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Generalized Reduced Mal'tsev Problem on Commutative Subalgebras of E_6 Type Chevalley Algebras over a Field

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Abstract. In 1905 I. Shur pointed out the largest dimension of commutative subgroups in the groups $SL(n, \mathbb{C})$ and proved that for $n > 3$ such the subgroups are automorphic to each other. In 1945 A.I. Mal'tsev investigated the problem of description of the largest dimension commutative subgroups in the simple complex Lie groups. He solved the problem by the transition to the complex Lie algebras and by the reduction to the same problem for the maximal nilpotent subalgebra. Let N be a niltriangular subalgebra of a Chevalley algebra. The article deals with the problem of describing the largest dimension commutative subalgebras of N over an arbitrary field. The solution of this problem is obtained for the subalgebra N of E_6 type Chevalley algebra.

Keywords: Chevalley algebra, niltriangular subalgebra, largest dimension commutative subalgebra.

1. Introduction

In 1905 I. Shur [9] pointed out the largest dimension of commutative subgroups in the group $SL(n, \mathbb{C})$ and proved that for $n > 3$ such the subgroups are automorphic to each other. In 1945 A. I. Mal'tsev investigated the problem of description of the largest dimension commutative subgroups in all finite dimension simple Lie groups using the transition to complex Lie algebras.

Let $L_\Phi(K)$ be a Chevalley algebra over an arbitrary field K associated with the root system Φ, [2]. The Chevalley base in this algebra consists of the elements e_r ($r \in \Phi$) and the appropriate base of Cartan subalgebra. The elements e_r ($r \in \Phi^+$) form the base on niltriangular subalgebra $N\Phi(K)$.

A. I. Mal'tsev solved his problem by reduction to analogous problem for the Lie algebras $N\Phi(\mathbb{C})$. Later his methods were modified and applied to description of maximal order commutative subgroups of a finite Chevalley group and to reduced problem for an unipotent radical U of its Borel subgroup [4]– [10], [1] – [7].

The following problems were stated in [6]:

Generalized Mal'tsev's problem: *Describe the largest dimension commutative subalgebras of Chevalley algebra over arbitrary field.*

Generalized reduced problem: *Describe the largest dimension commutative subalgebras in the subalgebra $N\Phi(K)$ of Chevalley algebra over arbitrary field.*

The generalized reduced problem was studied in [3] and such the hypotheses was confirmed: any largest dimension commutative ideal of the algebra $N\Phi(K)$ is its largest dimension commutative subalgebra. We proved (Theorem 1) that the algebra $N\Phi(K)$ of E_6 type has no another largest dimension commutative subalgebras.

2. Preliminary remarks

The structure constants of the algebra $N\Phi(K)$ are determined by Chevalley theorem on base:

if $r+s\in\Phi^+$ then $e_r * e_s = N_{rs}e_{r+s}$, $N_{sr}=-N_{rs}$;

if $r+s\notin\Phi^+$, then $e_r * e_s = 0$.

The coefficients N_{rs} in the algebra $N\Phi(K)$ of E_6 type are equal only ± 1.

The sum $\mathrm{ht}(r)$ of coefficients in the root r base decomposition is called the *root height*. The subset Ψ of the root system Φ is called *commutative* if for any two roots $r, s\in\Psi$ holds $r+s\notin\Phi$. Let $\{r\}^+$ be the set of roots $s\in\Phi^+$ such that the base decomposition of the root $s-r$ contains only non-negative coefficients. Let, further, $T(r)$ and $Q(r)$ be the subalgebras of $N\Phi(K)$ with the bases $\{e_s | s\in\{r\}^+\}$ and $\{e_s \mid s\in\{r\}^+\setminus\{r\}\}$, respectively.

If the condition $H\subseteq T(r_1)+T(r_2)+\cdots+T(r_m)$ holds and any substitution of $T(r_i)$ for $Q(r_i)$ leads to wrong inclusion then the set

$$\{r_1, r_2, \ldots, r_m\} = \mathcal{L}(H)$$

is called the *corner set for H*.

Let the regular ordering is determined for the positive root system Φ^+. Any element $a\in N\Phi(K)$ may be written as a sum $a=a_1e_{r_1}+a_2e_{r_2}+\ldots+a_ne_{r_n}$ $(a_i\neq 0)$, where $r_1, r_2, \ldots, r_n$ are the roots from Φ^+ which are ordered on increase. Then the root r_1 is called the *first corner* of the element a. If

$M \subseteq N\Phi(K)$ is arbitrary subalgebra then $\mathcal{L}_1(M)$ is the *set of first corners of all its elements.*

Further we will use the A. I. Mal'tsev's notation [8] for the base roots α_i $(i = 1, \ldots, 6)$ in the root system of E_6 type:

α_1	α_2	α_3	α_4	α_5	α_6
$\omega_1 - \omega_2$	$\omega_4 + \omega_5 + \omega_6$	$\omega_2 - \omega_3$	$\omega_3 - \omega_4$	$\omega_4 - \omega_5$	$\omega_5 - \omega_6$

The positive root system of E_6 type consists only of the roots $\omega_i - \omega_j$ $(1 \leq i < j \leq 6)$, $\omega_i + \omega_j + \omega_k$ $(i < j < k;\ i,\ j,\ k = 1, \ldots, 6)$ and $\omega_0 = \omega_1 + \ldots + \omega_6$. The height of roots is calculated as $j - i$ for $\omega_i - \omega_j$, as $16 - (i + j + k)$ for $\omega_i + \omega_j + \omega_k$. The root ω_0 is maximal and it has the height 11. See Pic. 1 for positive root system of E_6 type.

Further we denote $e_{\omega_i - \omega_j}$ and $e_{\omega_k + \omega_l + \omega_m}$ as e_{ij} and e_{klm}, respectively. It is easy to note that:
the product of e_{ij} and e_{kl} $(i < j \leq k < l)$ is non-zero only if the case $j = k$;
the product of e_{ij} and e_{klm} is non-zero only if $i \notin \{k, l, m\}$ and $j \in \{k, l, m\}$;
the product e_{ijk} and e_{lmn} is non-zero only in the case $\{i,\ j,\ k,\ l,\ m,\ n\} = \{1, 2, 3, 4, 5, 6\}$.

3. Largest dimension commutative subalgebras

Theorem 1. *A largest dimension commutative subalgebra of the algebra $N\Phi(K)$ of E_6 type over the field K coincides with either $T(\alpha_1)$ or $T(\alpha_6)$.*

Proof. It is proved [3] that a subalgebra M of the algebra $N\Phi(K)$ over the field K is a largest dimension commutative subalgebra if and only if the set of roots $\mathcal{L}_1(M)$ is a commutative root set of a maximal order in Φ.

According A. I. Mal'tsev, the maximal order commutative subsets in E_6^+ are only $\{\alpha_1\}^+$ and $\{\alpha_6\}^+$. Now we consider the largest dimension commutative subalgebras with such the sets as the sets of first corners for their elements.

Lemma 1. *Let M be a largest dimension commutative subalgebra of the algebra $N\Phi(K)$ over the field K. Then there exists a base in M consisting of the elements*

$$\gamma_s = e_s + A_s \ (s \in \mathcal{L}_1(M)),\ \textit{where } A_s \in \sum_{r \in \Phi^+ \setminus \mathcal{L}_1(M)}^{r > s} K e_r. \tag{3.1}$$

Proof. For any $s \in \mathcal{L}_1(M)$ there exists an element in M with the first corner s. Choose such the element α_s with the s-coordinate equal to one. It is of a form

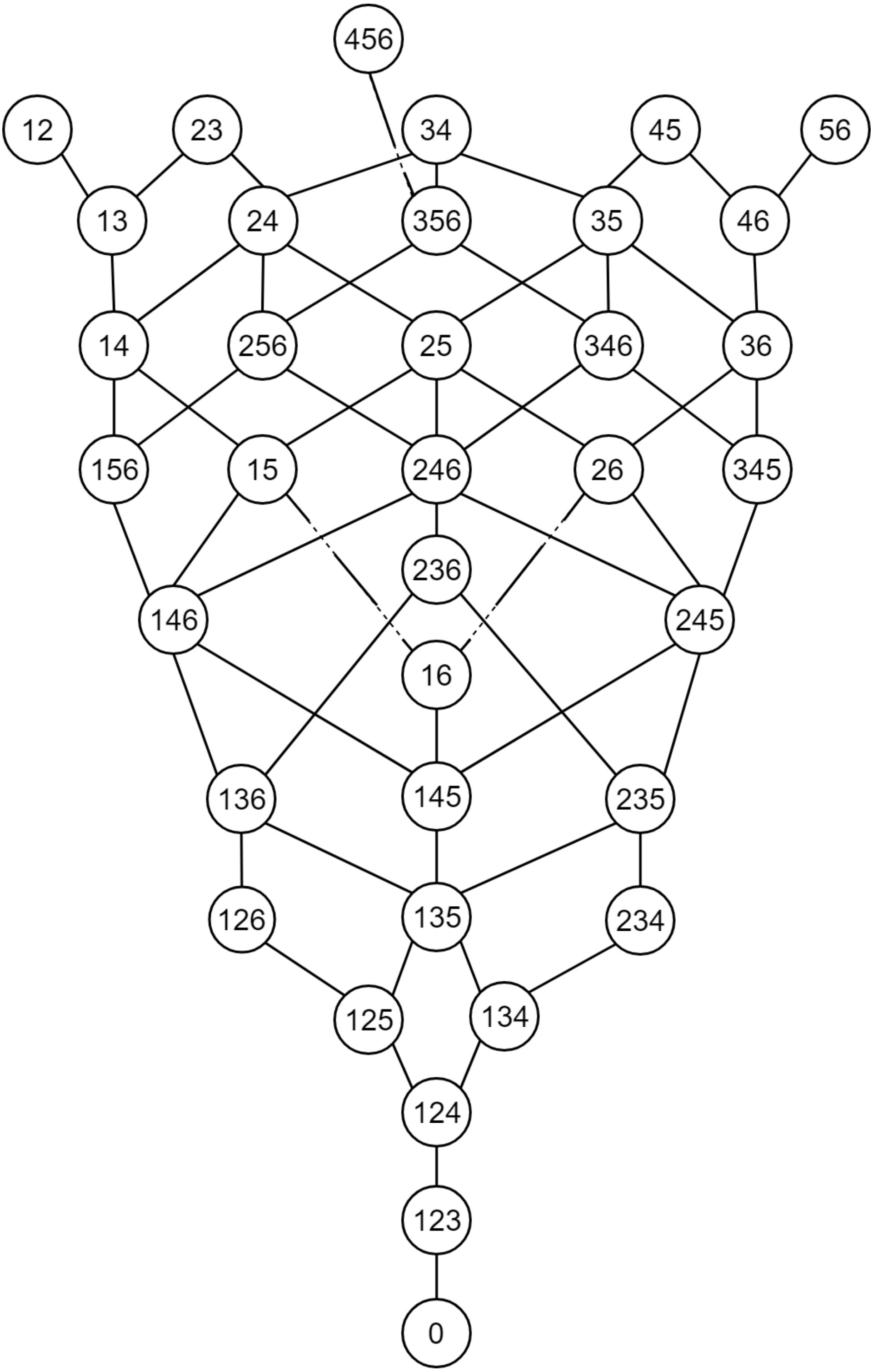

Figure 1. The root system of E_6 type

$$\alpha_s = e_s + \beta_s + A_s, \text{ where } \beta_s \in \sum_{r\in\mathcal{L}_1(M)}^{r>s} Ke_r, \; A_s \in \sum_{r\in\Phi^+\setminus\mathcal{L}_1(M)}^{r>s} Ke_r.$$

We can map to zero all items from β_s decomposition by adding the elements from $K\gamma_{s'}$ for all possible $s' > s$. □

Let's show that for Φ of E_6 type all A_s are zero. The elements A_s may be written as

$$A_s = c_s e_{q(s)} + \gamma'_s, \text{ where } \gamma'_s \in \sum_{p\in\Phi^+\setminus\mathcal{L}_1(M)}^{p>q(s)} Ke_p.$$

If $A_s \neq 0$ then we suppose that $c_s \neq 0$ and so $q(s)$ is a first corner in A_s. Let's choose such s and arbitrary $r \in \mathcal{L}_1(M)$. The product $\gamma_s * \gamma_r$ is equal to zero and is equal to

$$\begin{gathered} c_r(e_s * e_{q(r)}) + e_s * \gamma'_r + c_s(e_{q(s)} * e_r) + c_s c_r(e_{q(s)} * e_{q(r)}) + \\ + c_s(e_{q(s)} * \gamma'_r) + \gamma'_s * e_r + c_r(\gamma'_s * e_{q(r)}) + \gamma'_s * \gamma'_r \end{gathered} \tag{3.2}$$

Lemma 2. *If $r + q(s)$ and $s + q(r)$ are the equal roots then*

$$N_{s,q(r)} c_r + N_{q(s),r} c_s = 0.$$

If $r + q(s)$ is a root which is not equal to $s + q(r)$ then $r + q(s) - s$ is a root and corresponding coordinate of γ'_r is equal to $-(N_{q(s),r} N_{s,r+q(s)-s}) c_s$.

Proof. If $r + q(s)$ is a root then the third item in (3.2) is $c_s(e_{q(s)} * e_r) = \pm c_s e_{r+q(s)} \neq 0$. $r + q(s)$-coordinate in the product $\gamma_s * \gamma_r$ is non-zero, so the sum (3.2) contains, except the third item, another items with non-zero $r + q(s)$-coordinate. It is easy to prove that it may be either first or second items, but not simultaneously. Then $r + q(s)$-coordinate in (3.2) is a sum of corresponding coordinates of either first and third items or second and third ones. The first case is possible only if $r + q(s) = s + q(r)$, and in this case $N_{s,q(r)} c_r + N_{q(s),r} c_s = 0$. The second case γ'_r contains such the element ae_t that $ae_t * e_s = -N_{q(s),r} c_s e_{r+qs}$. So $ae_t * e_s = N_{t,s} a e_{t+s}$, and we have $t = r + q(s) - s$. Then $a = -(N_{q(s),r}/N_{r+q(s)-s,s}) c_s$. □

This lemma leads to

Corollary 1. *If $r + q(s)$ is a root then $r + q(s) - s$ is a root too.*

In the case $\Phi = E_6$ and $\mathcal{L}_1(M) = \{\alpha_1\}^+$ for arbitrary s and $q(s)$ there exists such the root r that $r + q(s)$ is a root but $r + q(s) - s$ is not a root. We will not consider $q(s)$ which does not commute with the roots t of the height > 8 (because A_t in (3.1) is equal to zero).

1) For the roots $s = \omega_1 - \omega_i$ and $q(s) = \omega_i - \omega_j$ we choose the root $r = \omega_1 + \omega_j + \omega_k$, where $k \neq i$.

2) For the roots $s = \omega_1 - \omega_i$ and $q(s) = \omega_j - \omega_k$ $(i \neq j)$ we choose the root $r = \omega_1 + \omega_i + \omega_k$.

3) For the roots $s = \omega_1 - \omega_i$ and $q(s) = \omega_i + \omega_j + \omega_k$ we choose the root $r = \omega_1 - \omega_j$.

4) For the roots $s = \omega_1 - \omega_i$ and $q(s) = \omega_j + \omega_k + \omega_l$ $(i \neq j, k, l)$ we choose the root $r = \omega_0 - q(s)$.

5) For the roots $s = \omega_1 + \omega_5 + \omega_6$ and $q(s) = \omega_2 - \omega_6$ we choose the root $r = \omega_1 - \omega_2$.

6) For the roots $s = \omega_1 + \omega_i + \omega_j$ and $q(s) = \omega_i + \omega_k + \omega_l$ we choose the root $r = \omega_1 - \omega_i$.

7) For the roots $s = \omega_1 + \omega_i + \omega_j$ and $q(s) = \omega_k + \omega_l + \omega_m$ $(s + q(s) = \omega_0)$ we choose the root $r = \omega_1 - \omega_k$.

The possibility of such a choice contradicts to the corollary, and, so, contradicts to the proposition $A_s \neq 0$. So, the largest dimension commutative subalgebra of the algebra $NE_6(K)$ with the first corners set $\{\alpha_1\}^+$ is only $T(\alpha_1)$. Acting to it by the graph automorphism we obtain the analogous result for the first corners set $\{\alpha_6\}^+$. So, the largest commutative subalgebras of $NE_6(K)$ are only $T(\alpha_1)$ and $T(\alpha_6)$. □

4. Conclusion

We proved that the list of largest dimension commutative subalgebras of the algebra $NE_6(K)$ completely coincides with the list of its largest dimension commutative ideals, which was obtained earlier. So, the generalized reduced Mal'tsev's problem is completely solved in this case. The problem of description of all maximal commutative ideals of the algebra $N\Phi(K)$ is written in [3] and now this problem is solved not for all root system types.

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Обобщённая редукционная задача Мальцева о коммутативных подалгебрах алгебр Шевалле типа E_6 над полем

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Аннотация. В 1905 г. И. Шур указал наивысшую размерность коммутативных подгрупп группы $SL(n, \mathbb{C})$ и доказал, что коммутативные подгруппы этой размерности при $n > 3$ автоморфны. В 1945 г. А. И. Мальцев исследовал задачу описания коммутативных подгрупп наивысшей размерности в комплексных простых группах Ли. Он получил решение, применив переход к комплексным алгебрам Ли и редуцирование к аналогичной задаче для максимальной нильпотентной подалгебры. Пусть N — нильтреугольная подалгебра алгебры Шевалле. Исследуется задача описания коммутативных подалгебр наибольшей размерности подалгебры N алгебры Шевалле, ассоциированной с системой корней типа E_6, над произвольным полем. Ранее при работе над этой задачей было получен полный список коммутативных идеалов наибольшей размерности подалгебры N типа E_6. В настоящей статье показано, что коммутативные подалгебры наивысшей размерности также исчерпываются этим списком; таким образом решена обобщённая редукционная задача Мальцева для алгебр Шевалле типа E_6.

Ключевые слова: алгебра Шевалле, нильтреугольная подалгебра, коммутативная подалгебра наибольшей размерности.

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