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## A Note on Commutative Nil-Clean Corners in Unital Rings

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**Abstract.** We shall prove that if  $R$  is a ring with a family of orthogonal idempotents  $\{e_i\}_{i=1}^n$  having sum 1 such that each corner subring  $e_i R e_i$  is commutative nil-clean, then  $R$  is too nil-clean, by showing that this assertion is actually equivalent to the statement established by Breaz-Călugăreanu-Danchev-Micu in *Lin. Algebra & Appl.* (2013) that if  $R$  is a commutative nil-clean ring, then the full matrix ring  $\mathbb{M}_n(R)$  is also nil-clean for any size  $n$ . Likewise, the present proof somewhat supplies our recent result in *Bull. Iran. Math. Soc.* (2018) concerning strongly nil-clean corner rings as well as it gives a new strategy for further developments of the investigated theme.

**Keywords:** nil-clean rings, nilpotents, idempotents, corners

### 1. Introduction and Background

Everywhere in the text of this short paper, our rings  $R$  into consideration are assumed to be associative, containing the identity element 1 which, in general, differs from the zero element 0 of  $R$ , and all proper subrings are unital (i.e., containing the same identity as that of the former ring) with the exception of the corners  $e R e$  having the identity  $e$ , where  $e$  is an arbitrary

idempotent of  $R$ . Our terminology and notations are at all standard being mainly in agreement with [11]. As usual,  $J(R)$  stands for the Jacobson radical of  $R$ , and  $\mathbb{M}_n(R)$  for the full  $n \times n$  matrix ring over  $R$  whenever  $n \in \mathbb{N}$ .

First of all, let us recall two more notions, namely that a ring  $R$  is called *boolean* whenever  $r^2 = r$  for every  $r \in R$ , and is called *strongly regular* whenever, for every  $r \in R$ , there exists an element  $a \in R$  such that  $r = r^2a$ . Evidently, boolean rings are always strongly regular, while the converse is false.

On the other side, a ring  $R$  is said to be *nil-clean* in [9] (see, e.g., [8] as well) if, for every element  $r \in R$ , there exist a nilpotent  $q$  and an idempotent  $e$ , both depending on  $r$ , with the property  $r = q + e$ ; such an element  $r$  is also called *nil-clean*. In [1] it was shown that the matrix ring  $\mathbb{M}_n(F)$  over a field  $F$  is nil-clean if, and only if,  $F \cong \mathbb{Z}_2$ . This was slightly extended to the matrix ring over a division ring in [10]. In particular, if  $R$  is a commutative nil-clean ring, then  $\mathbb{M}_n(R)$  remains nil-clean as well (see [2], too) as well as if  $R$  is a strongly regular ring, then  $\mathbb{M}_n(R)$  is nil-clean if, and only if,  $R$  is a boolean ring.

The aim that we pursue here is to give a brief note on nil-cleanness of corner rings. We shall demonstrate that the facts presented above are deducible without any matrix at hand; in fact, we shall use a few new tricks in terms of corners of an arbitrary ring (see [3], [4], [5] and [6] as well). As for the converse, it was asked in [9] of whether or not the nil-cleanness of  $R$  will also imply nil-cleanness of the corner subring  $eRe$  for any idempotent  $e$  of  $R$ , that question seems to be rather difficult and so it leaves unanswered yet.

## 2. The Main Result

Before proving up our main achievement, we need two technical claims as follows:

**Proposition 1.** *Every finitely generated subring of a commutative nil-clean ring is finite.*

*Proof.* Let  $R$  be a commutative nil-clean ring and let  $S$  be its subring generated by the elements  $a_1, \dots, a_n \in R$ . Write  $a_i = e_i + q_i$  for each index  $i$ , where  $e_i$  is an idempotent and  $q_i$  is a nilpotent, and consider the subring  $S_0$  of  $R$  generated by all such  $e_i$  and  $q_i$ . Choosing now an integer  $k$  such that  $q_i^k = 0$  for every index  $i$ , we observe that  $S_0$  as an additive group is generated by a finite set

$$\{e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n} q_1^{l_1} q_2^{l_2} \cdots q_n^{l_n} \mid 0 \leq k_i \leq 1; 0 \leq l_i \leq k - 1\}.$$

Since by [9] the  $\text{char}(R)$  is finite, it follows immediately that  $S_0$  must be a finitely generated torsion group and, therefore, it has to be finite. Hence  $S \subseteq S_0$  is finite, too.  $\square$

**Remark 1.** The above assertion could also be derived in the following more conceptual manner: It follows easily from [9] that a commutative ring  $R$  is nil-clean exactly when  $J(R)$  is nil and the quotient  $R/J(R)$  is boolean (see [7] as well). As any subring of a boolean ring is again boolean, it now plainly follows that a subring of a commutative nil-clean ring is also nil-clean. Thus being simultaneously finitely generated, we routinely obtain its finiteness.

Nevertheless, the illustrated above proof gives some further strategies to be developed.

The following technicality is pivotal for obtaining and proving our main result.

**Lemma 1.** *Let  $R$  be a ring with a family of pair-wise orthogonal idempotents  $e_1, \dots, e_n \in R$  with  $\sum_{i=1}^n e_i = 1$ . If each corner ring  $e_i R e_i$  is commutative nil-clean, then, for every  $a \in R$ , the subring of  $R$  generated by the set  $\{e_1, \dots, e_n, a\}$  is finite.*

*Proof.* We shall verify the statement by the usage of an induction on the number of idempotents  $n$ . In doing that, for  $n = 1$ , the conclusion follows appealing to Proposition 1. So, let us assume that  $n \geq 2$  and fix an element  $a \in R$ . For each  $i = 1, \dots, n$ , let  $R_i$  be the subring of the corner  $(1 - e_i)R(1 - e_i)$ , generated by the set  $\{e_1, e_2, \dots, e_{i-1}, e_{i+1}, \dots, e_n, (1 - e_i)a(1 - e_i)\}$ . Notice that  $R_i$  has to be finite by the induction hypothesis. Let us now  $S_i$  be the subring of  $e_i R e_i$ , generated by the union  $\{e_i, e_i a e_i\} \cup e_i a R_i a e_i$ . Since  $S_i$  is obviously a finitely generated subring of a commutative nil-clean ring, it is finite by Proposition 1. Note also that the element  $e_i a e_{i_1} a e_{i_2} \cdots a e_{i_k} a e_i$  lies in  $S_i$  for every  $1 \leq i_1, \dots, i_k \leq n$ .

Now, let  $T$  be the subring of  $R$ , generated by the set  $\{e_1, \dots, e_n, a\}$ . Then  $T$ , as an additive group, is generated by the set

$$\mathcal{T} = \{e_{i_1} a e_{i_2} \cdots a e_{i_k} \mid k \geq 1, 1 \leq i_j \leq n, 1 \leq j \leq k\}.$$

We assert that this set is finite. Indeed, take  $x = e_{i_1} a e_{i_2} \cdots a e_{i_k} \in \mathcal{T}$ . Choose  $j \geq 1$  such that  $i_j = i_1$  and  $j$  is maximal with this property. Consequently,  $x = r a s$ , where  $r = e_{i_1} a e_{i_2} \cdots a e_{i_j} \in S_{i_1}$  and

$$s = e_{i_{j+1}} a e_{i_{j+2}} \cdots a e_{i_k} \in R_{i_1}$$

provided  $j \neq k$ , or  $x \in S_{i_1}$  provided  $j = k$ . Thus  $\mathcal{T} \subseteq [\cup_{i=1}^n S_i] \cup [\cup_{i=1}^n S_i a R_i]$  is really finite, and hence  $T$  is finitely generated as an additive group. Note that, applying [9], the element  $2e_i$  is a nilpotent in the corner  $e_i R e_i$  for

each  $i$ , so that 2 is a nilpotent in  $R$ . Hence  $T$  is a finitely generated torsion group, whence finite, as expected.  $\square$

We now have all the ingredients necessary to proceed by proving our chief statement.

**Theorem 1.** *Suppose that  $R$  is a ring with a family of pair-wise orthogonal idempotents  $e_1, \dots, e_n \in R$  with  $\sum_{i=1}^n e_i = 1$ . If each corner ring  $e_i R e_i$  is commutative nil-clean, then  $R$  is also nil-clean.*

*Proof.* Assuming that the ring  $R$  satisfies the assumptions of the text, we take an arbitrary element  $a \in R$ . Let us now  $S$  be the subring of  $R$ , generated by the finite system  $\{e_1, \dots, e_n, a\}$ . We will prove now that  $S$  is necessarily nil-clean. In fact, with Lemma 1 at hand, the ring  $S$  is finite, whence by the well-known structure characterization of artinian rings accomplished with the classical Wedderburn's theorem (see cf. [11] for instance), one may write that

$$S/J(S) \cong \mathbb{M}_{n_1}(F_1) \times \cdots \times \mathbb{M}_{n_r}(F_r) = T,$$

where all  $n_j \geq 1$  and all  $F_j$  are finite fields ( $j \in [1, r]$ ,  $r \in \mathbb{N}$ ). It is not too difficult to observe that, under this isomorphism, the complete orthogonal set of idempotents  $e_1, \dots, e_n$  is mapped into the complete orthogonal set of idempotents  $\bar{e}_1, \dots, \bar{e}_n \in T$ , and the corresponding corner rings  $T_i = \bar{e}_i T \bar{e}_i$  are commutative nil-clean. Note also that

$$T_i \cong \mathbb{M}_{n_{1,i}}(F_1) \times \cdots \times \mathbb{M}_{n_{r,i}}(F_r),$$

where it is pretty obvious that  $0 \leq n_{j,i} \leq n_j$ . The commutativity of the ring  $T_i$  forces  $n_{j,i} \leq 1$  for any index  $j$ , whereas, in accordance with [1, Theorem 3], the nil-cleanness of  $T_i$  implies that  $F_j \cong \mathbb{Z}_2$  whenever  $n_{j,i} = 1$ . Hence, if  $F_j \not\cong \mathbb{Z}_2$  for some  $j$ , we would have  $n_{j,i} = 0$  for all  $i$ , thus contradicting the completeness of the family  $\{e_1, \dots, e_n\}$ . Therefore,  $F_j \cong \mathbb{Z}_2$  for all  $j$ . Furthermore, once again using [1, Theorem 3], we shall obtain that  $T$  is nil-clean. Since  $S$  is finite, one easily sees that  $J(S)$  is nil, whence with the aid of [9] it follows that  $S$  is nil-clean, indeed, as claimed.

That the ring  $R$  is also nil-clean follows rather elementary as the element  $a$  is already shown to be nil-clean, concluding the proof.  $\square$

As an immediate consequence, one deduces the following assertion which is actually the chief result in [1] (compare with [2] too). Besides, this manifestly shows that the above theorem and the next corollary are tantamount being deducible one of other.

**Corollary 1.** *If  $R$  is a commutative nil-clean ring, then  $\mathbb{M}_n(R)$  is nil-clean.*

*Proof.* It is principally known that there is a system of  $n$  pair-wise orthogonal idempotents of  $\mathbb{M}_n(R)$ , say  $f_1, \dots, f_n$ , with sum equal to the identity matrix  $E_n$  such that the isomorphisms  $R \cong f_1 \mathbb{M}_n(R) f_1 \cong \dots \cong f_n \mathbb{M}_n(R) f_n$  hold. Henceforth, the previous Theorem 1 successfully works to get the wanted claim.  $\square$

### 3. Concluding Discussion

As final comments, it is worthwhile noticing that we somewhat have obtain some advantage on the problem raised by Diesl in [9] of whether the full matrix ring of any size over a nil-clean ring is again nil-clean. As noticed above, this was partially settled in [1] when the former ring  $R$  is commutative nil-clean, that is,  $R/J(R)$  is boolean and  $J(R)$  is nil.

The further progress in the object to solve completely the aforementioned Diesl's problem seems to be not too quick by taking into account the very complicated structure of non-commutative nil-clean rings (see, e.g., [7]).

We end our work with the following query consisting of two questions of interest:

**Problem 1.** *What can be said for the ring  $R$  from Theorem 1, provided that its corners  $e_i R e_i$  are non-commutative nil-clean rings? Is this ring  $R$  still nil-clean?*

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## О коммутативных ниль-чистых угловых подкольцах в унитарных кольцах

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**Аннотация.** Мы доказали, что если  $R$  — кольцо с семейством ортогональных идемпотентов  $\{e_i\}_{i=1}^n$ , имеющее сумму 1, такую, что каждое угловое подкольцо  $e_i R e_i$  коммутативно ниль-чисто, тогда  $R$  также ниль-чисто, показывая, что это утверждение фактически эквивалентно утверждению, установленному Breaz S., Călugăreanu G., Danchev P., Micu T. в "Lin. Algebra & Appl." (2013), что если  $R$  — коммутативное ниль-чистое кольцо, то полное матричное кольцо  $M_n(R)$  также ниль — чисто для любого размера  $n$ . Настоящее доказательство в некоторой степени уточняет наш недавний результат, опубликованный в журнале "Bull. Iran. Math. Soc." (2018), касающийся сильно ниль-чистых угловых колец, а также дает новую стратегию для дальнейшего развития исследуемой темы.

**Ключевые слова:** ниль-чистые кольца, нильпотенты, идемпотенты, угловые подкольца.

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