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Ranks for Families of Theories of Abelian Groups *

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Abstract. The rank for families of theories is similar to Morley rank and can be considered as a measure for complexity or richness of these families. Increasing the rank by extensions of families we produce more rich families and obtaining families with the infinite rank that can be considered as "rich enough". In the paper, we realize ranks for families of theories of abelian groups. In particular, we study ranks and closures for families of theories of finite abelian groups observing that the set of theories of finite abelian groups in not totally transcendental, i.e., its rank equals infinity. We characterize pseudofinite abelian groups in terms of Szmielew invariants. Besides we characterize e-minimal families of theories of abelian groups both in terms of dimension, i.e., the number of independent limits for Szmielew invariants, and in terms of inequalities for Szmielew invariants. These characterizations are obtained both for finite abelian groups and in general case. Furthermore we give characterizations for approximability of theories of abelian groups and show the possibility to count Szmielew invariants via these parameters for approximations. We describe possibilities to form d-definable families of theories of abelian groups having given countable rank and degree.

Keywords: family of theories, abelian group, rank, degree, closure.

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1. Introduction

The rank [11] for families of theories, similar to Morley rank, can be considered as a measure for complexity or richness of these families. Thus increasing the rank by extensions of families we produce more rich families and obtaining families with the infinite rank that can be considered as “rich enough”. A series of such rich families, containing all families of given language, is described in [5]. Additional properties of families of theories defined by sets of sentences are studied in [6].

Links and closures for families of theories of abelian groups, as well as values of their e-spectra are described in [8].

In the present paper, we consider and realize ranks for families of theories of abelian groups. The paper is organized as follows. Preliminary notions, notations and related results for families of theories and for theories of abelian groups, including Szmielew invariants, are collected in Sections 2 and 3. In Section 4, we study closures and ranks for theories of finite abelian groups. In particular, we observe that the set of theories of finite abelian groups in not totally transcendental (Theorem 4.1) and characterize pseudofinite abelian groups in terms of Szmielew invariants (Theorem 4.2). In Section 5, we characterize e-minimal families of theories of abelian groups both in terms of dimension, i.e., the number of independent limits for Szmielew invariants, and in terms of inequalities for Szmielew invariants. These characterizations are obtained both for finite abelian groups (Theorems 5.1, 5.3, 5.4) and in general case (Theorem 5.5). Theorem 5.6 gives characterizations for approximability of theories of abelian groups, as well as it produce the possibility to count Szmielew invariants via these parameters for approximations. In Section 6, we describe possibilities to form d-definable families of theories of abelian groups having given countable rank and degree (Theorems 6.2 and 6.4).

2. Preliminaries

Throughout we consider families $\mathcal{T}$ of complete first-order theories of a language $\Sigma = \Sigma(\mathcal{T})$. For a sentence φ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$ being the *φ-neighbourhood* in $\mathcal{T}$.

Definition [12]. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called *$\mathcal{T}$-approximated*, or *approximated by $\mathcal{T}$*, or *$\mathcal{T}$-approximable*, or a *pseudo-$\mathcal{T}$-theory*, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [12] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Following [11] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [7], and a hierarchy with respect to these ranks in the following way.

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

Proposition 2.1 [11]. *If an infinite family $\mathcal{T}$ does not have e-minimal subfamilies $\mathcal{T}_\varphi$ then $\mathcal{T}$ is not totally transcendental.*

If $\mathcal{T}$ is totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Theorem 2.2 [11]. *For any family $\mathcal{T}$, $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(\mathrm{Cl}_E(\mathcal{T}))$, and if $\mathcal{T}$ is nonempty and e-totally transcendental then $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}(\mathrm{Cl}_E(\mathcal{T}))$.*

Theorem 2.3 [11]. *For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:*
(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) e-$\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$.

Definition [6]. Let $\mathcal{T}$ be a family of first-order complete theories in a language Σ. For a set Φ of Σ-sentences we put $\mathcal{T}_\Phi = \{T \in \mathcal{T} \mid T \models \Phi\}$. A family of the form $\mathcal{T}_\Phi$ is called *d-definable* (in $\mathcal{T}$). If Φ is a singleton $\{\varphi\}$ then $\mathcal{T}_\varphi = \mathcal{T}_\Phi$ is called *s-definable*.

Theorem 2.4 [6]. *Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, $\alpha \in \{0, 1\}$, $n \in \omega \setminus \{0\}$. Then there is a d-definable subfamily $\mathcal{T}_\Phi$ such that $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$ and $\mathrm{ds}(\mathcal{T}_\Phi) = n$.*

Recall that a subfamily $\mathcal{T}_0$ of $\mathcal{T}$ is called *d_∞-definable* if $\mathcal{T}_0$ is a union, possibly infinite, of d-definable subfamilies of $\mathcal{T}$.

Theorem 2.5 [6]. *Let $\mathcal{T}$ be a family of a countable language Σ and with* $\mathrm{RS}(\mathcal{T}) = \infty$, *$\alpha$ be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that* $\mathrm{RS}(\mathcal{T}^*) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}^*) = n$.

Similarly [7] and following [11], for a nonempty family $\mathcal{T}$, we denote by $\mathcal{B}(\mathcal{T})$ the Boolean algebra consisting of all subfamilies $\mathcal{T}_\varphi$, where φ are sentences in the language $\Sigma(\mathcal{T})$.

Theorem 2.6 [7; 11]. *A nonempty family $\mathcal{T}$ is e-totally transcendental if and only if the Boolean algebra $\mathcal{B}(\mathcal{T})$ is superatomic.*

Recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. We set $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal α does not exist then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0, points of rank 1 are those which are isolated in the subspace of all non-isolated points, and so on. For a non-empty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and C is closed in X then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in C and the number of such points is the CB_X-*degree* of C, denoted by $n_X(C)$.

If X is countable and compact then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree $n_X(X) \in \omega \setminus \{0\}$.

For any ordinal α the set $\{p \in X \mid \mathrm{CB}_X(p) \geq \alpha\}$ is called the α-th CB-*derivative* X_α of X.

Elements $p \in X$ with $\mathrm{CB}_X(p) = \infty$ form the *perfect kernel* X_∞ of X.

Clearly, $X_\alpha \supseteq X_{\alpha+1}$, $\alpha \in \mathrm{Ord}$, and $X_\infty = \bigcap\limits_{\alpha \in \mathrm{Ord}} X_\alpha$.

Similarly, for a nontrivial superatomic Boolean algebra $\mathcal{A}$ the characteristics $\mathrm{CB}_{\mathcal{A}}(A)$, $n_{\mathcal{A}}(A)$, and $\mathrm{CB}_{\mathcal{A}}(p)$, for $p \in A$, are defined [3] starting with atomic elements being isolated points. Following [3], $\mathrm{CB}_{\mathcal{A}}(A)$ and $n_{\mathcal{A}}(A)$ are called the *Cantor–Bendixson invariants*, or CB-*invariants* of $\mathcal{A}$.

Recall that by [3, Lemma 17.9], $\mathrm{CB}_{\mathcal{A}}(A) < |A|^+$ for any infinite $\mathcal{A}$, and the following theorem holds.

Theorem 2.7 [3, Theorem 17.11]. *Countable superatomic Boolean algebras are isomorphic if and only if they have the same* CB-*invariants.*

By Theorem 2.6 any e-totally transcendental family $\mathcal{T}$ defines a superatomic Boolean algebra $\mathcal{B}(\mathcal{T})$, and it is easy to observe step-by-step that $\mathrm{RS}(\mathcal{T}) = \mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $\mathrm{ds}(\mathcal{T}) = n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, i.e., the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ consists of CB-invariants for $\mathcal{B}(\mathcal{T})$.

In particular, by Theorem 2.7, for any countable e-totally transcendental family $\mathcal{T}$, $\mathcal{B}(\mathcal{T})$ is uniquely defined, up to isomorphism, by the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ of CB-invariants.

By the definition for any e-totally transcendental family $\mathcal{T}$ each theory $T \in \mathcal{T}$ obtains the CB-rank $\mathrm{CB}_{\mathcal{T}}(T)$ starting with $\mathcal{T}$-isolated points T_0, of $\mathrm{CB}_{\mathcal{T}}(T_0) = 0$. We will denote the values $\mathrm{CB}_{\mathcal{T}}(T)$ by $\mathrm{RS}_{\mathcal{T}}(T)$ as the rank for the point T in the topological space on $\mathcal{T}$ which is defined with respect to $\Sigma(\mathcal{T})$-sentences.

3. Theories of abelian groups

Let $\mathcal{A}$ be an abelian group in the language $\Sigma = \langle +^{(2)}, -^{(1)}, 0^{(0)} \rangle$. Then $k\mathcal{A}$ denotes its subgroup $\{ka \mid a \in A\}$ and $\mathcal{A}[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. Let P be the set of all prime numbers. If $p \in P$ and $pA = \{0\}$ then $\dim\mathcal{A}$ denotes the dimension of the group $\mathcal{A}$, considered as a vector space over a field with p elements. The following numbers, for arbitrary $p \in P$ and $n \in \omega \setminus \{0\}$ are called the *Szmielew invariants* for the group A [2; 13]:

$$\alpha_{p,n}(\mathcal{A}) = \min\{\dim((p^n\mathcal{A})[p]/(p^{n+1}\mathcal{A})[p]), \omega\},$$

$$\beta_p(\mathcal{A}) = \min\{\inf\{\dim((p^n\mathcal{A})[p] \mid n \in \omega\}, \omega\},$$

$$\gamma_p(\mathcal{A}) = \min\{\inf\{\dim((\mathcal{A}/\mathcal{A}[p^n])/p(\mathcal{A}/\mathcal{A}[p^n])) \mid n \in \omega\}, \omega\},$$

$$\varepsilon(\mathcal{A}) \in \{0,1\}, \text{ and } \varepsilon(\mathcal{A}) = 0 \Leftrightarrow (n\mathcal{A} = \{0\} \text{ for some } n \in \omega, n \neq 0).$$

It is known [2, Theorem 8.4.10] that two abelian groups are elementary equivalent if and only if they have same Szmielew invariants. Besides, the following proposition holds.

Proposition 3.1 [2, Proposition 8.4.12]. *Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, and $\varepsilon \in \{0,1\}$ be given. Then there is an abelian group $\mathcal{A}$ such that the Szmielew invariants $\alpha_{p,n}(\mathcal{A})$, $\beta_p(\mathcal{A})$, $\gamma_p(\mathcal{A})$, and $\varepsilon(\mathcal{A})$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:*

(1) *if for prime p the set $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;*

(2) *if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p,n\rangle \mid \alpha_{p,n} \neq 0\}$ is finite.*

We denote by $\mathbb{Q}$ the additive group of rational numbers, $\mathbb{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbb{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbb{Q}$, $\mathbb{Z}_{p^n}$, R_p, $\mathbb{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since abelian groups with same Szmielew invariants have same theories, any abelian group $\mathcal{A}$ is elementary equivalent to a group

$$\oplus_{p,n}\mathbb{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbb{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbb{Q}^{(\varepsilon)}, \tag{3.1}$$

where $\mathcal{B}^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group $\mathcal{B}$. Thus, any theory of an abelian group has a model being a direct sum of based groups. The groups of form (3.1) are called *standard*.

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas [2, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y), \tag{3.2}$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \tag{3.3}$$

where $m_i \in \mathbb{Z}$, $k \in \omega$, p is a prime number [1], [2, Lemma 8.4.7]. Formulas (3.2) and (3.3) allow to witness that Szmielew invariants defines theories of abelian groups modulo Proposition 3.1.

In view of Proposition 3.1 and equations (3.2) and (3.3) we have the following:

Remark 3.2. Theories of abelian groups are forced by sentences implied by formulas of form (3.2) and (3.3) and describing dimensions with respect to $\alpha_{p,n}$, β_p, γ_p, ε as well as bounds for orders p^k of elements and possibilities for divisions of elements by p^k. Moreover, distinct values of Szmielew invariants are separated by some sentences modulo Proposition 3.1. Hence, counting ranks of families of theories of abelian groups it suffices to consider sentences separating Szmielew invariants.

4. Closures and ranks for families of theories of finite abelian groups

Consider the family $\mathcal{T}_{A,\text{fin}}$ of all theories of finite abelian groups. Clearly, $\mathcal{T}_{A,\text{fin}}$ is countable corresponding to tuples of non-zero values of $\alpha_{p,n}$. By Proposition 3.1 the E-closure of $\mathcal{T}_{A,\text{fin}}$ produces theories, of infinite abelian groups, with some $\alpha_{p,n}$ and $\beta_p = \gamma_p = \omega$. Since $\beta_p = \gamma_p = \omega$ can be obtained independently with respect to distinct p, we have $|\mathrm{Cl}_E(\mathcal{T}_{A,\text{fin}})| = 2^\omega$. Applying Theorem 2.3 we have:

Theorem 4.1. $\mathrm{RS}(\mathcal{T}_{A,\text{fin}}) = \infty$.

Recall [4; 10] that an infinite structure $\mathcal{M}$ is *pseudofinite* if every sentence true in $\mathcal{M}$ has a finite model. Here the theory $\mathrm{Th}(\mathcal{M})$ is also called *pseudofinite*.

Now we consider Szmielew invariants of theories in $\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}})$. Since theories of finite groups can not generate new theories of finite groups and finite abelian groups have finitely many nonzero values $\alpha_{p,n}$, and $\beta_p = \gamma_p = \varepsilon = 0$ for any prime p, it suffices to study theories of pseudofinite groups, i.e., theories in $\mathcal{T}_{A,\mathrm{pf}} = \mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}}) \setminus \mathcal{T}_{A,\mathrm{fin}}$.

Notice by the way that by Theorem 4.1, $\mathrm{RS}(\mathcal{T}_{A,\mathrm{pf}}) = \infty$ since $|\mathcal{T}_{A,\mathrm{pf}}| = 2^\omega$ in view of $|\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}})| = 2^\omega$ and $|\mathcal{T}_{A,\mathrm{fin}}| = \omega$.

By Proposition 3.1 theories in $\mathcal{T}_{A,\mathrm{pf}}$ are exhausted by limit values $\alpha_{p,n} = \omega$, $\beta_p = \gamma_p = \omega$ and $\varepsilon = 1$ producing the following theorem.

Theorem 4.2. *For any theory T of abelian groups the following conditions are equivalent:*

(1) *$T \in \mathcal{T}_{A,\mathrm{pf}}$;*

(2) *T has some infinite $\alpha_{p,n}$, or some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$, moreover, for all nonzero values β_p and γ_p, $\beta_p = \gamma_p = \omega$;*

(3) *T has infinite models, and all nonzero values β_p and γ_p imply $\beta_p = \gamma_p = \omega$.*

Proof. (1) $\Rightarrow$ (2). Let $T \in \mathcal{T}_{A,\mathrm{pf}}$. Then T has infinite models and it is approximable by an e-minimal family $\mathcal{T} \subset \mathcal{T}_{A,\mathrm{fin}}$, see [12, Proof of Theorem 6.1]. Therefore T is unique accumulation point for $\mathcal{T}$. By the definition all theories in $\mathcal{T}$ has only finitely many positive values $\alpha_{p,n}$, all these values are natural numbers, and all Szmielew invariants β_p, γ_p, ε equal 0. Considering nonzero Szmielew invariants $\alpha_{p,n}$ for theories in $\mathcal{T}$, we have the following possibilities for each prime p:

(i) some value $\alpha_{p,n}$ unboundedly increases for theories in $\mathcal{T}$, with fixed n;

(ii) $\alpha_{p,n} \neq 0$ with unboundedly many n, for theories in $\mathcal{T}$;

(iii) values $\alpha_{p,n}$ are bounded for theories in $\mathcal{T}$, and $\{\langle p,n\rangle \mid \alpha_{p,n} \neq 0\}$ is infinite for T.

In the case (i), T has $\alpha_{p,n} = \omega$. In the case (ii), T has infinite $\{n \mid \alpha_{p,n} \neq 0\}$ producing $\beta_p = \gamma_p = \omega$ by Proposition 3.1, (1). In the case (iii), T has infinite $\{\langle p,n\rangle \mid \alpha_{p,n} \neq 0\}$ producing $\varepsilon = 1$ by Proposition 3.1, (2). Again by Proposition 3.1, T can not have finite positive β_p and γ_p, and if β_p or γ_p is positive then $\beta_p = \gamma_p = \omega$.

(2) $\Rightarrow$ (1). Let T have some infinite $\alpha_{p,n}$, or some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$, and all positive values β_p and γ_p imply $\beta_p = \gamma_p = \omega$. Now we construct step-by-step an e-minimal family $\mathcal{T} \subset \mathcal{T}_{A,\mathrm{fin}}$ of theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ of finite abelian groups $\mathcal{A}_i$, $i \in \omega$, with unique accumulation point T, satisfying the following conditions:

a) for any i, $\mathcal{A}_i$ is a subgroup of $\mathcal{A}_{i+1}$;

b) for any i-th prime number p_i, if T has positive $\alpha_{p_i,n}$, β_{p_i}, or γ_{p_i}, then $\mathcal{A}_j$, for $j \geq i$, have subgroups $\mathbb{Z}_{p_i^k}$;

c) if T has finite $\alpha_{p,n}$ then the theories T_i have this Szmielew invariant starting with some i;

d) if T has infinite $\alpha_{p,n} = \alpha_{p,n}(T)$ then T_i have monotone increasing Szmielew invariants $\alpha_{p,n} = \alpha_{p,n}(T_i)$ with $\lim\limits_{i\to\infty} \alpha_{p,n}(T_i) = \omega$;

e) if T has $\beta_p = \gamma_p = \omega$ then either $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite, for T, and all nonzero Szmielew invariants $\alpha_{p,n}$ for T_i are exhausted by nonzero Szmielew invariants $\alpha_{p,n}$ for T, or $\overline{\alpha}_p = \{n \mid \alpha_{p,n} \neq 0\}$ is finite and T_i have distinct positive $\alpha_{p,n}$, $n \notin \overline{\alpha}_p$;

f) if T has $\varepsilon = 1$ then either $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ is infinite, for T, and all nonzero Szmielew invariants $\alpha_{p,n}$ for T_i are exhausted by nonzero Szmielew invariants $\alpha_{p,n}$ for T, or $\overline{\alpha} = \{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ is finite and T_i have distinct positive $\alpha_{p,n}$, $\langle p, n\rangle \notin \overline{\alpha}$;

g) all nonzero Szmielew invariants $\alpha_{p,n}$ for T_i are described in the items b)–f).

The items c), d), g) guarantee the required invariants $\alpha_{p,n}$ for the accumulation point T of $\mathcal{T} = \{T_i \mid i \in \omega\}$, the items e), g) confirm the required invariants β_p and γ_p for T, and the items f), g) — the value ε of T. Thus, $T \in \mathcal{T}_{A,\mathrm{pf}}$.

(2) $\Leftrightarrow$ (3) immediately follows by Proposition 3.1. □

Notice that by Theorem 4.2 infinite standard groups

$$\oplus_{p,n}\mathbb{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbb{Z}_{p^\infty}^{(\omega)} \oplus \oplus_p R_p^{(\omega)} \oplus \mathbb{Q}^{(\varepsilon)}$$

and, in particular, the group $\mathbb{Q}$ are pseudofinite.

Theorem 4.2 immediately implies:

Corollary 4.3. *If a theory T of an abelian group has a positive natural value β_p or γ_p then models of T are not pseudofinite.*

Since $\mathrm{Th}(\mathbb{Z})$ has values $\gamma_p = 1$ [9], the group $\mathbb{Z}$ is not pseudofinite, as also noticed in [4].

Remark 4.4. Having Theorem 4.2 describing the set $\mathcal{T}_{A,\mathrm{pf}}$ we can study ranks for subfamilies $\mathcal{T} \subseteq \mathcal{T}_{A,\mathrm{fin}}$ and their closures $\mathrm{Cl}_E(\mathcal{T}) \subseteq \mathcal{T}_{A,\mathrm{fin}} \cup \mathcal{T}_{A,\mathrm{pf}}$. By Theorem 4.1 these subfamilies $\mathcal{T}$ admit $\mathrm{RS}(\mathcal{T}) = \infty$. Moreover, arguments for the proof of Theorem 4.1 show that one can choose 2^ω disjoint families $\mathcal{T} \subseteq \mathcal{T}_{A,\mathrm{fin}}$ with $\mathrm{RS}(\mathcal{T}) = \infty$. Indeed, we can define these families independently varying finite bounded values of some countably many $\alpha_{p,n}$ staying infinitely many prime p' free for arbitrary values of $\alpha_{p',n}$. Therefore the values $\alpha_{p,n}$ are responsible for 2^ω disjoint families $\mathcal{T}$ and $\alpha_{p',n}$ are responsible for $\mathrm{RS}(\mathcal{T}) = \infty$.

In view of Theorem 4.2 and Remark 4.4 we consider dynamics for $\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}(\mathcal{T})$, where $\mathcal{T} \subseteq \mathcal{T}_{A,\mathrm{fin}}$. The values $\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}(\mathcal{T})$ are defined by variations of possibilities for Szmielew invariants $\alpha_{p,n}$ of $T \in \mathcal{T}$.

By Proposition 3.1 and Theorem 4.2 this dynamics can be realized by unbounded increasing of values of some $\alpha_{p,n}$, for fixed p and n, or by infinitely many nonzero $\alpha_{p,n}$, for (non-)fixed p and unbounded n.

For instance, taking $\mathcal{T}_{A,\mathrm{fin},p} \subset \mathcal{T}_{A,\mathrm{fin}}$ consisting of theories of finite abelian groups whose positive Szmielew invariants are exhausted by $\alpha_{p,n}$, for chosen fixed p, we obtain 2^ω possibilities for $\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin},p})$ varying independently $\alpha_{p,n}$ for distinct n. Thus, $\mathrm{RS}(\mathcal{T}_{A,\mathrm{fin},p}) = \infty$.

Following Theorem 2.5, this rank value implies that $\mathcal{T}_{A,\mathrm{fin},p}$ has subfamilies $\mathcal{T}$ of arbitrary countable rank α and of arbitrary degree n. Below we show a mechanism to choose d-definable $\mathcal{T}$ with $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T})) = (\alpha, n)$.

5. e-Minimal families of theories and their accumulation points

In this section we consider both e-minimal families of theories of finite abelian groups and e-minimal subfamilies of the set $\mathcal{T}_A$ of all theories of abelian groups.

Let $\mathcal{T}$ be an e-minimal subfamily of $\mathcal{T}_{A,\mathrm{fin}}$. Then by the definition $\mathcal{T}$ has unique accumulation point T, models of T are pseudofinite, and Szmielew invariants are described in Theorem 4.2. It implies that, for T, some $\alpha_{p,n}$ is infinite, some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$.

The value $\alpha_{p,n} = \omega$ is obtained by unbounded increasing sequence of finite $\alpha_{p,n} = m_k$ for some theories $T_k \in \mathcal{T}$ with these values. Thus,

$$\alpha^T_{p,n} = \lim_{k\to\infty} \alpha^{T_k}_{p,n}, \tag{5.1}$$

where $\alpha^T_{p,n}$ is the value $\alpha_{p,n}$ for T, $\alpha^{T_k}p, n$ are the values $\alpha_{p,n}$ for T_k.

Following Proposition 3.1 the case $\beta_p = \gamma_p = \omega$ corresponds infinitely many n with $\alpha_{p,n} \neq 0$ for $\mathcal{T}$, i.e., by e-minimality, for infinitely many n there are infinitely many theories in $\mathcal{T}$ with $\alpha_{p,n} \neq 0$. Again by e-minimality it means that for each considered n there are finitely many theories in $\mathcal{T}$ with $\alpha_{p,n} = 0$.

Again by Proposition 3.1 the case $\varepsilon = 1$ implies infinitely many pairs $\langle p, n\rangle$ with $\alpha_{p,n} \neq 0$ for $\mathcal{T}$, i.e., by e-minimality, for infinitely many $\langle p, n\rangle$ there are infinitely many theories in $\mathcal{T}$ with $\alpha_{p,n} \neq 0$, that is, for each considered $\langle p, n\rangle$ there are finitely many theories in $\mathcal{T}$ with $\alpha_{p,n} = 0$.

Similarly (5.1), in the latter two cases the values $\beta_p = \gamma_p = \omega$ and/or $\varepsilon = 1$ can be interpreted as limits $\lim\limits_n \alpha_{p,n}$ and/or $\lim\limits_{p,n} \alpha_{p,n}$ in the set of all Szmielew invariants, where $\alpha_{p,n}$ are Szmielew invariants for some theories in $\mathcal{T}$. Since $\beta_p = \gamma_p \in \{0, \omega\}$ and $\varepsilon \in \{0, 1\}$ for any theory in $\mathrm{Cl}_E(\mathcal{T})$, we can assume that $\lim\limits_n \alpha_{p,n} \in \{0, \omega\}$ and $\lim\limits_{p,n} \alpha_{p,n} \in \{0, 1\}$.

Additionally, since e-minimal families have unique accumulation points, these limits are unique, too, i.e., there are no independent possibilities to obtain distinct limit values using distinct sequences of theories in $\mathcal{T}$. In such a case we say that $\mathcal{T}$ does not have *independent limit values*.

It means that if we have some values for $\lim\limits_{k\to\infty}\alpha^{T_k}_{p,n}$, $\lim\limits_{n}\alpha^{T_k}_{p,n}$, $\lim\limits_{p,n}\alpha^{T_k}_{p,n}$ for an infinite sequence of theories T_k in $\mathcal{T}$, we can not find another infinite sequence of theories T'_k in $\mathcal{T}$ producing different limit values.

Clearly, assuming that $\mathcal{T}$ does not have independent limit values we conversely obtain the e-minimality of $\mathcal{T}$.

Thus, we have:

Theorem 5.1. *For any infinite family $\mathcal{T}\subseteq\mathcal{T}_{A,\rm fin}$ the following conditions are equivalent:*
(1) *$\mathcal{T}$ is e-minimal;*
(2) *$\mathcal{T}$ does not have independent limit values.*

Considering an arbitrary infinite family $\mathcal{T}\subseteq\mathcal{T}_{A,\rm fin}$ we define the finite or infinite number of independent limit values for $\mathcal{T}$. This number is called the *dimension* of $\mathcal{T}$ and denoted by $\dim(\mathcal{T})$.

Remark 5.2. Let $\mathcal{T}_{p,n}$ be an infinite family consisting of theories $T\in\mathcal{T}_{A,\rm fin}$ with unbounded $\alpha_{p,n}$ and fixed $\alpha_{p',n'}$ for $\langle p',n'\rangle\neq\langle p,n\rangle$. Clearly, $\dim(\mathcal{T}_{p,n})=1$. Besides,

$$\dim\left(\bigcup_{\langle p,n\rangle\in X}\mathcal{T}_{p,n}\right)=|X| \tag{5.2}$$

for any finite set X of some pairs $\langle p,n\rangle$. The equation (5.2) stays valid for infinite X if $\mathcal{T}_{p,n}$ are defined uniformly, say, if $\alpha_{p',n'}=\mathrm{Const}$ for $\langle p',n'\rangle\neq\langle p,n\rangle$. Otherwise, if fixed $\alpha_{p',n'}$ are random, one can generate continuum many pseudofinite theories $T\in\mathrm{Cl}_E\left(\bigcup\limits_{\langle p,n\rangle\in X}\mathcal{T}_{p,n}\right)$. Indeed, we can form $\mathcal{T}_{p,1}$ for some infinite and co-infinite set Z of prime numbers p such that $\alpha_{p',1}$, for theories in $\bigcup\limits_{\langle p,1\rangle\in X}\mathcal{T}_{p,1}$, are independently bounded and unbounded for prime $p'\notin Z$. Thus, there are continuum many possibilities for the sequences of finite/infinite values $\alpha_{p',1}$ for theories in $\mathrm{Cl}_E\left(\bigcup\limits_{\langle p,1\rangle\in X}\mathcal{T}_{p,1}\right)$.

Clearly, the value $\dim(\mathcal{T})$ equals the e-spectrum of $\mathcal{T}$:

$$\dim(\mathcal{T})=e\text{-}\mathrm{Sp}(\mathcal{T}). \tag{5.3}$$

Remind that here $\dim(\mathcal{T})$ is defined in terms of Szmielew invariants and their ordinal limits whereas e-$\mathrm{Sp}(\mathcal{T})$ is a model-theoretic value for the topology with respect to E-closure.

Theorem 5.1 immediately implies the following reformulation:

Theorem 5.3. *For any infinite family $\mathcal{T}\subseteq\mathcal{T}_{A,\rm fin}$ the following conditions are equivalent:*

(1) $\mathcal{T}$ *is e-minimal;*
(2) $\dim(\mathcal{T}) = 1$.

Remind [11] that e-minimality of $\mathcal{T}$ means that $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = 1$. Thus, these values for rank and degree are characterized by $\dim(\mathcal{T}) = 1$.

Similarly, finite $\dim(\mathcal{T}) = n$ means that $\mathcal{T}$ has n accumulation points producing $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = n$ as well as a representation of $\mathcal{T}$ as a disjoint union of n e-minimal subfamilies.

The following theorem gives an additional criterion for e-minimality of a family $\mathcal{T} \subseteq \mathcal{T}_{A,\mathrm{fin}}$.

Theorem 5.4. *An infinite family $\mathcal{T} \subseteq \mathcal{T}_{A,\mathrm{fin}}$ of theories of abelian groups is e-minimal if and only if for any upper bound $\alpha_{p,n} \geq m$ or lower bound $\alpha_{p,n} \leq m$, for $m \in \omega$, there are finitely many theories in $\mathcal{T}$ satisfying this bound. Having finitely many theories with $\alpha_{p,n} \geq m$, there are infinitely many theories in $\mathcal{T}$ with a fixed value $\alpha_{p,n} < m$.*

Proof. Let $\mathcal{T}$ be e-minimal. Consider a bound $\alpha_{p,n} \geq m$ (respectively $\alpha_{p,n} \leq m$). By Remark 3.2 there is a sentence separating theories in $\mathcal{T}$ with $\alpha_{p,n} \geq m$ ($\alpha_{p,n} \leq m$). Since $\mathcal{T}$ is e-minimal, exactly one of the conditions $\alpha_{p,n} \geq m$, $\alpha_{p,n} < m$ satisfies infinitely many theories in $\mathcal{T}$. Thus, there are only finitely many theories in $\mathcal{T}$ satisfying $\alpha_{p,n} \leq m - 1$ or $\alpha_{p,n} \geq m$. In the latter case, with infinitely many theories for $\alpha_{p,n} < m$, the invariant $\alpha_{p,n}$ should be repeated infinitely many times for distinct theories in $\mathcal{T}$.

Conversely, we again apply Remark 3.2: taking an arbitrary sentence φ in the group language, we can describe only finitely many bounds $\alpha_{p,n} \geq m$ and $\alpha_{p,n} \leq m$. Since each bound forces finite or cofinite subfamily of $\mathcal{T}$ we have finite $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$, i.e., $\mathcal{T}$ is e-minimal. □

Now we extend the context characterizing the e-minimality of subfamilies $\mathcal{T}$ in $\mathcal{T}_A$.

Following Proposition 3.1 we again notice that all dependencies between values of Szmielew invariants in a given theory of an abelian group are exhausted by ones given by infinite $\{n \mid \alpha_{p,n} \neq 0\}$ implying $\beta_p = \gamma_p = \omega$ as well as by infinite $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ implying $\varepsilon = 1$. It means that Szmielew invariants, for a fixed theory and for a family, can not force positive values $\alpha_{p,n}$, β_p, γ_p using positive values for different prime p' and/or ε. Besides, values $\alpha_{p,n}$ and natural β_p, γ_p do not forced by other Szmielew invariants. Moreover, finite values $\alpha_{p,n}$, β_p, γ_p, for theories in $\mathrm{Cl}_E(\mathcal{T})$, can not be forced by other finite or infinite values of these invariants. Thus, all dependencies between distinct Szmielew invariants $\alpha^T_{p,n}$, β^T_p, γ^T_p, ε^T, for theories $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$, are exhausted by the following ones for sequences $(T_k)_{k\in\omega}$ of theories in $\mathcal{T}$:

1) $\alpha^T_{p,n} = \lim\limits_{k\to\infty} \alpha^{T_k}_{p,n}$,
2) $\beta^T_p = \lim\limits_{k\to\infty} \beta^{T_k}_p$,

3) $\gamma_p^T = \lim\limits_{k\to\infty} \gamma_p^{T_k}$,

3) $\varepsilon^T = \lim\limits_{k\to\infty} \varepsilon^{T_k}$,

4) $\beta_p^T = \gamma_p^T = \omega = \lim\limits_{n} \alpha_{p,n}^{T_k}$,

5) $\varepsilon^T = 1 = \lim\limits_{p,n} \alpha_{p,n}^{T_k}$.

The items 1)–5) show that limit values for Szmielew invariants are independent modulo $\alpha_{p,n}^{T_k}$, i.e., the limits of $\beta_p^{T_k}$, $\gamma_p^{T_k}$, ε^{T_k} can produce only β_p^T, γ_p^T, ε^T, respectively, whereas $\alpha_{p,n}^{T_k}$ can generate both $\alpha_{p,n}^T$, $\beta_p^T = \gamma_p^T = \omega$ and $\varepsilon^T = 1$.

Thus, we can extend the notion of dimension $\dim(\mathcal{T})$ till arbitrary families $\mathcal{T} \subseteq \mathcal{T}_A$ as the number of independent limit values for $\mathcal{T}$. Notice that the equation (5.3) stays valid for the general case.

We denote by **Szm** the set $\{\alpha_{p,n} \mid p \in P, n \in \omega \setminus \{0\}\} \cup \{\beta_p \mid p \in P\} \cup \{\gamma_p \mid p \in P\} \cup \{\varepsilon\}$.

Theorem 5.5. *For any infinite family $\mathcal{T}$ of theories of abelian groups the following conditions are equivalent:*

(1) *$\mathcal{T}$ is e-minimal;*

(2) $\dim(\mathcal{T}) = 1$*;*

(3) *for any upper bound $\xi \geq m$ or lower bound $\xi \leq m$, for $m \in \omega$, of a Szmielew invariant $\xi \in$ **Szm**, there are finitely many theories in $\mathcal{T}$ satisfying this bound; having finitely many theories with $\xi \geq m$, there are infinitely many theories in $\mathcal{T}$ with a fixed value $\alpha_{p,n} < m$, if $\xi = \alpha_{p,n}$, with a fixed value $\beta_p < m$, if $\xi = \beta_p$, with a fixed value $\gamma_p < m$, if $\xi = \gamma_p$, and with a fixed value $\varepsilon < m$, if $\xi = \varepsilon$.*

Proof repeats arguments for Theorems 5.3 and 5.4 replacing $\alpha_{p,n}$ by ξ. $\square$

It is known [12, Theorem 7.3] that e-minimal families have unique accumulation points. Thus, Theorem 5.5 characterizes the possibilities of approximations of theories of abelian groups by families with unique accumulation points. Since theories of finite groups are isolated by complete sentences, the possibilities for the approximations are exhausted by theories of infinite groups. As the equations (3.2) and (3.3) can not produce complete sentences for infinite abelian groups, theories T of abelian groups $\mathcal{A}$ are approximable if and only if $\mathcal{A}$ are infinite. Moreover, in view of Proposition 3.1 these theories T can be approximated by e-minimal families as follows.

If T has finitely many prime p with positive Szmielew invariants and $\varepsilon = 0$ then by Proposition 3.1, T is approximated by a family of theories T_i, $i \in \omega$, of finite groups and with fixed $\alpha_{p,n}$, if $\alpha_{p,n}$ is finite for T, and with strictly increasing $\alpha_{p,n}$, if $\alpha_{p,n} = \omega$ for T.

If T has finitely many prime p with positive Szmielew invariants and $\varepsilon = 1$ then T is approximated by a family of theories T_i, $i \in \omega$, with $\varepsilon = 1$ and same positive values of Szmielew invariants for infinitely many prime q, replacing p by q, including given p, and forming sets Q_i such that $Q_i \supset Q_{i+1}$, $i \in \omega$, and $\bigcap\limits_{i\in\omega} Q_i$ consists of all p with positive Szmielew invariants for T.

If T has infinitely many prime p with positive Szmielew invariants then T is approximated by a family of theories T_i, $i \in \omega$, with same positive values of Szmielew invariants for finitely many prime p, and forming sets Q'_i such that $Q'_i \subset Q'_{i+1}$, $i \in \omega$, and $\bigcup\limits_{i\in\omega} Q'_i$ consists of all p with positive Szmielew invariants for T.

Thus, the following theorem holds.

Theorem 5.6. *For any theory T of an abelian group $\mathcal{A}$ the following conditions are equivalent:*
(1) *T is approximated by some family of theories;*
(2) *T is approximated by some e-minimal family;*
(3) *$\mathcal{A}$ is infinite.*

Remark 5.7. Theorem 5.6 characterizes accumulation points in the set of theories of abelian groups. Items 1)–5) allows to define Szmielew invariants for accumulation points via correspondent Szmielew invariants of given theories. Thus, these items give possibilities to control theories of abelian groups by their approximations in terms of Szmielew invariants.

Remark 5.8. Theorems 4.2 and 5.6 describes accumulation points for the set of theories of finite abelian groups being theories of pseudofinite abelian groups. Thus the ranks for subfamilies $\mathcal{T}$ of $\mathcal{T}_{A,\mathrm{fin}}$ are controlled by cardinalities and links in $\mathrm{Cl}_E(\mathcal{T}) \cap \mathcal{T}_{A,\mathrm{pf}}$.

Now we can consider some particular possibilities for e-minimal approximations taking subfamilies $\mathcal{T}$ of the family $\mathcal{T}_A$ of all theories of abelian groups as follows.

If all Szmielew invariants except one of $\xi \neq \varepsilon$ are fixed for theories in infinite $\mathcal{T} \subset \mathcal{T}_A$ then $\mathcal{T}$ is e-minimal with unique accumulation point having $\xi = \omega$.

Since there are continuum many sequences of Szmielew invariants $\neq \xi$, we have continuum many pairwise disjoint e-minimal subfamilies of $\mathcal{T}_A$ obtaining the following:

Proposition 5.9. *The family $\mathcal{T}_A$ contains continuum many pairwise disjoint e-minimal subfamilies.*

Again, in particular, if $\xi \neq \varepsilon$ is unique nonzero Szmielew invariant for an infinite $\mathcal{T} \subset \mathcal{T}_A$ then $\mathcal{T}$ is e-minimal whose unique accumulation point has exactly one nonzero Szmielew invariant $\xi = \omega$ modulo ε. And there are countably many such e-minimal families.

6. Ranks for families of theories

Since there are continuum many theories of abelian groups, varying Szmielew invariants, Theorem 2.3, as well as Theorem 4.1 and monotony of rank imply the following proposition for the family $\mathcal{T}_A$.

Proposition 6.1. $\mathrm{RS}(\mathcal{T}_A) = \infty$.

Having RS $= \infty$ for the family of all theories of (finite or infinite) abelian groups, we also notice that there are continuum many theories of divisible or, respectively, torsion free abelian groups that again produces RS $= \infty$ for the families of all theories of divisible/torsion free abelian groups.

By Theorem 2.4 and 4.1 there are many e-minimal d-definable subfamilies of $\mathcal{T}_A$. Below we generalize Theorem 2.4 for $\mathcal{T}_A$ and arbitrary pair (α, n), where α is a countable ordinal and $n \in \omega \setminus \{0\}$.

Theorem 6.2. *Let α be at most countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d-definable subfamily $(\mathcal{T}_A)_\Phi$ such that $\mathrm{RS}((\mathcal{T}_A)_\Phi) = \alpha$ and $\mathrm{ds}((\mathcal{T}_A)_\Phi) = n$.*

Proof. If $\alpha \leq 1$ we can apply arguments for [6, Theorem 4.6] obtaining the assertion. If $\alpha \geq 2$ we consider a countable family of countable subsets X_i^Δ of the set P, where Δ is a sequence of indexes for some $\beta < \alpha$, $i \in \omega$, as follows. Take an arbitrary countable family $\mathcal{T} \subset \mathcal{T}_A$ with $\mathrm{RS}(\mathcal{T}) = \alpha$ and $\mathrm{ds}(\mathcal{T}) = n$. By [6, Proposition 4.4] there are countably many countable s-definable subfamilies $\mathcal{T}_\varphi$ with $\mathrm{RS}(\mathcal{T}_\varphi) = \beta \leq \alpha$, $\beta > 0$, and $\mathrm{ds}(\mathcal{T}_\varphi) = 1$ witnessing $\mathrm{RS}(\mathcal{T}) = \alpha$ and $\mathrm{ds}(\mathcal{T}) = n$ such that distinct subfamilies with $\mathrm{RS} = \beta$ are disjoint. We can code these subfamilies by $\mathcal{T}_i^\Delta$, where Δ is a sequence coding the rank of taken subfamily as well as containing codes of all $\mathcal{T}_\varphi$ containing $\mathcal{T}_i^\Delta$, $i \in \omega$. Now we replace $\mathcal{T}_i^\Delta$ by countable $X_i^\Delta \subset P$ such that:

1) if Δ witnesses a rank β then $X_i^\Delta \cap X_j^\Delta = \emptyset$ for $i \neq j$;

2) if Δ and Δ' witness ranks β and γ, $\beta < \gamma$, and Δ contains the code for $\mathcal{T}_i^{\Delta'}$ then $X_j^\Delta \subset X_i^{\Delta'}$ for each j, moreover, $X_i^{\Delta'} = \bigcup\limits_{\Delta, j} X_j^\Delta$, where the union is taken with respect to all admissible codes Δ, for the rank β and inclusions $X_j^\Delta \subset X_i^{\Delta'}$, and indexes j enumerating families X_j^Δ.

We denote by Y the union of all sets X_i^Δ and assume that $P \setminus Y$ is infinite. Prime numbers in $P \setminus Y$ are called *free* and will be used to mark s-definable subfamilies of theories in $\mathcal{T}_A$. Now we construct, in the following way, the required subfamily $(\mathcal{T}_A)_\Phi$ consisting of theories in families $(\mathcal{T}_A)_i^{\Delta'}$ correspondent to the families X_i^Δ and satisfying the conditions:

1′) if Δ' and Δ'' witness a rank β then $(\mathcal{T}_A)_i^{\Delta'} \cap (\mathcal{T}_A)_j^{\Delta''} = \emptyset$ for $i \neq j$;

2′) if Δ' and Δ'' witness ranks β and γ, $\beta < \gamma$, and Δ' contains the code $\langle \Delta'', i \rangle$ then $(\mathcal{T}_A)_j^{\Delta'} \subset (\mathcal{T}_A)_i^{\Delta''}$ for each j, moreover, $(\mathcal{T}_A)_i^{\Delta''} = \bigcup\limits_{\Delta', j} (\mathcal{T}_A)_j^{\Delta'}$.

We assume that all theories in $(\mathcal{T}_A)_\Phi$ satisfy $\alpha_{p,1}\in\{0,1\}$, $p\in P$, $\alpha_{p,n}=0$, for $n\geq 2$ and $p\in P$, $\beta_p=\gamma_p=0$ for $p\in P$, $\varepsilon=1$. If Δ codes the rank 1 then X_i^Δ defines the e-minimal, in view of Theorem 5.5, family $(\mathcal{T}_A)_i^\Delta$ of all theories T of abelian groups with unique positive $\alpha_{p,1}$, where $p\in X_i^\Delta$, and with some positive $\alpha_{p',1}$ for β free prime numbers p' marking X_i^Δ and all $X_j^{\Delta'}$ containing X_i^Δ such that writing some $\alpha_{p'',1}=1$ we can separate $(\mathcal{T}_A)_i^\Delta$ both from bigger families $(\mathcal{T}_A)_j^{\Delta'}$, collecting e-minimal families $(\mathcal{T}_A)_k^{\Delta''}$ with $\alpha_{p',1}=1$, and $\mathrm{RS}((\mathcal{T}_A)_j^{\Delta'})>\mathrm{RS}((\mathcal{T}_A)_i^\Delta)$ and from distinct marked families of the same rank. Now we unite $(\mathcal{T}_A)_i^\Delta$ by values $\alpha_{p',1}\in\{0,1\}$ forming the families $(\mathcal{T}_A)_j^{\Delta'}$, and the required d-definable subfamily $\mathcal{T}=\mathrm{Cl}_E\left(\bigcup\limits_{\Delta,i}(\mathcal{T}_A)_i^\Delta\right)$ of $\mathcal{T}_A$ satisfies $\mathrm{RS}(\mathcal{T})=\alpha$ and $\mathrm{ds}(\mathcal{T})=n$. □

Remark 6.3. Varying values $\alpha_{p,n}$ for the families in the construction for the proof of Theorem 6.2 we obtain continuum many disjoint, modulo ε, families $(\mathcal{T}_A)_\Phi$ such that $\mathrm{RS}((\mathcal{T}_A)_\Phi)=\alpha$ and $\mathrm{ds}((\mathcal{T}_A)_\Phi)=n$.

Notice that the arguments stay valid replacing $\alpha_{p,n}$ by β_p or/and γ_p.

Applying the construction for the proof of Theorem 6.2 we can form a family $\mathcal{T}$ of theories of finite abelian groups for given rank and degree, with a d-definable closure $\mathrm{Cl}_E(\mathcal{T})\subset\mathcal{T}_{A,\mathrm{fin}}\cup\mathcal{T}_{A,\mathrm{pf}}$. Thus, the following theorem holds:

Theorem 6.4. *Let α be a countable ordinal, $n\in\omega\setminus\{0\}$. Then there is a subfamily $\mathcal{T}\subset\mathcal{T}_{A,\mathrm{fin}}$ such that $\mathrm{RS}(\mathcal{T})=\alpha$, $\mathrm{ds}(\mathcal{T})=n$, and $\mathrm{Cl}_E(\mathcal{T})\subset\mathcal{T}_{A,\mathrm{fin}}\cup\mathcal{T}_{A,\mathrm{pf}}$ is d-definable with $(\mathrm{RS}(\mathrm{Cl}_E(\mathcal{T})),\mathrm{ds}(\mathrm{Cl}_E(\mathcal{T}))=(\alpha,n)$.*

Remark 6.5. Theorems 2.6, 2.7, and 6.2 allow to form countable superatomic Boolean algebras $\mathcal{B}_{\alpha,n}$, unique up to isomorphism, for E-closed, d-definable families of abelian groups with arbitrary countable CB-invariants (α,n). Additionally, Theorem 6.4 produces similar realizations using E-closed, d-definable subsets of $\mathcal{T}_{A,\mathrm{fin}}\cup\mathcal{T}_{A,\mathrm{pf}}$.

7. Conclusion

We found ranks for families of theories of abelian groups. In particular, we studied closures and ranks for theories of finite abelian groups observing that the set of theories of finite abelian groups in not totally transcendental. We characterized pseudofinite abelian groups in terms of Szmielew invariants. e-minimal families of theories of abelian groups are characterized both in terms of dimension and in terms of inequalities for Szmielew invariants. These characterizations were obtained both for finite abelian groups and in general case. Furthermore we gave characterizations for approximability of theories of abelian groups and show the possibility to count Szmielew invariants via these parameters for approximations. We described possibilities to form d-definable families of theories of abelian groups having given countable rank and degree.

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Ранги семейств теорий абелевых групп

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Аннотация. Ранг семейства теорий подобен рангу Морли и может служить мерой сложности или богатства данного семейства. Увеличивая ранг расширениями семейства, мы получаем более богатые семейства, которые в случае достижения бесконечности могут рассматриваться как “достаточно богатые”. В данной статье реализуются ранги для семейств теорий абелевых групп. В частности, изучаются ранги и замыкания для семейств теорий конечных абелевых групп. Показано, что множество теорий конечных абелевых групп не является тотально трансцендентным, т.е. его ранг равен бесконечности. В терминах шмелевских инвариантов характеризуются псевдоконечные абелевы группы. Кроме того, характеризуются e-минимальные семейства теорий абелевых групп как на языке размерности, т.е. числа независимых пределов шмелевских инвариантов, так и в терминах неравенств для шмелевских инвариантов. Эти характеризации получены для конечных абелевых групп и в общем случае. Найдены характеризации аппроксимируемости теорий абелевых групп и показаны возможности подсчета шмелевских инвариантов через параметры аппроксимаций. Описаны возможности построения d-определимых семейств теорий абелевых групп, имеющих данный счетный ранг и данную степень.

Ключевые слова: семейство теорий, абелева группа, ранг, степень, замыкание.

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