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On Exact Multidimensional Solutions of a Nonlinear System of First Order Partial Differential Equations*

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Abstract. This study is concerned with a system of two nonlinear first order partial differential equations. The right-hand sides of the system contain the squares of the gradients of the unknown functions. Such type of Hamilton-Jacobi like equations are considered in mechanics and control theory. In the paper, we propose to search a solution in the form of an ansatz, the latter containing a quadratic dependence on the spatial variables and arbitrary functions of time. The use of this ansatz allows us to decompose the search of the solution's components depending on the spatial variables and time. In order to find the dependence on the spatial variables one needs to solve an algebraic system of some matrix and vector equations and of a scalar equation. A general solution of this system of equations is found in a parametric form. To find the time-dependent components of the solution of the original system, we are faced with a system of nonlinear differential equations. The existence of exact solutions of a certain kind for the original system is established. A number of examples of the constructed exact solutions, including periodic in time and anisotropic in the spatial variables ones, are given. The spatial structure of the solutions is analyzed revealing that it depends on the rank of the matrix of the quadratic form entering the solution.

Keywords: nonlinear system, Hamilton-Jacobi type equations, exact solutions.

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1. Introduction

The aim of the present work is to construct exact multidimensional solutions of the following nonlinear system of first order partial differential equations:

$$\begin{cases} u_t = \beta_1(t)|\nabla u|^2 + \alpha_1(t)v + f(t), \\ v_t = \beta_2(t)|\nabla v|^2 + \alpha_2(t)u + g(t). \end{cases} \tag{1.1}$$

Here, $u \triangleq u(\mathbf{x},t)$, $v \triangleq v(\mathbf{x},t)$, $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $u_t \triangleq \dfrac{\partial u}{\partial t}$, ∇ is the nabla operator; $\alpha_i(t)$, $\beta_i(t)$, $i = 1,2$, $f(t)$, $g(t)$ are given functions of time t. First order partial differential equations of the kind (1.1) are considered in mechanics and control theory, in particular, when $u = v$, $\alpha_1(t) = \alpha_2(t) = \alpha(t)$, $\beta_1(t) = \beta_2(t) = \beta(t)$, $f(t) = g(t)$ the system (1.1) is reduced to the following multidimensional Hamilton-Jacobi type equation [7; 16]

$$u_t = \beta(t)|\nabla u|^2 + \alpha(t)u + f(t). \tag{1.2}$$

When $\beta(t) \equiv -1/(2m)$, $m \in \mathbb{R}$, $\alpha(t) = f(t) \equiv 0$ from (1.2) we obtain the Hamilton-Jacobi equation for a free particle [1]:

$$u_t + \frac{1}{2m}|\nabla u|^2 = 0, \tag{1.3}$$

where the function $u(\mathbf{x},t)$ represents the action, m is a constant (the particle's mass). We note that the maximal local (pointwise) invariance group of Eq. (1.3) is also given and some of its exact solutions are provided in [1]. Therefore, the construction of exact multidimensional solutions of nonlinear systems of the form (1.1) is a quite interesting problem. In general, the construction of exact solutions of nonlinear differential systems is important in a number of other applications, for instance, in the theory of kinetic systems [8; 13; 14].

In this paper, we search exact multidimensional solutions of the system (1.1) in the form

$$u(\mathbf{x},t) = \psi_1(t)\Big[W(\mathbf{x}) + \varphi_1(t)\Big], \quad v(\mathbf{x},t) = \psi_2(t)\Big[W(\mathbf{x}) + \varphi_2(t)\Big], \tag{1.4}$$

$$W(\mathbf{x}) = \frac{1}{2}(A\mathbf{x},\mathbf{x}) + (\mathbf{B},\mathbf{x}) + C, \tag{1.5}$$

where $\psi_i(t)$, $\varphi_i(t)$, $i = 1,2$ are unknown functions of time; a nonzero numerical symmetric matrix A of dimension $n \times n$, a constant vector $\mathbf{B} \in \mathbb{R}^n$ and a constant $C \in \mathbb{R}$ will be defined later. Here and in what follows $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$. Formulas similar to (1.4)–(1.5) were successfully used by the authors in several previous papers [10–12] to construct exact solutions of the equation of nonlinear heat conduction and of nonlinear parabolic systems with exponential nonlinearities [5; 6]. Note that the ideology to construct solutions in the form (1.4) corresponds to the method of generalized separation of variables [2; 3; 9; 15; 16].

2. Reduction to a system of ODE

After the substitution of the functions (1.4) into the system of equations (1.1) and simple calculations, we obtain the equations

$$\psi_1' W(\mathbf{x}) + (\psi_1\varphi_1)' = \beta_1(t)\psi_1^2 |\nabla W(\mathbf{x})|^2 + \alpha_1(t)\psi_2 \Big[W(\mathbf{x}) + \varphi_2\Big] + f(t), \quad (2.1)$$

$$\psi_2' W(\mathbf{x}) + (\psi_2\varphi_2)' = \beta_2(t)\psi_2^2 |\nabla W(\mathbf{x})|^2 + \alpha_2(t)\psi_1 \Big[W(\mathbf{x}) + \varphi_1\Big] + g(t). \quad (2.2)$$

Here, $\psi_i = \psi_i(t)$, $\varphi_i = \varphi_i(t)$, $\psi_i' = \dfrac{d\psi_i}{dt}$, $\varphi_i' = \dfrac{d\varphi_i}{dt}$, $i = 1, 2$. From (1.5) we immediately find

$$|\nabla W(\mathbf{x})|^2 = (A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2.$$

In view of these relations and (1.5) the equalities (2.1), (2.2) can be rewritten as

$$\big(\psi_1' - \alpha_1(t)\psi_2\big)\left[\frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + C\right] + (\psi_1\varphi_1)' =$$

$$\beta_1(t)\psi_1^2\left[(A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2\right] + \alpha_1(t)\psi_2\varphi_2 + f(t), \quad (2.3)$$

$$\big(\psi_2' - \alpha_2(t)\psi_1\big)\left[\frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + C\right] + (\psi_2\varphi_2)' =$$

$$\beta_2(t)\psi_2^2\left[(A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2\right] + \alpha_2(t)\psi_1\varphi_1 + g(t). \quad (2.4)$$

It is easy to verify that if the symmetric matrix A, the vector $\mathbf{B}$ and the constant C satisfy the following system of algebraic equations (SAE):

$$A = 2\sigma\, A^2, \quad \mathbf{B} = 2\sigma\, A\mathbf{B}, \quad C = \sigma\, |\mathbf{B}|^2, \quad (2.5)$$

where $\sigma \neq 0$ is the separation constant, then the equalities (2.3), (2.4) are reduced to the following system of ordinary differential equations (ODE):

$$\psi_1' - \alpha_1(t)\psi_2 - \frac{\beta_1(t)}{\sigma}\,\psi_1^2 = 0, \quad \psi_2' - \alpha_2(t)\psi_1 - \frac{\beta_2(t)}{\sigma}\,\psi_2^2 = 0, \quad (2.6)$$

$$(\psi_1\varphi_1)' - \alpha_1(t)\psi_2\varphi_2 - f(t) = 0, \quad (\psi_2\varphi_2)' - \alpha_2(t)\psi_1\varphi_1 - g(t) = 0. \quad (2.7)$$

Hence, the following statement is valid.

Theorem 1. *The nonlinear system of first order partial differential equations* (1.1) *has exact solutions* (1.4)*, where the function* $W(\mathbf{x})$ *can be an arbitrary polynomial of the form* (1.5) *with the coefficients satisfying the* SAE (2.5)*, and the functions* $\psi_i(t)$*,* $\varphi_i(t)$*,* $i = 1, 2$ *are solutions of* ODE *systems* (2.6)*,* (2.7)*.*

Remark 1. Theorem 1 allows us to decompose the construction of the solution's components depending on spatial variables and time. Such a decomposition essentially simplifies the problem in that it replaces the original nonlinear system of first order partial differential equations by systems of algebraic and ordinary differential equations. This may prove useful in the development of numerical methods and algorithms for constructing approximate solutions of the corresponding boundary value problems.

The algebro-differential systems (2.5)–(2.7) consist of a block of algebraic equations (2.5) and a block of ODE (2.6), (2.7). We will investigate each block separately.

3. Solvability of algebraic equations

First of all, we consider the solvability of the matrix equation of the system (2.5). We note that it always has the trivial solution $A = 0$. Hence, in the sequel we consider only nontrivial solutions of this equation. It is easy to verify that a solution of the matrix equation of system (2.5) is of the form $A = \dfrac{1}{2\sigma} P$, where P is an arbitrary idempotent matrix, i.e. a matrix satisfying the equality $P^2 = P$. It is known [4] that any idempotent matrix P can be written as $P = ME_mM^{-1}$, where M is an arbitrary nondegenerate matrix of order n, E_m is a diagonal matrix with the diagonal having $m \in \{1, 2, \dots, n\}$ units and $n - m$ zeros arranged in an arbitrary order; E_m is also an idempotent matrix: $E_m^2 = E_m$. Since we are interested only in symmetric matrices A, we need to choose idempotent matrices P to be symmetric as well, i.e. $P = SE_mS^T$, where S is an arbitrary orthogonal matrix. Therefore, the matrix

$$A = \frac{1}{2\sigma} SE_mS^T \tag{3.1}$$

is a solution of the matrix equation (2.5). In this case, we have

$$\operatorname{tr} A = \frac{m}{2\sigma}, \quad m \le n, \quad n \in \mathbb{N}, \quad n \ge 2. \tag{3.2}$$

The vector equation (2.5) is a system of n linear homogeneous algebraic equations with respect to the components $b_1, \dots, b_n$ of the unknown vector $\mathbf{B}$. For any fixed matrix (3.1) with the rank $\operatorname{rank} A = m < n$ there always exists a nontrivial solution to the linear homogeneous system. Moreover, the components $b_1, \ldots, b_m$ of the vector $\mathbf{B}$ can be chosen arbitrarily from an m –dimensional linear manifold. In the case when $\operatorname{rank} A = m \equiv n$, i.e. if $E_m \equiv E$, the linear homogeneous system of equations has a solution, which is an arbitrary vector $\mathbf{B} \in \mathbb{R}^n$. Once the solutions of the matrix and vector equations are successively found, the constant C is uniquely determined from the scalar equation of the system (2.5).

Example 1. Let $n = 3$ and square 3×3 matrices E_m have ranks 3, 2, and 1, respectively. An arbitrary orthogonal matrix S of dimension 3×3 is represented by the formula

$$S = \begin{bmatrix} a_{11} & -\sin(s_1)\cos(s_2) & a_{13} \\ a_{21} & \cos(s_1)\cos(s_2) & a_{23} \\ \cos(s_2)\sin(s_3) & -\sin(s_2) & \cos(s_2)\cos(s_3) \end{bmatrix}.$$

Here, for simplicity we introduce the following notation

$$a_{11} = \cos(s_1)\cos(s_3) - \sin(s_1)\sin(s_2)\sin(s_3),$$

$$a_{21} = \cos(s_1)\sin(s_2)\sin(s_3) + \sin(s_1)\cos(s_3),$$

$$a_{13} = -\cos(s_1)\sin(s_3) - \sin(s_1)\sin(s_2)\cos(s_3),$$

$$a_{23} = \cos(s_1)\sin(s_2)\cos(s_3) - \sin(s_1)\sin(s_3).$$

To construct specific examples, we give the parameters s_1, s_2, s_3 certain numerical values. Let, for instance, $s_1 = -\dfrac{\pi}{4}$, $s_2 = \arcsin\left(\dfrac{1}{3}\right)$, $s_3 = \dfrac{3\pi}{4}$, then the orthogonal matrix S will have the form

$$S = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}.$$

The matrix E_m of rank 3 coincides with the identity matrix and, in this case, we obtain the form $\dfrac{1}{2}(A\mathbf{x}, \mathbf{x}) = \dfrac{1}{4\sigma}\left(x^2 + y^2 + z^2\right)$. For the matrix A of this quadratic form a solution of the vector equation of the system (2.5) is an arbitrary vector $\mathbf{B} = (b_1, b_2, b_3)$. In addition, formula (1.5) will have the form

$$W_0(x, y, z) = \frac{1}{4\sigma}\left(x^2 + y^2 + z^2\right) + b_1 x + b_2 y + b_3 z + \sigma\left(b_1^2 + b_2^2 + b_3^2\right) \quad (3.3)$$

and we obtain solutions radially symmetric with respect to the spatial variables x, y, z.

Now, take the matrices E_m of rank 2:

$$E_{1m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{2m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{3m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For these matrices, from (3.1) we have

$$A_1 = \frac{1}{18\sigma} \begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}, \quad A_2 = \frac{1}{18\sigma} \begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix},$$

$$A_3 = \frac{1}{18\sigma} \begin{bmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}.$$

From the vector equation (2.5) we obtain the corresponding vectors $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$:

$$\mathbf{B}_1 = (l_1, 2l_1 + 2l_3, l_3), \quad \mathbf{B}_2 = (l_1, l_2, 2l_1 + 2l_2), \quad \mathbf{B}_3 = (2l_2 + 2l_3, l_2, l_3),$$

where l_1, l_2, l_3 are arbitrary constants. Hence, the constants C_1, C_2, C_3 are given by the formulas

$$C_1 = \sigma \left(5l_1^2 + 8l_1 l_3 + 5l_3^2 \right), \quad C_2 = \sigma \left(5l_1^2 + 8l_1 l_2 + 5l_2^2 \right),$$

$$C_3 = \sigma \left(5l_2^2 + 8l_2 l_3 + 5l_3^2 \right).$$

Consequently, we obtain

$$W_1(x, y, z) = \frac{1}{180\sigma} \left[(5x + 2y - 4z)^2 + 9(2y + z)^2 \right] + \tag{3.4}$$

$$+ l_1 x + (2l_1 + 2l_3) y + l_3 z + \sigma \left(5l_1^2 + 8l_1 l_3 + 5l_3^2 \right),$$

$$W_2(x, y, z) = \frac{1}{180\sigma} \left[(5x - 4y + 2z)^2 + 9(y + 2z)^2 \right] + \tag{3.5}$$

$$+ l_1 x + l_2 y + (2l_1 + 2l_2)\, z + \sigma \left(5l_1^2 + 8l_1 l_2 + 5l_2^2 \right),$$

$$W_3(x, y, z) = \frac{1}{72\sigma} \left[(4x + y + z)^2 + 9(y - z)^2 \right] + \tag{3.6}$$

$$+ (2l_2 + 2l_3)\, x + l_2 y + l_3 z + \sigma \left(5l_2^2 + 8l_2 l_3 + 5l_3^2 \right).$$

Therefore, in this case, we have solutions anisotropic with respect to the spatial variables x, y, z.

Finally, consider the matrices E_m of rank 1:

$$E_{1m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{2m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{3m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For these matrices, from (3.1) we have

$$A_1 = \frac{1}{18\sigma}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \quad A_2 = \frac{1}{18\sigma}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix},$$

$$A_3 = \frac{1}{18\sigma}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

From the vector equation (2.5) we obtain the corresponding vectors $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$:

$$\mathbf{B}_1 = (l, -2l, -2l), \quad \mathbf{B}_2 = \left(l, l, -\frac{1}{2}\, l\right), \quad \mathbf{B}_3 = \left(l, -\frac{1}{2}\, l, l\right),$$

where l is an arbitrary constant. Hence, the constants C_1, C_2, C_3 are given by the formulas $C_1 = 9\sigma l^2$, $C_2 = \dfrac{9}{4}\,\sigma l^2$, $C_3 = \dfrac{9}{4}\,\sigma l^2$. Finally, we get

$$\begin{aligned} W_1(x,y,z) &= \frac{1}{36\sigma}\,\omega_1^2 + l\omega_1 + 9\sigma l^2, \quad \text{where} \quad \omega_1 = x - 2y - 2z, \\ W_2(x,y,z) &= \frac{1}{36\sigma}\,\omega_2^2 + \frac{l}{2}\,\omega_2 + \frac{9}{4}\,\sigma l^2, \quad \text{where} \quad \omega_2 = 2x + 2y - z, \\ W_3(x,y,z) &= \frac{1}{36\sigma}\,\omega_3^2 + \frac{l}{2}\,\omega_3 + \frac{9}{4}\,\sigma l^2, \quad \text{where} \quad \omega_3 = 2x - y + 2z. \end{aligned} \tag{3.7}$$

In this case, we will actually obtain solutions ¡¡pseudo-multidimensional¿¿ with respect to the variables ω_1, ω_2, ω_3, i.e. solutions containing a linear combination of spatial variables.

For $n = 2$, square 2×2 matrices E_m of unit rank also provide ¡¡pseudo-multidimensional¿¿ solutions. And the identity matrix E_m of rank 2 for any 2×2 orthogonal matrix S gives $W(x,y) = \dfrac{1}{4\sigma}\left(x^2 + y^2\right) + b_1 x + b_2 y + \sigma\left(b_1^2 + b_2\right)$, where b_1, b_2 are arbitrary constants. In this case, we have solutions radially symmetric with respect to the spatial variables x and y. Therefore, in the two-dimensional coordinate space, the nonlinear parabolic system (1.1) does not have multidimensional exact solutions anisotropic with respect to the spatial variables of the form (1.4) with the function (1.5).

4. Solvability of the ODE system (2.6), (2.7)

In this section, we construct exact solutions of the ODE system (2.6), (2.7) under the assumption that the functions $\alpha_i(t)$, $\beta_i(t)$ are identically constant: $\alpha_i(t) \equiv \alpha_i \in \mathbb{R}$, $\alpha_i \neq 0$, $\beta_i(t) \equiv \beta_i \in \mathbb{R}$, $\beta_i \neq 0$, $i = 1, 2$. Such an assumption is useful when searching solutions of the system of equations (2.6) in the case when $\psi_1(t)$, $\psi_2(t)$ are constant and when constructing periodic in time solutions of the system (1.1).

4.1. The case when the functions $\psi_1(t)$, $\psi_2(t)$ are constant

We start with the integration of the nonlinear ODE system (2.6) with respect to the unknown functions $\psi_1(t)$, $\psi_2(t)$. But, first, we note that nontrivial constant solutions of the ODE system (2.6) are equally interesting for our study. They are determined from the following system of nonlinear algebraic equations:

$$\frac{\beta_1}{\sigma}\,\psi_1^2 + \alpha_1\psi_2 = 0, \quad \frac{\beta_2}{\sigma}\,\psi_2^2 + \alpha_2\psi_1 = 0.$$

Whence we find a nontrivial solution

$$\psi_{10} = \sigma\left(-\frac{\alpha_1^2\alpha_2}{\beta_1^2\beta_2}\right)^{1/3}, \quad \psi_{20} = -\frac{\sigma\beta_1}{\alpha_1}\left(-\frac{\alpha_1^2\alpha_2}{\beta_1^2\beta_2}\right)^{2/3}. \tag{4.1}$$

For these constants ψ_{10}, ψ_{20} the ODE system (2.7) will have the form

$$\psi_{10}\varphi_1' - \alpha_1\psi_{20}\varphi_2 = f(t), \quad \psi_{20}\varphi_2' - \alpha_2\psi_{10}\varphi_1 = g(t), \tag{4.2}$$

Hence, taking into account the above results and using Theorem 1 we see that the following statement holds.

State 1. *The nonlinear system of first order partial differential equations* (1.1) *has an exact multidimensional solution*

$$u(\mathbf{x}, t) = \psi_{10}\left[W(\mathbf{x}) + \varphi_1(t)\right], \quad v(\mathbf{x}, t) = \psi_{20}\left[W(\mathbf{x}) + \varphi_2(t)\right],$$

where the constants ψ_{10}*,* ψ_{20} *are given by* (4.1)*, and the functions* $\varphi_1(t)$*,* $\varphi_2(t)$ *are determined from the linear* ODE *system* (4.2)*.*

Example 2. Let $\alpha_1\alpha_2 > 0$, then the nonlinear system of first order partial differential equations of the form

$$u_t = \beta_1|\nabla u|^2 + \alpha_1 v + \sin(t), \;\; v_t = \beta_2|\nabla v|^2 + \alpha_2 u + \cos(t),$$

in the three-dimensional coordinate space, has a parametric family of particular exact solutions anisotropic in the spatial variables

$$u_k(x, y, z, t) = \psi_{10}\left[W_k(x, y, z) + \varphi_1(t)\right], \quad k = 1, 2, 3,$$

$$v_k(x, y, z, t) = \psi_{20}\left[W_k(x, y, z) + \varphi_2(t)\right], \quad k = 1, 2, 3,$$

where the constants ψ_{10}, ψ_{20} are given by (4.1), $W_k(x,y,z)$, $k=1,2,3$ are defined by the formulas (3.4)–(3.6), and the functions $\varphi_1(t)$, $\varphi_2(t)$ have the form

$$\varphi_1(t) = C_1 e^{-\sqrt{\alpha_1\alpha_2}\,t} + C_2 e^{\sqrt{\alpha_1\alpha_2}\,t} - \frac{\alpha_1+1}{(\alpha_1\alpha_2+1)\psi_{10}}\cos(t),$$

$$\varphi_2(t) = \frac{\sqrt{\alpha_1\alpha_2}}{\alpha_1}\,\frac{\psi_{10}}{\psi_{20}}\left(C_2 e^{\sqrt{\alpha_1\alpha_2}\,t} - C_1 e^{-\sqrt{\alpha_1\alpha_2}\,t}\right) + \frac{1-\alpha_2}{\alpha_1\alpha_2+1}\,\frac{1}{\psi_{20}}\sin(t).$$

Here, C_1, C_2 are arbitrary constants.

Example 3. (**Periodic in time solutions**)
Let $\alpha_1=-p^2$, $\alpha_2=q^2$, $p\neq 0$, $q\neq 0$ be arbitrary parameters, $f(t)=g(t)\equiv 0$. Then, the nonlinear system (1.1), in the three-dimensional coordinate space, has a parametric family of particular periodic in time exact solutions anisotropic in the spatial variables

$$u_k(x,y,z,t) = \psi_{10}\left[W_k(x,y,z)+\varphi_1(t)\right], \quad k=1,2,3,$$

$$v_k(x,y,z,t) = \psi_{20}\left[W_k(x,y,z)+\varphi_2(t)\right], \quad k=1,2,3,$$

where the constants ψ_{10}, ψ_{20} are given by (4.1), $W_k(x,y,z)$, $k=1,2,3$ are defined by the formulas (3.4)–(3.6), and the functions $\varphi_1(t)$, $\varphi_2(t)$ have the form

$$\varphi_1(t)=C_1\sin(pq\,t)+C_2\cos(pq\,t),\;\; \varphi_2(t)=\frac{q}{p}\,\frac{\psi_{10}}{\psi_{20}}\left(C_2\sin(pq\,t)-C_1\cos(pq\,t)\right).$$

Here, C_1, C_2 are arbitrary constants.

4.2. The case of non-constant functions $\psi_1(t)$, $\psi_2(t)$

The ODE system (2.6) is easily reduced to one nonlinear ODE of the second order with respect to the function $\psi_1(t)$ of the following form:

$$\psi_1'' - \frac{\beta_2}{\alpha_1\sigma}\,\psi_1'^2 + \frac{2\beta_1}{\sigma}\left(\frac{\beta_2}{\alpha_1\sigma}\,\psi_1^2-\psi_1\right)\psi_1' - \frac{\beta_1^2\beta_2}{\alpha_1\sigma^3}\,\psi_1^4 - \alpha_1\alpha_2\psi_1 = 0.$$

By the substitution $\psi_1(t)' = y(\psi_1)$ this ODE is reduced to the following Abel equation of the second kind

$$y\frac{dy}{d\psi_1} - \frac{\beta_2}{\alpha_1\sigma}\,y^2 + \frac{2\beta_1}{\sigma}\left(\frac{\beta_2}{\alpha_1\sigma}\,\psi_1^2-\psi_1\right)y - \frac{\beta_1^2\beta_2}{\alpha_1\sigma^3}\,\psi_1^4 - \alpha_1\alpha_2\psi_1 = 0.$$

Let the parameters of the original systems are related by the equality $\beta_1^2\alpha_1+\beta_2^2\alpha_2=0$. Then, the Abel equation has a particular exact solution $y(\psi_1)=$

$\dfrac{\beta_1}{\sigma}\psi_1^2 + \dfrac{\beta_1\alpha_1}{\beta_2}\psi_1 + \dfrac{\beta_1\alpha_1^2\sigma}{\beta_2^2}$. Whence it is easy to obtain a particular exact solution of the system (2.6) having the form

$$\psi_1^*(t) = \frac{\alpha_1\sigma}{2\beta_2}\left(\sqrt{3}\tan(T) - 1\right), \quad \psi_2^*(t) = \frac{\beta_1\alpha_1\sigma}{2\beta_2^2}\left(\sqrt{3}\tan(T) + 1\right). \quad (4.3)$$

Here, we denote $T = \dfrac{\beta_1\alpha_1\sqrt{3}}{2\beta_2}(t - t_0)$, where t_0 is an arbitrary constant. Hence, given the results above invoking Theorem 1 we have the following statement.

State 2. *The nonlinear system of first order partial differential equations* (1.1) *with the parameters related by the equality* $\beta_1^2\alpha_1 + \beta_2^2\alpha_2 = 0$ *has an exact multidimensional solution*

$$u(\mathbf{x}, t) = \psi_1^*(t)\left[W(\mathbf{x}) + \varphi_1(t)\right], \quad v(\mathbf{x}, t) = \psi_2^*(t)\left[W(\mathbf{x}) + \varphi_2(t)\right],$$

where the functions $\psi_1^*(t)$, $\psi_2^*(t)$ *are given by* (4.3), *and the functions* $\varphi_1(t)$, $\varphi_2(t)$ *are determined from the system of linear non-autonomous and inhomogeneous* ODE (2.7).

We note that in Statements 1, 2, in the case $n = 3$ ($\mathbf{x} \in \mathbb{R}^3$), as $W(x, y, z)$ we can take any of the functions given in Example 1. Thus, for the functions defined by (3.4)–(3.6) we obtain exact solutions anisotropic with respect to the spatial variables, and for the function (3.3) we can write down an exact solution radially symmetric with respect to the spatial variables x, y, z

Example 4. The nonlinear system of first order partial differential equations of the form

$$u_t = \beta_1|\nabla u|^2 + v + \mathrm{sh}(t), \quad v_t = \beta_2|\nabla v|^2 - u - \mathrm{ch}(t),$$

with the parameters such that $\beta_1^2 - \beta_2^2 = 0$ has a parametric family of particular exact solutions anisotropic with respect to the spatial variables

$$u_k(x, y, z, t) = \mu_1\left(\sqrt{3}\tan(T) - 1\right)\left[W_k(x, y, z) + \varphi_1(t)\right], \quad k = 1, 2, 3,$$

$$v_k(x, y, z, t) = \mu_2\left(\sqrt{3}\tan(T) + 1\right)\left[W_k(x, y, z) + \varphi_2(t)\right], \quad k = 1, 2, 3,$$

where $\mu_1 = \dfrac{\sigma}{2\beta_2}$, $\mu_2 = \dfrac{\beta_1\sigma}{2\beta_2^2}$, $W_k(x, y, z)$, $k = 1, 2, 3$ are defined by (3.4)–(3.6), and the functions $\varphi_1(t)$, $\varphi_2(t)$ have the form

$$\varphi_1(t) = \frac{C_1\sin t + C_2\cos t}{\sqrt{3}\,\mathrm{tg}(T) - 1}, \quad \varphi_2(t) = \frac{\mu_1\left(C_1\cos t - C_2\sin t\right) - \mathrm{sh}\,t}{\mu_2\left(\sqrt{3}\,\mathrm{tg}(T) + 1\right)}.$$

Here, $\mathrm{sh}\,t$, $\mathrm{ch}\,t$ are the hyperbolic sine and cosine, respectively, t_0, C_1, C_2 are arbitrary constants, $T = \dfrac{\beta_1\sqrt{3}}{2\beta_2}(t - t_0)$.

5. Exact solutions of the Hamilton-Jacobi type equation

In addition, we construct exact multidimensional solutions of the Hamilton-Jacobi type equation (1.2). Namely, for the equation (1.2) we have

State 3. *The Hamilton-Jacobi type equation* (1.2) *has an exact multidimensional solution*

$$u(\mathbf{x},t)=\psi(t)\left[\frac{1}{2}(A\mathbf{x},\mathbf{x})+(\mathbf{B},\mathbf{x})+C+\varphi(t)\right], \tag{5.1}$$

where the functions $\psi(t)$, $\varphi(t)$ *satisfy the ODE system*

$$\psi'-\alpha(t)\psi-\frac{\beta(t)}{\sigma}\,\psi^2=0,\quad (\psi\varphi)'=\alpha(t)\psi\varphi+f(t), \tag{5.2}$$

and the symmetric matrix A*, the constant vector* $\mathbf{B}\in\mathbb{R}^n$ *and the constant* $C\in\mathbb{R}$ *satisfy SAE* (2.5) *with an arbitrary constant* $\sigma\neq 0$*.*

The validity of this statement can be verified by the direct substitution of the function (5.1) to the equation (1.2). After elementary calculations in view of the equalities (2.5) we arrive at the ODE system (5.2). Note that the function (5.1) is also a solution of Eq. (1.3). Some of the exact solutions of Eq. (1.3) given in [1] are obtained for the function (5.1) as a particular case.

Example 5. The Hamilton-Jacobi type equation of the form

$$u_t=\beta t|\nabla u|^2+\frac{\alpha}{t}\,u+t^{\lambda},$$

has a parametric family of exact solutions anisotropic with respect to the spatial variables:

$$u_k(x,y,z,t)=\psi(t)\left[W_k(x,y,z)+\varphi(t)\right],\quad k=1,2,3,$$

where $W_k(x,y,z)$, $k=1,2,3$ are defined by (3.4)–(3.6), and the functions $\psi(t)$, $\varphi(t)$ have the form

$$\psi(t)=\frac{\alpha+2}{C_1(\alpha+2)t^{-\alpha}-\beta t^2},$$

$$\varphi(t)=\left(C_1(\alpha+2)-\beta t^{\alpha+2}\right)\left(C_2-\frac{t^{\lambda+1-\alpha}}{(\alpha+2)(\alpha-\lambda-1)}\right).$$

Here, $\alpha\neq-2$, $\beta\neq 0$, $\lambda\neq\alpha-1$ are arbitrary parameters of the equation, C_1, C_2 are the constants of integration.

6. Conclusion

In the present paper, we obtain formulas for new exact multidimensional solutions of the nonlinear system of first order partial differential equations (1.1) using the construction (1.5), which has never been used before by other authors. From the point of view of finding exact multidimensional solutions, a useful feature of the construction (1.5) is that along with the linear summands it contains the quadratic form of n variables $\frac{1}{2}(A\mathbf{x}, \mathbf{x})$. From our results it follows that the spatial structure of the solutions is determined by the rank of the matrix A. If $\operatorname{rank} A = 1$, then we have ¡¡pseudo-multidimensional¿¿ exact solutions, i.e. solutions with a linear combination of the spatial variables. If $1 < \operatorname{rank} A < n$, then we obtain exact solutions anisotropic with respect to the spatial variables. Finally, if $\operatorname{rank} A = n$, then we have exact solutions radially symmetric with respect to the spatial variables. It is shown that in the two-dimensional coordinate space the nonlinear system of first order partial differential equations (1.1) does not have multidimensional exact solutions anisotropic with respect to the spatial variables of the form (1.4) with the function $W(\mathbf{x})$ of the form (1.5). Exact multidimensional solutions of the Hamilton-Jacobi type equation (1.2) are constructed. The explicit expressions of the exact multidimensional solutions obtained in the article, which are expressed in terms of elementary functions, are important both from the theoretical and applied points of view, as they can be used for testing, adjusting and adapting numerical methods and algorithms for constructing approximate solutions of boundary value problems for nonlinear systems of first order partial differential equations of a large dimension.

References

1. Fushhich V.I., Shtelen V.M., Serov N.I. *Simmetrijnyj analiz i tochnye reshenij nelinejnyx uravnenij matematicheskoj fiziki* [Symmetry analysis and exact solutions of nonlinear equations of mathematical physics]. Kiev, Naukova dumka, 1989, 336 p. (in Russian)
2. Galaktionov V.A. , Posashkov S.A. New exact solutions of parabolic equations with quadratic nonlinearities. *Comput. Math. Math. Phys.,* 1989, vol. 29, no. 2, pp. 112–119.
3. Galactionov V.A., Svirshchevskii S.R. *Subspaces of nonlinear partial differential equations in mechanics and physics*. Chapman & Hall/CRC, 2007, 493 p.
4. Gantmaxer F.R. *Teoriya matric* [Matrix Theory]. Moscow, Nauka Publ., 1988, 552 p. (in Russian)
5. Kosov A.A., Semenov E.I. Multidimensional exact solutions to the reaction-diffusion system with power-law nonlinear terms. *Siberian Math. J.,* 2017, vol. 58, no. 4, pp. 619-632. https://doi.org/10.1134/S0037446617040085

6. Kosov A.A., Semenov E.I. On Exact Multidimensional Solutions of a Nonlinear System of Reaction–Diffusion Equations/*Differential Equations,* 2018, vol.54, no 1, pp. 106-120. https://doi.org/10.1134/S0012266118010093
7. Kurzhanski A.B. Comparison principle for equations of the Hamilton-Jacobi type in control theory. *Proceedings of the Steklov Institute of Mathematics,* 2006, vol. 253, supplement 1, pp. 185-195. https://doi.org/10.1134/S0081543806050130
8. Markov Y., Rudykh G., Sidorov N., Sinitsyn A., Tolstonogov D. Steady state solutions of the Vlasov-Maxwell system and their stability. *Acta Appl. Math.,* 1992, vol. 28, no. 3. pp. 253-293.
9. Polyanin A.D., Zajcev V.F., Zhurov A.I. *Metody resheniya nelinejnyx uravnenij matematicheskoj fiziki i mexaniki* [Methods for solving nonlinear equations of mathematical physics and mechanics]. Moscow, FIZMATLIT, 2005, 256 p. (in Russian)
10. Rudykh G.A., Semenov E.I. The construction of exact solutions of the multidimensional quasilinear heat-conduction equation. *Comput. Math. Math. Phys.,* 1993, vol. 33, no. 8, pp. 1087–1097.
11. Rudykh G.A., Semenov E.I. Exact nonnegative solutions of the multidimensional nonlinear diffusion equation. *Siberian Math. J.,* 1998, vol. 39, no. 5, pp. 977-985. https://doi.org/10.1007/BF02672920
12. Rudykh G.A., Semenov E.I. Non-self-similar solutions of multidimensional nonlinear diffusion equations. *Math. Notes,* 2000, vol. 67, no. 2, pp. 200–206. https://doi.org/10.1007/BF02686247
13. Sidorov N.A., Sinitsyn A.V. Stacionarnaya sistema Vlasova – Maksvella v ogranichennyx oblastyax. [The stationary Vlasov – Maxwell system in bounded domains]. *Nelinejnyj analiz i nelinejnye differencialnye uravneniya* [Nonlinear analysis and nonlinear differential equations]. Moscow, Fizmatlit Publ., 2003, pp. 50-88. (in Russian)
14. Skubachevskii A.L. Vlasov – Poisson equations for a two-component plasma in homogeneous magnetic field. *Russian Math. Surveys,* 2014, vol. 69, no. 2, pp. 291-335. http://dx.doi.org/10.1070/RM2014v069n02ABEH004889
15. Titov S.S. Metod konechnomernyh kolec dlya resheniya nelinejnyh uravnenij matematicheskoj fiziki [The method of finite-dimensional rings for solving nonlinear equations of mathematical physics] *Aerodinamika*, Saratov, Saratovskij Univ. Publ., 1988, pp. 104-110. (in Russian)
16. Zajcev V.F., Polyanin A.D. *Spravochnik po differencialnym uravneniyam s chastnymi proizvodnymi pervogo poryadka* [Handbook of differential equations with partial derivatives of the first order]. Moscow, Fizmatlit Publ., 2003, 416 p. (in Russian)

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О точных многомерных решениях одной нелинейной системы уравнений с частными производными первого порядка.

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Аннотация. В статье изучается система двух нелинейных уравнений в частных производных первого порядка. Правые части системы уравнений содержат квадраты градиентов искомых функций. Такого рода уравнения, близкие к уравнению Гамильтона – Якоби, встречаются в задачах механики и теории управления. В статье предлагается искать решение в виде анзаца, содержащего квадратичную зависимость от пространственных переменных и произвольные функции от времени. Использование предложенного анзаца позволяет декомпозировать процесс отыскания компонент решения зависящих от пространственных переменных и от времени. Для отыскания зависимости от пространственных переменных необходимо решать алгебраическую систему матричных, векторных и скалярного уравнения. Найдено общее решение этой системы уравнений в параметрическом виде. Для отыскания компонент решения исходной системы, зависящих от времени, возникает система нелинейных дифференциальных уравнений. Установлено существование точных решений определенного вида у исходной системы. Приводится ряд примеров построенных точных решений, в том числе периодические по времени и анизотропные по пространственным переменным. Проведен анализ пространственной структуры решений, установлено, что она зависит от ранга матрицы квадратичной формы, входящей в решение.

Ключевые слова: нелинейная система, уравнения типа Гамильтона – Якоби, точные решения.

Список литературы

1. Фущич В. И., Штелень В. М., Серов Н. И. Симметрийный анализ и точные решений нелинейных уравнений математической физики. Киев : Наукова думка, 1989. 336 с.
2. Галактионов В. А., Посашков С. С. О новых точных решениях параболических уравнений с квадратичными нелинейностями // Журн. вычисл. математики и мат. физики. 1989. Т. 29, № 4. С. 497–506.
3. Galactionov V. A., Svirshchevskii S. R. Subspaces of nonlinear partial differential equations in mechanics and physics. Chapman & Hall/CRC, 2007. 493 p.
4. Гантмахер Ф. Р. Теория матриц. М. : Наука, 1988. 552 с.
5. Косов А. А., Семенов Э. И. О точных многомерных решениях системы уравнений реакции-диффузии со степенными нелинейностями // Сиб. мат. журн. 2017. Т. 58, № 4. С. 796–812. https://doi.org/10.17377/smzh.2017.58.408
6. Косов А. А., Семенов Э. И. О точных многомерных решениях одной нелинейной системы уравнений реакции-диффузии // Дифференц. уравнения. 2018. Т. 54, № 1. С. 108–122. https://doi.org/10.1134/S0374064118010090
7. Куржанский А. Б. Принцип сравнения для уравнений типа Гамильтона — Якоби в теории управления // Тр. ИММ УрО РАН. 2006. Т. 12, № 1. С. 173—183.
8. Markov Y., Rudykh G., Sidorov N., Sinitsyn A., Tolstonogov D. Steady state solutions of the Vlasov-Maxwell system and their stability // Acta Appl. Math. 1992. Vol. 28, N 3. P. 253–293.
9. Полянин А. Д., Зайцев В. Ф., Журов А. И. Методы решения нелинейных уравнений математической физики и механики. М. : Физматлит, 2005. 256 с.
10. Рудых Г. А., Семенов Э. И. Построение точных решений многомерного уравнения нелинейной теплопроводности // Журн. вычисл. математики и мат. физики. 1993. Т. 33, № 8. С. 1228–1239.
11. Рудых Г. А., Семенов Э. И. Точные неотрицательные решения многомерного уравнения нелинейной диффузии // Сиб. мат. журн. 1998. Т. 39, № 5. С. 1131–1140. https://doi.org/10.1007/BF02672920
12. Рудых Г. А., Семенов Э. И. Неавтомодельные решения многомерного уравнения нелинейной диффузии // Мат. заметки. 2000. Т. 67. № 2. С. 250–256. https://doi.org/10.4213/mzm833
13. Сидоров Н. А., Синицын А. В. Стационарная система Власова-Максвелла в ограниченных областях // Нелинейный анализ и нелинейные дифференциальные уравнения. М. : Физматлит, 2003. С. 50-88.
14. Скубачевский А. Л. Уравнения Власова – Пуассона для двухкомпонентной плазмы в однородном магнитном поле // Успехи мат. наук. 2014. Т. 69, № 2. С. 107–148. https://doi.org/10.4213/rm9579
15. Титов С. С. Метод конечномерных колец для решения нелинейных уравнений математической физики // Аэродинамика. Саратов : Саратов. ун-т, 1988. С. 104–110.
16. Зайцев В. Ф., Полянин А. Д. Справочник по дифференциальным уравнениям с частными производными первого порядка. М. : Физматлит, 2003. 416 с.

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