиоперации представляют отображения, задаваемые на конечном множестве и возвращающие в качестве своих значений все подмножества рассматриваемого множества. Оператор суперпозиции приводит к континууму замкнутых множеств. Поэтому возникает необходимость рассмотрения операторов замыкания, которые наряду с суперпозицией содержат другие операции. В работе рассматривается замыкание мультиопераций, полученное применением оператора суперпозиции, основанной на объединении, оператора разветвления по предикату равенства. Для мультиопераций, задаваемых на двухэлементном множестве, указаны все предполные множества, сформулирован и доказан критерий полноты. Приведена диаграмма включений замкнутых классов для одного из предполных классов.

Ключевые слова: замыкание, предикат равенства, мультиоперация, замкнутое множество, суперпозиция, критерий полноты.

Список литературы

1. Doroslovački R., Pantović J., Vojvodić G. One Interval in the Lattice of Partial Hyperclone // Chechoslovak Mathematical Journal. 2005. N 55(3). P. 719–724. https://doi.org/10.1007/s10587-005-0059-0
2. Lau D. Function Algebras on Finite Sets. A basic course on many-valued logic and clone theory. Berlin : Springer-Verlag, 2006. 668 p.
3. Ло Джукай. Максимальные замкнутые классы в множестве частичных функций многозначной логики // Кибернетический сборник. Новая серия. М. : Мир, 1988. Вып. 25. С. 131–141.
4. Ло Джукай. Теория полноты для частичных функций многозначной логики // Кибернетический сборник. Новая серия. М. : Мир, 1988. Вып. 25. С. 142–157.

Известия Иркутского государственного университета.
2019. Т. 29. Серия «Математика». С. 68–85

5. Machida H. Hyperclones on a Two-Element Set // Multiple-Valued Logic. An International Journal. 2002. N 8(4). P. 495–501. https://doi.org/10.1080/10236620215294
6. Machida H., Pantovic J. On Maximal Hyperclones on $\{0, 1\}$ — a new approach // Proceedings of 38th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2008). 2008. P. 32–37.
7. Марченков С. С. О выразимости функций многозначной логики в некоторых логико-функциональных языках // Дискретная математика. 1999. Т. 11, вып. 4. С. 110–126. https://doi.org/10.4213/dm400
8. Марченков С. С. Операторы замыкания с разветвлением по предикату // Вестник МГУ. Сер. 1, Математика и механика. 2003. № 6. С. 37–39.
9. Марченков С. С. Оператор замыкания с разветвлением по предикату равенства на множестве частичных булевых функций // Дискретная математика. 2008. Т. 20, вып. 3. С. 80–88. https://doi.org/10.4213/dm1015
10. Марченков С. С. Оператор E-замыкания на множестве частичных функций многозначной логики // Математические вопросы кибернетики. М. : Физматлит, 2013. Вып. 19. С. 227–238.
11. Матвеев С. А. Построение всех E-замкнутых классов частичных булевых функций // Математические вопросы кибернетики. М. : Физматлит, 2013. Вып. 18. С. 239–244.
12. Пантелеев В. И. Критерий полноты для доопределяемых булевых функций // Вестник Самарского государственного университета. Естественнонаучная серия. 2009. № 2 (68). С.60–79.
13. Пантелеев В. И. Критерий полноты для недоопределенных частичных булевых функций // Вестник НГУ. Сер. Математика, механика, информатика. 2009. Т. 9, вып. 3. С. 95–114.
14. Пантелеев В. И., Рябец Л. В. Оператор замыкания с разветвлением по предикату равенства на множестве гиперфункций ранга 2 // Известия Иркутского государственного университета. Сер. Математика. 2014. Т. 10. С. 93–105.
15. Romov B. A. Hyperclones on a Finite Set // Multiple-Valued Logic. An International Journal. 1998. Vol. 3(2). P. 285–300.

Владимир Иннокентьевич Пантелеев, доктор физико-математических наук, профессор, Иркутский государственный университет, Российская Федерация, 664000, Иркутск, ул. К. Маркса, 1
тел.: +7(3952)242214 (e-mail: vl.panteleyev@gmail.com)

Леонид Владимирович Рябец, кандидат физико-математических наук, доцент, Иркутский государственный университет, Российская Федерация, 664000, Иркутск, ул. К. Маркса, 1 тел.: +7(3952)242214 (e-mail: l.riabets@gmail.com)

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